MATH 2551-D FINAL EXAM
VERSION S
SPRING 2023
COVERS SECTIONS 12.1-12.6, 13.1-13.4, 14.1-14.8, 15.1-15.8, 16.1-16.8

Full name: ______________________________ **GT ID:**____________________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

I will not discuss the exam with anyone until Friday May 5.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 170 minutes to take the exam.
- You may not use aids of any kind.
- Answers with little or no work shown will receive little or no credit.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points	Question	Points
1	2	10	4
2	2	11	10
3	2	12	10
4	2	13	10
5	2	14	10
6	4	15	10
7	4	16	10
8	4	17	10
9	4	Total:	100

Choose whether the following statements are true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false.

1. (2 points) Any three points in 3 space determine a unique plane.

   ◯ **TRUE** ◯ **FALSE**

2. (2 points) The vector field $\mathbf{F}$ is defined everywhere in a volume D bounded by a surface S. If $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma > 0$, then $\nabla \cdot \mathbf{F} > 0$ at some points in D.

   ◯ **TRUE** ◯ **FALSE**

3. (2 points) If $w = f(x, y)$ and $x = g(s, t), y = h(s, t)$ such that $g(1, 0) = 2, h(1, 0) = 1$, then to compute $\frac{\partial w}{\partial s}$ at the point $(s, t) = (1, 0)$ we need to know $f_x(2, 1), f_y(2, 1), g_s(1, 0)$, and $h_s(1, 0)$.

   ◯ **TRUE** ◯ **FALSE**

4. (2 points) If the Death Star is in an elliptical orbit around the forest moon of Endor and is moving at a constant speed of 5770 km/h then its acceleration is zero.

   ◯ **TRUE** ◯ **FALSE**

5. (2 points) If the limit of $f(x, y)$ as (x, y) approaches $(0, 0)$ along every straight line through the origin is 5, then

$$\lim_{(x,y)\to(0,0)} f(x, y) = 5.$$

   ◯ **TRUE** ◯ **FALSE**

6. (4 points) Which transformation can be used to simplify the integral below?

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy$$

Hint: the domain of integration is the region bounded by $y = 2x, y = 2x - 4, y = 0, y = 2$

- ○ **A)** $u = x, v = y$
- ○ **B)** $u = \sqrt{x^2 + y^2}, v = \arctan(y/x)$
- ○ **C)** $u = 2x - y, v = y^7$
- ○ **D)** $u = 2x - y, v = y$
- ○ **E)** None of the above.

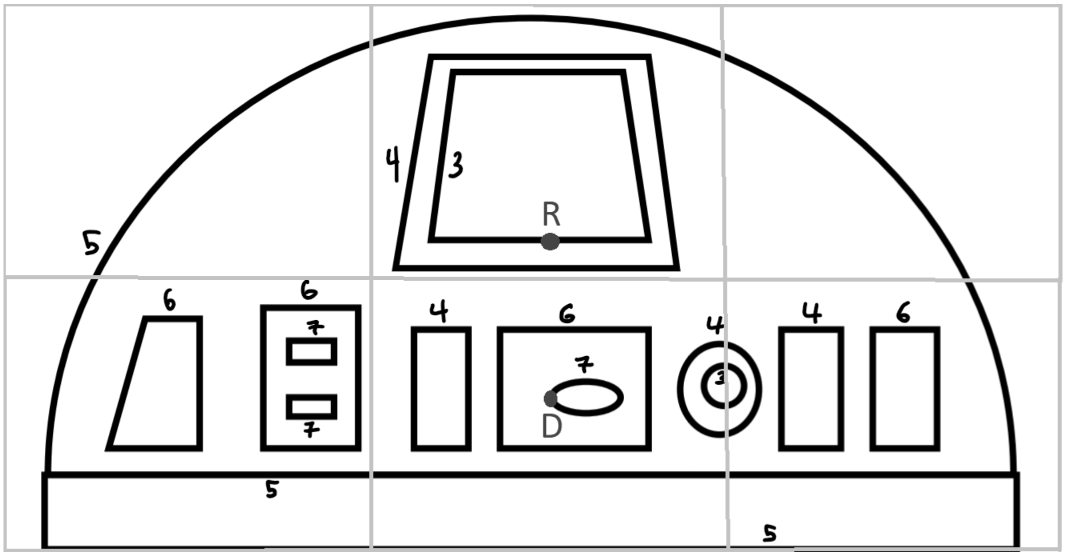
7. (4 points) The parameterization $\mathbf{r}(u, v) = \langle 2\sin(u), 3\cos(u), v\rangle, 0 \le u \le 2\pi, 0 \le v \le 2$ describes part of which surface?

