MATH 2551-D MIDTERM 3
VERSION A
SPRING 2023
COVERS SECTIONS 15.1-15.8

Full name: ______________________________ **GT ID:**____________________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Answers with little or no work shown will receive little or no credit.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	2
4	3
5	8
6	9
7	14
8	10
Total:	50

For problems 1-3 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points)

$$\int_5^6 \int_7^8 x^3 e^y \, dy \, dx = \left(\int_5^6 x^3 \, dx \right) \left(\int_7^8 e^y \, dy \right)$$

√ **TRUE** ○ **FALSE**

2. (2 points) If $g(x, y)$ is strictly positive at all points in $\mathbb{R}^2$, then

$$\iint_{R_1} g(x, y) \, dA > \iint_{R_2} g(x, y) \, dA$$

if the area of R_1 is larger than the area of R_2.

○ **TRUE** √ **FALSE**

3. (2 points) The integral below might compute the total mass of a solid sphere of radius 1 centered at the origin which has constant density $\delta = 2$.

$$\int_0^{2\pi} \int_0^1 2r \, dr \, d\theta$$

○ **TRUE** √ **FALSE**

4. (3 points) Suppose the contour plot shown gives the height of a pile of dirt in feet. Which of the following is the **best** lower bound for the volume of dirt in the hill?

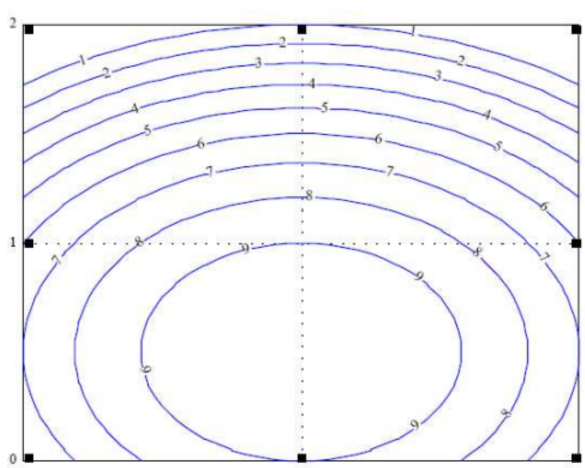
√ **A) 0*1+0*1+6*1+6*1**

○ **B)** 9*1+9*1+9*1+9*1

○ **C)** 9

○ **D)** 6*1+6*1+9*1+9*1

○ **E)** 0



5. The temperature in degrees Celsius of a thin rectangular metal plate located on the region $[0,1] \times [1,4]$ is given by

$$T(x,y) = (x^2 + \sqrt{y})\sin(x^2y^2).$$

Finding the average temperature on the plate exactly is very difficult, but we can find an upper bound for it.

(a) (2 points) Find the area of the plate.

Solution: The plate is a rectangular of width 1 and height 3, so its area is 3.

(b) (2 points) Write an expression including a double integral for the average temperature on the plate.

Solution:

$$T_{avg} = \frac{\int_0^1 \int_1^4 (x^2 + \sqrt{y})\sin(x^2y^2)\ dy\ dx}{3}$$

(c) (4 points) Show that the average temperature on the plate is at most 3 degrees Celsius. *Hint: It may be useful to consider the maximum temperature on the plate.*

Solution: Since $x \leq 1$ on the plate, $x^2 \leq 1$ and since $y \leq 4$ on the plate, $\sqrt{y} \leq 2$. Further, $\sin(x^2y^2) \leq 1$, so $T(x,y) \leq (1+2)(1) = 3$ everywhere on the plate.

