**MATH 2551-D FINAL EXAM**
**VERSION S**
**SPRING 2023**
**COVERS SECTIONS 12.1-12.6, 13.1-13.4, 14.1-14.8, 15.1-15.8, 16.1-16.8**

**Full name:** ______________________________ **GT ID:**____________________

**Honor code statement:** I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

**I will not discuss the exam with anyone until Friday May 5.**

(     ) I attest to my integrity.

**Read all instructions carefully** before beginning.

- Print your name and GT ID neatly above.
- You have 170 minutes to take the exam.
- You may not use aids of any kind.
- Answers with little or no work shown will receive little or no credit.
- Good luck! Write yourself a message of encouragement on the front page!

| Question | Points |
|---|---|
| 1 | 2 |
| 2 | 2 |
| 3 | 2 |
| 4 | 2 |
| 5 | 2 |
| 6 | 4 |
| 7 | 4 |
| 8 | 4 |
| 9 | 4 |

| Question | Points |
|---|---|
| 10 | 4 |
| 11 | 10 |
| 12 | 10 |
| 13 | 10 |
| 14 | 10 |
| 15 | 10 |
| 16 | 10 |
| 17 | 10 |
| Total: | 100 |

Choose whether the following statements are true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false.

1. (2 points) Any three points in 3 space determine a unique plane.

   ○ **TRUE** √ **FALSE**

2. (2 points) The vector field $\mathbf{F}$ is defined everywhere in a volume $D$ bounded by a surface $S$. If $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma > 0$, then $\nabla \cdot \mathbf{F} > 0$ at some points in $D$.

   √ **TRUE** ○ **FALSE**

3. (2 points) If $w = f(x, y)$ and $x = g(s, t), y = h(s, t)$ such that $g(1, 0) = 2, h(1, 0) = 1$, then to compute $\frac{\partial w}{\partial s}$ at the point $(s, t) = (1, 0)$ we need to know $f_x(2, 1), f_y(2, 1), g_s(1, 0)$, and $h_s(1, 0)$.

   √ **TRUE** ○ **FALSE**

4. (2 points) If the Death Star is in an elliptical orbit around the forest moon of Endor and is moving at a constant speed of 5770 km/h then its acceleration is zero.

   ○ **TRUE** √ **FALSE**

5. (2 points) If the limit of $f(x, y)$ as $(x, y)$ approaches $(0, 0)$ along every straight line through the origin is 5, then

$$\lim_{(x,y)\to(0,0)} f(x, y) = 5.$$

   ○ **TRUE** √ **FALSE**

6. (4 points) Which transformation can be used to simplify the integral below?

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy$$

*Hint: the domain of integration is the region bounded by* $y = 2x, y = 2x - 4, y = 0, y = 2$

○ **A)** $u = x, v = y$

○ **B)** $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$

○ **C)** $u = 2x - y, v = y^7$

√ **D)** $u = 2x - y, v = y$

○ **E)** None of the above.

7. (4 points) The parameterization $\mathbf{r}(u,v) = \langle 2\sin(u), 3\cos(u), v\rangle, 0 \le u \le 2\pi, 0 \le v \le 2$ describes part of which surface?

○ **A)** A sphere of radius $\sqrt{13}$

○ **B)** A circular cylinder centered on the $z$-axis

√ **C) An elliptical cylinder centered on the $z$-axis**

○ **D)** An elliptical paraboloid centered on the $z$-axis

○ **E)** An elliptical paraboloid centered on the $x$-axis

8. (4 points) Suppose $\mathbf{F}$ is the velocity field in a pool of swirling swamp water on Dagobah. If one of the round o-rings (forming a curve $C$) from Luke's X-wing falls into the swamp and $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$ is positive, which statement below is correct, assuming standard orientations?

○ **A)** The net flow of the water around the loop is counterclockwise.