- ○ **A)** A sphere of radius $\sqrt{13}$
- ○ **B)** A circular cylinder centered on the z-axis
- ○ **C)** An elliptical cylinder centered on the z-axis
- ○ **D)** An elliptical paraboloid centered on the z-axis
- ○ **E)** An elliptical paraboloid centered on the x-axis

8. (4 points) Suppose $\mathbf{F}$ is the velocity field in a pool of swirling swamp water on Dagobah. If one of the round o-rings (forming a curve C) from Luke's X-wing falls into the swamp and $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$ is positive, which statement below is correct, assuming standard orientations?

- ○ **A)** The net flow of the water around the loop is counterclockwise.
- ○ **B)** The net flow of the water around the loop is clockwise.
- ○ **C)** The net flux of the water across the loop is from the inside to the outside.
- ○ **D)** The net flux of the water across the loop is from the outside to the inside.
- ○ **E)** None of the above.

The next two questions refer to the contour plot below of an unknown function f.



9. (4 points) What is the sign of the directional derivative of f at the point R in the direction from R to D?

   ◯ **A)** Positive

   ◯ **B)** Negative

   ◯ **C)** Zero

   ◯ **D)** Undefined

   ◯ **E)** Cannot be determined.

10. (4 points) In the contour plot above, the six grey squares are all 1 unit by 1 unit. Which of the following expressions is certainly larger than the volume under the graph of f in the pictured region?

    ◯ **A)** $1*5+1*5+1*5+1*7+1*7+1*6$

    ◯ **B)** $1*4+1*3+1*4+1*5+1*3+1*3$

    ◯ **C)** $1*5+1*5+1*5+1*5+1*5+1*5$

    ◯ **D)** 0

    ◯ **E)** None of the above.

11. (a) (4 points) Is there a unique line through the point $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$? Explain why or why not.

(b) (6 points) Find parametric equations for the line through the point $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$ and perpendicular to the line

$$\ell : x = 1 + t \quad y = 1 - t \quad z = 2t.$$

12. (10 points) Find the extreme values of the function $f(x, y) = xy^2$ on the domain $R = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$.

13. (10 points) Consider the volume D in the first octant of $\mathbb{R}^3$ which is bounded above by the sphere $x^2 + y^2 + z^2 = 12$ and below by the paraboloid $z = x^2 + y^2$. If D has a constant density function $\delta = 16$, compute M_{xy}, the first moment of D about the xy-plane. Fully simplify your answer.

    Hint: Be careful with your choice of coordinate system.

14. In this problem, you will work with the curve C that consists of the quarter of the circle $x^2 + y^2 = 9$ from $(3, 0)$ to $(0, 3)$, followed by the straight line segment back to $(3, 0)$.

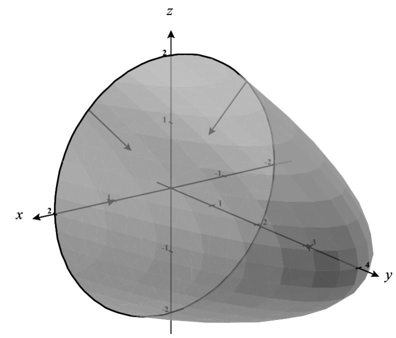
(a) (6 points) Compute $\oint_C y\, ds$.

(b) (4 points) Compute $\oint_C y\mathbf{j} \cdot d\mathbf{r}$.

15. (a) (6 points) Show that the vector field $\mathbf{F} = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k}$ is conservative by computing a potential function for it.

(b) (4 points) Compute $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$, where C is the curve $\mathbf{r}(t) = \langle\sqrt{t}, t+1, t^2\rangle$ for $0 \le t \le 1$.

16. In this problem, you will compute the flux of the vector field $\mathbf{F} = \langle x - 2xy, 1 - y, 2zy - 2xy \rangle$ across the surface S consisting of the potion of the paraboloid $y = 4 - x^2 - z^2$ with $y \geq 0$, oriented with normal vectors towards the origin.



(a) (2 points) Show that $\mathbf{F}$ is the curl of the vector field $\mathbf{G} = \langle xy^2 + z, 2xyz, xy \rangle$.

(b) (3 points) Find a parameterization of the boundary curve C of S, oriented compatibly with S.

Hint: The boundary is the part of S in the plane $y = 0$.