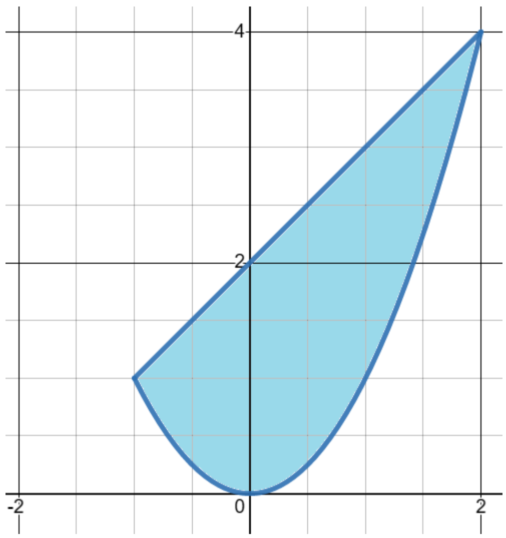
Then our integral is at most

$$\frac{\int_0^1 \int_1^4 3\ dy\ dx}{3} = 3.$$

6. (a) (4 points) Sketch the region of integration for and give an interpretation of the value of the integral expression below.

$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy$$

Solution: The integral expression computes the area of the region below.



(b) (3 points) Write a new integral expression with one or more integrals for the same value, written in the opposite order of integration.

Solution:

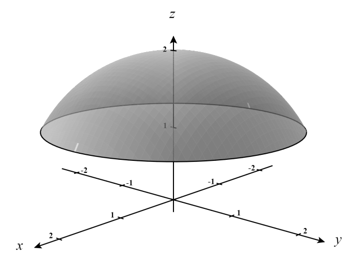
$$\int_{-1}^2 \int_{x^2}^{x+2} dy\, dx$$

(c) (2 points) Explain which expression you would rather evaluate. You do **not** need to actually evaluate either integral.

Solution: Many possible answers here. I prefer the new expression since it involves only one integral and polynomials rather than square roots.

Your interpretation and your explanation should each be a full sentence.

7. Let D be the smaller cap cut from a solid ball of radius 2 units by the plane $z = 1$.



(a) (4 points) Express the volume of D as an iterated triple integral in spherical coordinates. Show all of your work in setting up this integral. *You do not need to evalute the integral.*

Solution: The equation of the bounding sphere in spherical coordinates is $\rho = 2$ and the lower plane is $\rho = \sec(\varphi)$. These intersect when $2 = \sec(\varphi)$, or $\varphi = \pi/3$. Therefore we have

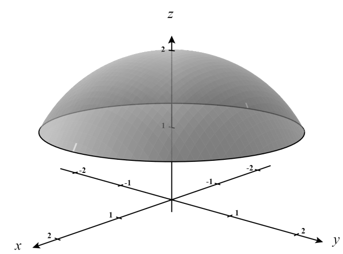
$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec(\varphi)}^{2} \rho^2 \sin(\varphi)\ d\rho\ d\varphi\ d\theta.$$

(b) (4 points) Express the volume of D as an iterated triple integral in cylindrical coordinates. Show all of your work in setting up this integral. *You do not need to evalute the integral.*

Solution: The equation of the bounding sphere in cylindrical coordinates is $r^2 + z^2 = 4$ and the lower plane is $z = 1$. These intersect when $r^2 = 3$ or $r = \sqrt{3}$. Therefore we have

$$V = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r\ dz\ dr\ d\theta.$$

Let D be the smaller cap cut from a solid ball of radius 2 units by the plane $z = 1$.



(c) (4 points) Express the volume of D as an iterated triple integral in Cartesian coordinates. Show all of your work in setting up this integral. *You do not need to evalute the integral.*

Solution: In Cartesian coordinates the bounding sphere is $x^2 + y^2 + z^2 = 4$ and the lower plane is $z = 1$. These intersect in the plane $z = 1$ along the circle $x^2 + y^2 = 3$, so the shadow of the region on the xy-plane is the disk $x^2 + y^2 \le 3$. Therefore we have

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{1}^{\sqrt{4-x^2-y^2}} \, dz \; dy \; dx.$$

(d) (2 points) Use one of parts (a), (b), or (c) to find the volume of D. Fully simplify your answer.