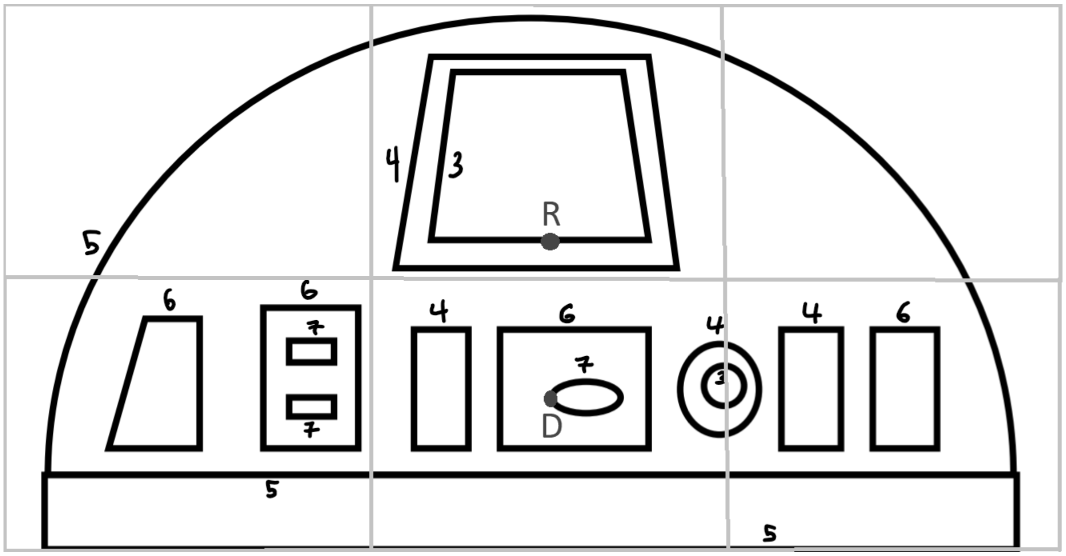
○ **B)** The net flow of the water around the loop is clockwise.

√ **C) The net flux of the water across the loop is from the inside to the outside.**

○ **D)** The net flux of the water across the loop is from the outside to the inside.

○ **E)** None of the above.

The next two questions refer to the contour plot below of an unknown function $f$.




9. (4 points) What is the sign of the directional derivative of $f$ at the point $R$ in the direction from $R$ to $D$?

   √ **A) Positive**

   ◯ **B)** Negative

   ◯ **C)** Zero

   ◯ **D)** Undefined

   ◯ **E)** Cannot be determined.

10. (4 points) In the contour plot above, the six grey squares are all 1 unit by 1 unit. Which of the following expressions is certainly larger than the volume under the graph of $f$ in the pictured region?

    √ **A)** $1*5+1*5+1*5+1*7+1*7+1*6$

    ◯ **B)** $1*4+1*3+1*4+1*5+1*3+1*3$

    ◯ **C)** $1*5+1*5+1*5+1*5+1*5+1*5$

    ◯ **D)** 0

    ◯ **E)** None of the above.

11. (a) (4 points) Is there a unique line through the point $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$? Explain why or why not.

**Solution:** No. There is an entire plane through $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$ (the plane $x + y + z = 3$), and this plane contains infinitely many lines through $(0, 1, 2)$. Any of those lines is a line through the given point parallel to the given plane.

(b) (6 points) Find parametric equations for the line through the point $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$ and perpendicular to the line

$$\ell : x = 1 + t \quad y = 1 - t \quad z = 2t.$$

**Solution:** If $\ell_2$ is a line parallel to $x + y + z = 2$, its direction vector must be orthogonal to the normal vector $\langle 1, 1, 1\rangle$ of the plane. We are also told that the desired line is perpendicular to the line $\ell$, so its direction vector is orthogonal to $\langle 1, -1, 2\rangle$. Thus the direction vector is a multiple of

$$\langle 1, 1, 1\rangle \times \langle 1, -1, 2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \langle 3, -1, -2\rangle.$$

Hence $\ell_2$ is the line with

$$x(t) = 3t, \quad y(t) = -t + 1, \quad z(t) = -2t + 2.$$

12. (10 points) Find the extreme values of the function $f(x, y) = xy^2$ on the domain $R = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$.

**Solution:** First, we find the critical points of $f$ inside $R$. $Df(x, y) = \begin{bmatrix} y^2 & 2xy \end{bmatrix}$, so we need $y^2 = 0$ and $2xy = 0$. Thus all points $(x, 0)$ in $R$ are critical points of $f$ and at all such points we have $f(x, 0) = 0$.

We already considered the boundary $y = 0$ of $R$ above. On $x = 0$, we have $f = 0$ again at all points.

Finally, on $x^2 + y^2 = 3$, we have $y^2 = 3 - x^2$, so $f(x, y) = x(3 - x^2) = 3x - x^3$ and so $f'(x) = 3 - 3x^2$. This has a critical point at $x = 1$, so we have $y = \sqrt{3 - 1} = \sqrt{2}$. $f(1, \sqrt{2}) = 2$, so comparing all values shows that the minimum value of $f$ on $R$ is 0 and the maximum value of $f$ on $R$ is 2.