(c) (5 points) Apply Stokes' Theorem to find the flux of $\mathbf{F}$ across S.

17. (10 points) Compute the flux of the vector field $\mathbf{F}(x, y, z) = 3xy^2\mathbf{i} + xe^z\mathbf{j} + z^3\mathbf{k}$ across the surface S of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$ using any method.

    Hint: A variant of cylindrical coordinates may be useful.

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FORMULA SHEET

- $\langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$
- $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$
- $\langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin(\theta)|$
- $L = \int_a^b |\mathbf{r}'(t)|\ dt$
- $s(t) = \int_{t_0}^t |\mathbf{r}'(T)|\ dT$
- $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{r}}{ds}$
- $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$
- $\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$
- Trig identities:
  $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$,
  $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$
- Polar coordinates:
  $x = r\cos(\theta), \quad y = r\sin(\theta)$,
  $dA = r\ dr\ d\theta$
- Cylindrical coordinates:
  $x = r\cos(\theta), \quad y = r\sin(\theta)$,
  $z = z$,
  $dV = r\ dz\ dr\ d\theta$
- Spherical coordinates:
  $x = \rho\sin(\phi)\cos(\theta)$,
  $y = \rho\sin(\phi)\sin(\theta)$,
  $z = \rho\cos(\phi)$,
  $dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$
- For $\mathbf{f}(x_1,\ldots,x_n) = \langle f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n)\rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Near $\mathbf{a}$, $L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x}-\mathbf{a})$
- If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$
- Implicit Differentiation: $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$ and $\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$.
- If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P)\cdot\mathbf{u}$
- The tangent line to a level curve of $f(x,y)$ at (a,b) is $0 = \nabla f(a,b)\cdot\langle x-a, y-b\rangle$
- The tangent plane to a level surface of $f(x,y,z)$ at (a,b,c) is

$$0 = \nabla f(a,b,c)\cdot\langle x-a, y-b, z-c\rangle.$$

- For $f(x,y)$, $Hf(x,y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$
- If (a,b) is a critical point of $f(x,y)$ then
  1. If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) < 0$ then f has a local maximum at (a,b)
  2. If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) > 0$ then f has a local minimum at (a,b)
  3. If $\det(Hf(a,b)) < 0$ then f has a saddle point at (a,b)
  4. If $\det(Hf(a,b)) = 0$ the test is inconclusive
- Area/volume: $\text{area}(R) = \iint_R dA$, $\text{volume}(D) = \iiint_D dV$
- $f_{avg} = \frac{\iint_R f(x,y)dA}{\text{area of } R}$, or $f_{avg} = \frac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$
- Mass: $M = \iint_D \delta\ dA$ or $M = \iiint_D \delta\ dV$

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- First moments (2D plate): $M_y = \iint_R x\delta \, dA, \quad M_x = \iint_R y\delta \, dA$
- Center of mass (2D plate): $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
- First moments (3D solid): $M_{yz} = \iiint_D x\delta \, dV$, $M_{xz} = \iiint_D y\delta \, dV$, $M_{xy} = \iiint_D z\delta \, dV$
- Center of mass (3D solid): $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$
- Substitution for double integrals: If R is the image of G under a coordinate transformation $\mathbf{T}(u, v) = \langle x(u, v), y(u, v)\rangle$ then
$$\iint_R f(x, y) \, dx \, dy = \iint_G f(\mathbf{T}(u, v))|\det D\mathbf{T}(u, v)| \, du \, dv.$$
- Scalar line integral: $\int_C f(x, y, z) \, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- Tangential vector line integral: $\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$
- Normal vector line integral: $\int_C \mathbf{F}(x, y) \cdot \mathbf{n} \, ds = \int_C P \, dy - Q \, dx = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t)\rangle \, dt$.
- Fundamental Theorem of Line Integrals: $\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Mixed Partials Test: $\mathbf{F} = \nabla f$ if $P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.
- $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ div $\mathbf{F} = \nabla \cdot \mathbf{F}$ curl $\mathbf{F} = \nabla \times \mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then
$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA \qquad \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R (\nabla \cdot \mathbf{F}) \, dA.$$
- Surface Area=$\iint_S 1 \, d\sigma$
- Scalar surface integral: $\iint_S f(x, y, z) \, d\sigma = \iint_R f(\mathbf{r}(u, v)) \, |\mathbf{r}_u \times \mathbf{r}_v| dA$
- Flux surface integral: $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S \mathbf{F} \cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$
- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then
$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma.$$
- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then
$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV.$$