MATH 2551-K MIDTERM 1
VERSION A
FALL 2023
COVERS SECTIONS 12.1-12.6, 13.1-13.4, 14.1

Full name: ______________________________ **GT ID:** __________________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	2
4	4
5	10
6	10
7	10
8	10
Total:	50

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) Any plane has only two distinct normal vectors.

   ○ **TRUE** √ **FALSE**

2. (2 points) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, then $\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{u} = 0$.

   √ **TRUE** ○ **FALSE**

3. (2 points) Order the following curves in increasing order of curvature.

   A) A helix of curvature 2

   B) A line segment of length 10

   C) A circle of radius 20

Solution: $B < C < A$

4. (4 points) Which of the following vectors could be the principal unit normal vector at time $t = 2$ to a curve whose tangent line at $t = 2$ is given by

$$\ell(p) = \langle 1, 0, -1 \rangle + p\langle 1, 2, 1 \rangle.$$

You must justify your answer to receive full credit.

A) $\langle 1, 1, 1 \rangle$

B) $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

C) $\langle -2, 1, 0 \rangle$

D) $\left\langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right\rangle$

Solution: The correct answer is D). Only B) and D) are unit vectors, and only D) is normal to the curve, since the dot product between it and the tangent vector $\langle 1, 2, 1 \rangle$ at $t = 2$ is 0, while the dot product with B is non-zero.

5. (10 points) Tobert, a hard-working documentarian, is hiking through the African savannah for two days and his path is given by the curve $\mathbf{r}(t) = \langle t^2, \frac{1}{3}t^3, 2\rangle$ for $0 \leq t < 48$, where t is measured in hours and $\mathbf{r}(t)$ in meters.

(a) What is Tobert's position two hours into his hike?

Solution: $\mathbf{r}(2) = \langle 4, 8/3, 2\rangle$

(b) Find a function $s(t)$ that gives the total distance Tobert has hiked for any time t since he started at $t = 0$.

Solution:

$$\begin{aligned} s(t) &= \int_0^t |\mathbf{r}'(\tau)| d\tau \\ &= \int_0^t |\langle 2\tau, \tau^2, 0\rangle| d\tau \\ &= \int_0^t \sqrt{4\tau^2 + \tau^4 + 0}\, d\tau \\ &= \int_0^t \tau\sqrt{4 + \tau^2}\, d\tau \end{aligned}$$

Let $u = 4 + \tau^2$, $du = 2\tau\ d\tau$. Then we have

$$\begin{aligned} s(t) &= \int_4^{4+t^2} \frac{1}{2}\sqrt{u}\, du \\ &= \frac{1}{3}u^{3/2}|_4^{4+t^2} \\ &= \frac{(4+t^2)^{3/2} - 8}{3} \end{aligned}$$

(c) How far has Tobert hiked when he is at the point $(4, 8/3, 2)$?

Solution: From a) and b), this is just $s(2)$.

$$s(2) = \frac{8^{3/2} - 8}{3} = \frac{8}{3}(2\sqrt{2} - 1)$$

6. In this problem, you will work with the differential equation

$$\mathbf{r}''(t) = 3\mathbf{i} + e^t\mathbf{j} + 49e^{7t}\mathbf{k}, \quad -\infty < t < \infty.$$

(a) (6 points) Find all vector-valued functions $\mathbf{r}(t)$ with $-\infty < t < \infty$ which are solutions to this equation.

Hint: Your answer to this part should include some undetermined constant vectors.

Solution: From the differential equation we get (via integration with respect to t):

$$\mathbf{r}'(t) = 3t\mathbf{i} + e^t\mathbf{j} + 7e^{7t}\mathbf{k} + \mathbf{C_1} \tag{1}$$

$$\mathbf{r}(t) = \frac{3}{2}t^2\mathbf{i} + e^t\mathbf{j} + e^{7t}\mathbf{k} + \mathbf{C_1}t + \mathbf{C_2}, \tag{2}$$

where $\mathbf{C_1}$ and $\mathbf{C_2}$ are constant vectors.

(b) (2 points) Find all vector-valued functions $\mathbf{r}(t)$ with $-\infty < t < \infty$ which are solutions to this equation and also have the property that $\mathbf{r}'(0) = 3\mathbf{j} + 7\mathbf{k}$.

Solution: Applying this condition to equation (1) gives

$$3\mathbf{j} + 7\mathbf{k} = \mathbf{r}'(0) = \mathbf{j} + 7\mathbf{k} + \mathbf{C_1}.$$

So we have $\mathbf{C_1} = 2\mathbf{j}$ and thus

$$\mathbf{r}(t) = \frac{3}{2}t^2\mathbf{i} + (e^t + 2t)\mathbf{j} + e^{7t}\mathbf{k} + \mathbf{C_2}. \tag{3}$$

(c) (2 points) Find all vector-valued functions $\mathbf{r}(t)$ with $-\infty < t < \infty$ which are solutions to this equation, have the property that $\mathbf{r}'(0) = 3\mathbf{j} + 7\mathbf{k}$ and also have the property that $\mathbf{r}(0) = \langle 1, 1, 5\rangle$.

Solution: Applying the new condition to equation (3) gives

$$\langle 1, 1, 5\rangle = \mathbf{r}(0) = \langle 0, 1, 1\rangle + \mathbf{C_2}.$$

So we have $\mathbf{C_2} = \langle 1, 0, 4\rangle$ and thus

$$\mathbf{r}(t) = (\frac{3}{2}t^2 + 1)\mathbf{i} + (e^t + 2t)\mathbf{j} + (e^{7t} + 4)\mathbf{k}.$$

7. Let p_1 be the plane defined by the equation $2x - 3y + z = -10$ and let Q be the point $(3, 1, 1)$.

   (a) (2 points) Find an equation for the plane p_2 which contains the point Q and is parallel to p_1.