13. (10 points) Consider the volume $D$ in the first octant of $\mathbb{R}^3$ which is bounded above by the sphere $x^2 + y^2 + z^2 = 12$ and below by the paraboloid $z = x^2 + y^2$. If $D$ has a constant density function $\delta = 16$, compute $M_{xy}$, the first moment of $D$ about the $xy$-plane. Fully simplify your answer.

*Hint: Be careful with your choice of coordinate system.*

**Solution:** We have $M_{xy} = \iiint_D z\delta\ dV$, so we compute $\iiint_D 16z\ dV$. This is easiest in cylindrical coordinates, in which $D$ is described by $r^2 \leq z \leq \sqrt{12 - r^2}$, $0 \leq \theta \leq \pi/2$. To find bounds on $r$, we solve $r^2 = \sqrt{12 - r^2}$ to get $r^4 = 12 - r^2$, which then gives $(r^2 + 4)(r^2 - 3) = 0$. Thus we have $0 \leq r \leq \sqrt{3}$.

Now we integrate:

$$\begin{aligned} M_{xy} = \iiint_D 16z\ dV &= \int_0^{\pi/2} \int_0^{\sqrt{3}} \int_{r^2}^{\sqrt{12-r^2}} 16zr\ dz\ dr\ d\theta \\ &= \frac{\pi}{2} \int_0^{\sqrt{3}} (8rz^2)\Big|_{r^2}^{\sqrt{12-r^2}}\ dr = 4\pi \int_0^{\sqrt{3}} r(12 - r^2 - r^4)\ dr \\ &= 4\pi \int_0^{\sqrt{3}} 12r - r^3 - r^5\ dr \\ &= 4\pi(6r^2 - \frac{1}{4}r^4 - \frac{1}{6}r^6)\Big|_0^{\sqrt{3}} \\ &= 4\pi(18 - \frac{9}{4} - \frac{27}{6}) = \boxed{45\pi} \end{aligned}$$

14. In this problem, you will work with the curve $C$ that consists of the quarter of the circle $x^2 + y^2 = 9$ from $(3, 0)$ to $(0, 3)$, followed by the straight line segment back to $(3, 0)$.

(a) (6 points) Compute $\oint_C y\ ds$.

**Solution:** We first parameterize the two portions of $C$. The quarter circle is parameterized by $\mathbf{r}_1(t) = \langle 3\cos(t), 3\sin(t)\rangle$ for $0 \leq t \leq \pi/2$ and the straight line segment by $\mathbf{r}_2(t) = \langle 3t, 3 - 3t\rangle$ for $0 \leq t \leq 1$.

The scaling factors for each of these parameterizations are

$$|\mathbf{r}_1'(t)| = |\langle -3\sin(t), 3\cos(t)\rangle| = 3 \text{ and } |\mathbf{r}_2'(t)| = |\langle 3, -3\rangle| = 3\sqrt{2}.$$

Thus we have

$$\begin{aligned}\oint_C y\ ds &= \int_{C_1} y\ ds + \int_{C_2} y\ ds \\ &= \int_0^{\pi/2} 3\sin(t)(3)\ dt + \int_0^1 (3 - 3t)(3\sqrt{2})\ dt \\ &= -9\cos(t)|_0^{\pi/2} + 3\sqrt{2}(3t - \frac{3}{2}t^2)|_0^1 \\ &= 9 + \frac{9}{\sqrt{2}}\end{aligned}$$

(b) (4 points) Compute $\oint_C y\mathbf{j} \cdot d\mathbf{r}$.

**Solution:** We could compute this directly with the parameterization above. We could also note that $\mathbf{F} = y\mathbf{j}$ is a conservative vector field, since it is the gradient of $f(x, y) = \frac{1}{2}y^2$ and therefore this integral over a closed curve is 0. We could also apply Green's Theorem: the scalar curl of $\mathbf{F}$ is 0 and so the integral is 0.

15. (a) (6 points) Show that the vector field $\mathbf{F} = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k}$ is conservative by computing a potential function for it.