Solution:

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r \; dz \; dr \; d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} r\sqrt{4-r^2} - r \; dr \; d\theta \\ &= \int_0^{2\pi} -\frac{1}{3}(4-r^2)^{3/2} - \frac{1}{2}r^2\Big|_0^{\sqrt{3}} \; d\theta \\ &= \int_0^{2\pi} -\frac{1}{3} - \frac{3}{2} + \frac{8}{3} + 0 \; d\theta \\ &= \frac{5\pi}{3} \end{aligned}$$

8. (10 points) Make an appropriate change of variables and evaluate the integral

$$\iint_R (x+y)e^{x^2-y^2}\ dA,$$

where R is the rectangle enclosed by the lines $x-y=0, x-y=2, x+y=0$, and $x+y=3$. Include a sketch of the new region of integration G in the uv-plane.

Hint: Think about what you would like $u(x,y)$ and $v(x,y)$ to be and work backwards to find the transformation $\mathbf{T}(u,v)$.

Solution: The given rectangle and integrand suggest the substitution $u = x-y, v = x+y$. From this we see that G is the region in the uv-plane bounded by $u=0, u=2, v=0$, and $v=3$. Inverting this system of equations,

$$\begin{cases} u = x-y \\ v = x+y \end{cases} \Rightarrow \begin{cases} u+v = 2x \\ u-v = -2y \end{cases} \Rightarrow \begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{v-u}{2} \end{cases}$$

we find $\mathbf{T}(u,v) = \langle \frac{1}{2}(u+v), \frac{1}{2}(v-u)\rangle$. Thus the Jacobian of this transformation is

$$|\det(D\mathbf{T}(u,v))| = |\det\left(\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\right)| = |1/4+1/4| = 1/2.$$

Finally, the integrand becomes ve^{uv}. Applying the change of variables theorem, we have

$$\begin{aligned}
\iint_R (x+y)e^{x^2-y^2}\ dA &= \int_0^3\int_0^2 ve^{uv}\frac{1}{2}\ du\ dv \\
&= \frac{1}{2}\int_0^3 e^{uv}|_{u=0}^{u=2}\ dv \\
&= \frac{1}{2}\int_0^3 e^{2v}-1\ dv \\
&= \frac{1}{4}e^{2v}-\frac{1}{2}v|_0^3 \\
&= \frac{1}{4}e^6-\frac{3}{2}-(\frac{1}{4}-0) \\
&= \frac{e^6-7}{4}
\end{aligned}$$

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FORMULA SHEET

- Area/volume: area$(R) = \iint_R dA$, volume$(D) = \iiint_D dV$
- Trig identities: $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$, $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$
- $f_{avg} = \dfrac{\iint_R f(x,y)dA}{\text{area of } R}$, or $f_{avg} = \dfrac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$
- Mass: $M = \iint_D \delta\ dA$ or $M = \iiint_D \delta\ dV$
- First moments (2D plate): $M_y = \iint_R x\delta\ dA$, $M_x = \iint_R y\delta\ dA$
- Center of mass (2D plate): $(\bar{x}, \bar{y}) = \left(\dfrac{M_y}{M}, \dfrac{M_x}{M}\right)$
- First moments (3D solid): $M_{yz} = \iiint_D x\delta\ dV$, $M_{xz} = \iiint_D y\delta\ dV$, $M_{xy} = \iiint_D z\delta\ dV$
- Center of mass (3D solid): $(\bar{x}, \bar{y}, \bar{z}) = \left(\dfrac{M_{yz}}{M}, \dfrac{M_{xz}}{M}, \dfrac{M_{xy}}{M}\right)$
- Polar coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $dA = r\ dr\ d\theta$
- Cylindrical coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$, $dV = r\ dz\ dr\ d\theta$
- Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$
- Substitution for double integrals: If R is the image of G under a coordinate transformation $\mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle$ then

$$\iint_R f(x,y)\ dx\ dy = \iint_G f(\mathbf{T}(u,v))|\det D\mathbf{T}(u,v)|\ du\ dv.$$

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