   Solution: The normal vector to p_2 needs to be parallel to the normal vector of p_1, so we can take the same vector. Since the plane must contain Q, we use that as the reference point:

   $$2(x-3) - 3(y-1) + (z-1) = 0 \quad \text{or} \quad 2x - 3y + z = 4.$$

   (b) (2 points) Find an equation for the line ℓ which passes through Q and is orthogonal to both planes.

   Solution: Since ℓ is orthogonal to both planes, its direction vector is parallel to the normal of both planes, so we again take the same vector and use Q as our reference point.

   $$\ell(t) = \langle 2, -3, 1\rangle t + \langle 3, 1, 1\rangle.$$

   (c) (3 points) Find the point R where the line ℓ intersects the plane p_1.

   Solution: To find R, we plug the line equation into the plane equation.

   $$\begin{aligned} 2(2t+3) - 3(-3t+1) + (t+1) &= -10 \\ 4t + 6 + 9t - 3 + t + 1 &= -10 \\ 14t &= -14 \\ t &= -1 \end{aligned}$$

   So R is $\ell(-1) = \langle 1, 4, 0\rangle$.

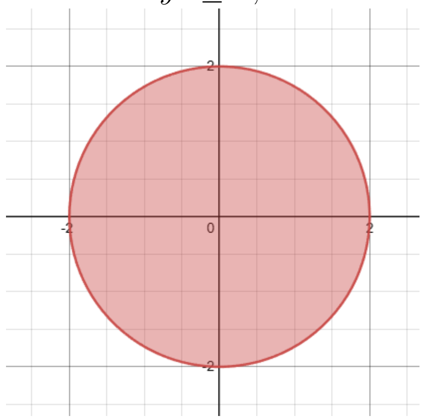
   (d) (3 points) Compute the distance between the two planes using your work in parts (a)-(c) above.

   Solution: The distance between the two planes is the length of any line segment orthogonal to both planes. Thus the segment of ℓ between Q and R gives the distance:

   $$|\vec{QR}| = \sqrt{(3-1)^2 + (1-4)^2 + (1-0)^2} = \sqrt{4+9+1} = \sqrt{14}.$$

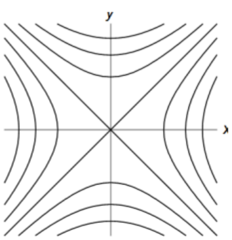
8. (a) (4 points) Find and sketch the domain of the function $f(x, y) = \sqrt{4 - x^2 - y^2}$. Show your work and label your axes clearly.

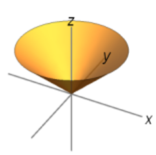
Solution: We need $4 - x^2 - y^2 \geq 0$, so the domain is all (x, y) inside or on the circle $x^2 + y^2 = 4$.

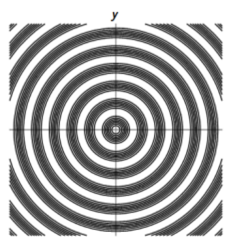


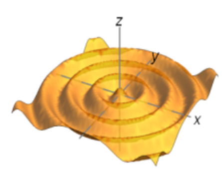
(b) (6 points) Match the contour plots and graphs below with the given functions of two variables.

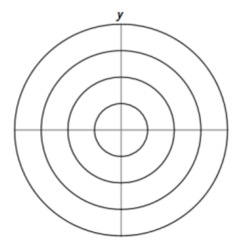
Function	Graph	Contour Plot
$f(x, y) = \cos\left(\sqrt{x^2 + y^2}\right)$	________	________
$g(x, y) = x^2 - y^2$	________	________
$h(x, y) = x^2 + 2y^2$	________	________

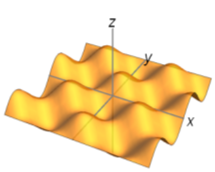

A

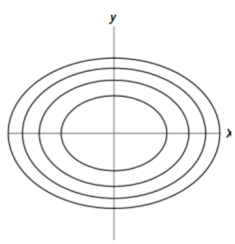

B

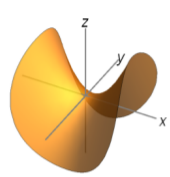

C

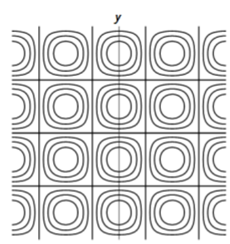

D

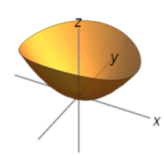

E


F


G


H


I


J

Solution: f: Graph D, Contour Plot C.
g: Graph H, Contour Plot A.
h: Graph J, Contour Plot G.

SCRATCH PAPER - PAGE LEFT BLANK

FORMULA SHEET

- $\langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$
- $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$
- $\langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin(\theta)|$
- $L = \displaystyle\int_a^b |\mathbf{r}'(t)|\ dt$
- $s(t) = \displaystyle\int_{t_0}^t |\mathbf{r}'(T)|\ dT$
- $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{d\mathbf{r}}{ds}$
- $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$
- $\mathbf{N} = \dfrac{1}{\kappa}\dfrac{d\mathbf{T}}{ds} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

SCRATCH PAPER - PAGE LEFT BLANK