**Solution:** We want to compute $f$ such that $\nabla f = \mathbf{F}$. We have $f_x = y^2z + 2xz^2$, so $f(x, y) = \int y^2z + 2xz^2\ dx = xy^2z + x^2z^2 + g(y, z)$.
Taking the partial derivative with respect to $y$ and comparing to $\mathbf{F}$ gives

$$2xyz = f_y = 2xyz + g_y(y, z).$$

From this we see $g_y(y, z) = 0$ and so $g(y, z) = 0 + h(z)$. Finally, we repeat with the partial derivative with respect to $z$:

$$xy^2 + 2x^2z = f_z = xy^2 + 2x^2z + h'(z).$$

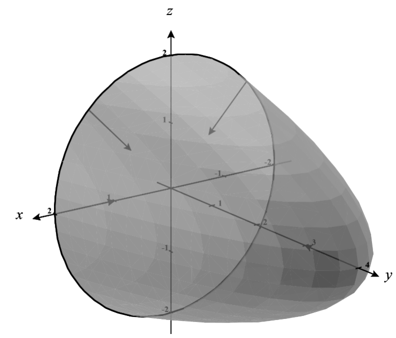
So $h'(z) = 0$ and therefore a potential function for $\mathbf{F}$ is $f(x, y) = xy^2z + x^2z^2$.

(b) (4 points) Compute $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$, where $C$ is the curve $\mathbf{r}(t) = \langle \sqrt{t}, t+1, t^2 \rangle$ for $0 \le t \le 1$.

**Solution:** Since $\mathbf{F}$ is conservative, we can apply the fundamental theorem of line integrals.

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T}\ ds &= f(\mathbf{r}(1)) - f(\mathbf{r}(0)) \\ &= f(1, 2, 1) - f(0, 1, 0) \\ &= (4 + 1) - (0 + 0) \\ &= 5\end{aligned}$$

16. In this problem, you will compute the flux of the vector field $\mathbf{F} = \langle x - 2xy, 1 - y, 2zy - 2xy \rangle$ across the surface $S$ consisting of the potion of the paraboloid $y = 4 - x^2 - z^2$ with $y \geq 0$, oriented with normal vectors towards the origin.




(a) (2 points) Show that $\mathbf{F}$ is the curl of the vector field $\mathbf{G} = \langle xy^2 + z, 2xyz, xy \rangle$.

**Solution:**

$$\nabla \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 + z & 2xyz & xy \end{vmatrix} = \langle x - 2xy, -(y-1), 2yz - 2xy \rangle = \mathbf{F}.$$

(b) (3 points) Find a parameterization of the boundary curve $C$ of $S$, oriented compatibly with $S$.

*Hint: The boundary is the part of $S$ in the plane $y = 0$.*

**Solution:** $C$ is the circle $0 = 4 - x^2 - z^2$ in the plane $y = 0$, oriented from the $x$-axis towards the $z$-axis.

One suitable parameterization is therefore $\mathbf{r}(t) = \langle 2\cos(t), 0, 2\sin(t) \rangle$ for $0 \leq t \leq 2\pi$.

(c) (5 points) Apply Stokes' Theorem to find the flux of $\mathbf{F}$ across $S$.

**Solution:** Note $\mathbf{r}'(t) = \langle -2\sin(t), 0, 2\cos(t) \rangle$. Applying Stokes' Theorem and parts a) and b), we have:

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \iint_S (\nabla \times \mathbf{G}) \cdot \mathbf{n}\, d\sigma \\
&= \int_C \mathbf{G} \cdot \mathbf{T}\, ds \\
&= \int_0^{2\pi} \langle 2\sin(t), 0, 0 \rangle \cdot \langle -2\sin(t), 0, 2\cos(t) \rangle\, dt \\
&= \int_0^{2\pi} -4\sin^2(t)\, dt \\
&= \int_0^{2\pi} -2 + 2\cos(2t)\, dt \\
&= -2t + \sin(2t)\big|_0^{2\pi} \\
&= -4\pi
\end{aligned}$$

17. (10 points) Compute the flux of the vector field $\mathbf{F}(x,y,z) = 3xy^2\mathbf{i} + xe^z\mathbf{j} + z^3\mathbf{k}$ across the surface $S$ of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$ using any method.

*Hint: A variant of cylindrical coordinates may be useful.*

**Solution:** The surface $S$ is closed since it bounds a solid and thus we can apply the divergence theorem. The solid in question is most easily integrated over using a cylindrical coordinate system where $r$ is the radial distance from the $x$-axis and $\theta$ is the angle in the $yz$-plane from the positive $y$-axis.

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \iiint_D \nabla\cdot\mathbf{F}\,dV \\
&= \iiint_D 3y^2 + 3z^2\,dV \\
&= \int_0^{2\pi}\int_0^1\int_{-1}^2 3r^2(r)\,dx\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 3r^3x|_{-1}^2\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 9r^3\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{9}{4}r^4|_0^1\,d\theta \\
&= \frac{9\pi}{2}
\end{aligned}$$

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# FORMULA SHEET

- $\langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$
- $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$
- $\langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin(\theta)|$
- $L = \int_a^b |\mathbf{r}'(t)|\ dt$
- $s(t) = \int_{t_0}^t |\mathbf{r}'(T)|\ dT$
- $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{r}}{ds}$
- $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$
- $\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$
- Trig identities:
  $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$,
  $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$
- Polar coordinates:
  $x = r\cos(\theta), \quad y = r\sin(\theta)$,
  $dA = r\ dr\ d\theta$
- Cylindrical coordinates:
  $x = r\cos(\theta), \quad y = r\sin(\theta)$,
  $z = z$,
  $dV = r\ dz\ dr\ d\theta$
- Spherical coordinates:
  $x = \rho\sin(\phi)\cos(\theta)$,
  $y = \rho\sin(\phi)\sin(\theta)$,
  $z = \rho\cos(\phi)$,
  $dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$
- For $\mathbf{f}(x_1,\ldots,x_n) = \langle f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n)\rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Near $\mathbf{a}$, $L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x}-\mathbf{a})$
- If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$
- Implicit Differentiation: $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$ and $\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$.
- If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P)\cdot\mathbf{u}$
- The tangent line to a level curve of $f(x,y)$ at $(a,b)$ is $0 = \nabla f(a,b)\cdot\langle x-a, y-b\rangle$
- The tangent plane to a level surface of $f(x,y,z)$ at $(a,b,c)$ is

$$0 = \nabla f(a,b,c)\cdot\langle x-a, y-b, z-c\rangle.$$

- For $f(x,y)$, $Hf(x,y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$
- If $(a,b)$ is a critical point of $f(x,y)$ then
  1. If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) < 0$ then $f$ has a local maximum at $(a,b)$
  2. If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) > 0$ then $f$ has a local minimum at $(a,b)$
  3. If $\det(Hf(a,b)) < 0$ then $f$ has a saddle point at $(a,b)$
  4. If $\det(Hf(a,b)) = 0$ the test is inconclusive
- Area/volume: $\text{area}(R) = \iint_R dA, \quad \text{volume}(D) = \iiint_D dV$
- $f_{avg} = \frac{\iint_R f(x,y)dA}{\text{area of } R}$, or $f_{avg} = \frac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$
- Mass: $M = \iint_D \delta\ dA$ or $M = \iiint_D \delta\ dV$

## FORMULA SHEET

- First moments (2D plate): $M_y = \iint_R x\delta \, dA, \quad M_x = \iint_R y\delta \, dA$
- Center of mass (2D plate): $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
- First moments (3D solid): $M_{yz} = \iiint_D x\delta \, dV$, $M_{xz} = \iiint_D y\delta \, dV$, $M_{xy} = \iiint_D z\delta \, dV$
- Center of mass (3D solid): $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$
- Substitution for double integrals: If $R$ is the image of $G$ under a coordinate transformation $\mathbf{T}(u, v) = \langle x(u, v), y(u, v)\rangle$ then
$$\iint_R f(x, y) \, dx \, dy = \iint_G f(\mathbf{T}(u, v))|\det D\mathbf{T}(u, v)| \, du \, dv.$$
- Scalar line integral: $\int_C f(x, y, z) \, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- Tangential vector line integral: $\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$
- Normal vector line integral: $\int_C \mathbf{F}(x, y) \cdot \mathbf{n} \, ds = \int_C P \, dy - Q \, dx = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t)\rangle \, dt$.
- Fundamental Theorem of Line Integrals: $\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$ if $C$ is a path from $A$ to $B$
- Mixed Partials Test: $\mathbf{F} = \nabla f$ if $P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.
- $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$ $\quad$ div $\mathbf{F} = \nabla \cdot \mathbf{F}$ $\quad$ curl $\mathbf{F} = \nabla \times \mathbf{F}$
- Green's Theorem: If $C$ is a simple closed curve with positive orientation and $R$ is the simply-connected region it encloses, then
$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA \qquad \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R (\nabla \cdot \mathbf{F}) \, dA.$$
- Surface Area=$\iint_S 1 \, d\sigma$
- Scalar surface integral: $\iint_S f(x, y, z) \, d\sigma = \iint_R f(\mathbf{r}(u, v)) \, |\mathbf{r}_u \times \mathbf{r}_v|dA$
- Flux surface integral: $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S \mathbf{F} \cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$
- Stokes' Theorem: If $S$ is a piecewise smooth oriented surface bounded by a piecewise smooth curve $C$ and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing $S$, then
$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma.$$
- Divergence Theorem: If $S$ is a piecewise smooth closed oriented surface enclosing a volume $D$ and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on $D$, then
$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV.$$