# MATH 2551-D MIDTERM 2
# VERSION A
# SPRING 2023
# COVERS SECTIONS 14.1-14.8

**Full name:** ______________________ **GT ID:**______________

**Honor code statement:** I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(     ) I attest to my integrity.

**Read all instructions carefully** before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

| Question | Points |
|---|---|
| 1 | 2 |
| 2 | 2 |
| 3 | 2 |
| 4 | 3 |
| 5 | 9 |
| 6 | 8 |
| 7 | 7 |
| 8 | 7 |
| 9 | 10 |
| Total: | 50 |

For problems 1-3 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) The domain of the function $f(x, y, z) = \frac{1}{x-2} + yz^2$ is $(-\infty, 2) \cup (2, \infty)$.

   ○ **TRUE** √ **FALSE**

2. (2 points) There does not exist a function $f(x, y)$ which is continuous and has continuous partial derivatives such that $f_x(x, y) = x^2 + 2xy$ and $f_y(x, y) = 3xy + y^2$.

   √ **TRUE** ○ **FALSE**

3. (2 points) A function of two variables is differentiable at the point $(a, b)$ if the surface $z = f(x, y)$ has a unique tangent plane at the point $(a, b, f(a, b))$.

   √ **TRUE** ○ **FALSE**

4. (3 points) Which of the following is NOT a valid approach to finding the extreme values of the function $f(x, y) = 3x^2 - y^2 + 4$ on the unit disk $x^2 + y^2 \leq 1$?

   ○ **A)** Find all critical points of $f$ inside the disk, use the formula $x^2 + y^2 = 1$ to rewrite $f$ as function of only $x$ and use this to find the critical points of $f$ on the boundary, check for endpoints, and then evaluate $f$ at all the points you found.

   ○ **B)** Find all critical points of $f$ inside the disk, use the formula $x^2 + y^2 = 1$ to rewrite $f$ as function of only $y$ and use this to find the critical points of $f$ on the boundary, check for endpoints, and then evaluate $f$ at all the points you found.

   ○ **C)** Find all critical points of $f$ inside the disk, parameterize the unit circle to rewrite $f$ as function of only $t$ and use this to find the critical points of $f$ on the boundary, check for endpoints, and then evaluate $f$ at all the points you found.

   ○ **D)** Find all critical points of $f$ inside the disk, use Lagrange multipliers with the constraint $g(x, y) = 1 - x^2 - y^2 = 0$ to find possible extreme points on the boundary, check for endpoints, and then evaluate $f$ at all the points you found.

   √ **E) None of the above.**

5. (a) (4 points) By considering the paths $x = 0$ and $x = y^4$, show that the following limit does not exist. Your final answer should be a sentence that includes the test you are using.

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8}$$

**Solution:** Along $x = 0$, the limit becomes

$$\lim_{y\to 0} \frac{(0)y^4}{0 + y^8} = \lim_{y\to 0} 0 = 0.$$

On the other hand, along $x = y^4$, the limit becomes

$$\lim_{y\to 0} \frac{(y^4 y^4}{(y^4)^2 + y^8} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since these values are different, the **two-path test** says that this limit does not exist.

(b) (5 points) Let $w = xz + y^2$, with $x = 3s - t, y = s^2$, and $z = 4st$. Use the Chain Rule, either via total derivatives or a tree diagram, to compute the partial derivatives $\frac{\partial w}{\partial s}(1, 1)$ and $\frac{\partial w}{\partial t}(1, 1)$.

**Solution:** The total derivative of $w(x, y, z)$ is

$$Dw(x, y, z) = \begin{bmatrix} z & 2y & x \end{bmatrix}.$$

The total derivative of $\mathbf{r}(s, t) = \langle x(s, t), y(s, t), z(s, t)\rangle$ is

$$D\mathbf{r}(s, t) = \begin{bmatrix} 3 & -1 \\ 2s & 0 \\ 4t & 4s \end{bmatrix}$$

.

At $(s, t) = (1, 1)$, $(x, y, z) = (2, 1, 4)$. The Chain Rule then tells us

$$\begin{bmatrix} \frac{\partial w}{\partial s}(1, 1) & \frac{\partial w}{\partial t}(1, 1) \end{bmatrix} = Dw(2, 1, 4)D\mathbf{r}(1, 1) = \begin{bmatrix} 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 24 & 4 \end{bmatrix}.$$

6. Suppose you are on a landscape whose elevation can be modeled by the function $f(x, y) = e^{xy} - xy^2$, and you are standing at the point where $(x, y) = (1, 2)$.

   (a) (3 points) Find the rate of change of your elevation if you were to walk north (positive $y$-direction) or east (positive $x$-direction).

   **Solution:** The rate of change if we walk east is $f_x(1, 2)$ and the rate of change is we walk north is $f_y(1, 2)$.
   $f_x = ye^{xy} - y^2$ and $f_y = xe^{xy} - 2xy$, so the rate of change walking east is $2e^2 - 4$ and the rate of change walking north is $e^2 - 4$.

   (b) (3 points) Find the rate of change of your elevation if you were to walk in the direction 3 units east and 4 unit south.

   **Solution:** The unit vector in this direction is $\mathbf{u} = \frac{\langle 3, -4\rangle}{|\langle 3, -4\rangle|} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$.
   The rate of change of elevation in this direction is the directional derivative

   $$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = \langle 2e^2 - 4, e^2 - 4\rangle \cdot \langle \frac{3}{5}, -\frac{4}{5} \rangle = \frac{2}{5}e^2 + \frac{4}{5}.$$

   (c) (2 points) What is the relationship between the direction of maximum decrease of elevation from $(1, 2)$ and the contour line of $f$ passing through $(1, 2)$?

   **Solution:** This is the direction which is exactly opposite of the gradient of $f$ at $(1, 2)$, and therefore will be perpendicular to the contour line of $f$ passing through $(1, 2)$.

7. (7 points) Use Lagrange multipliers to find the point on the plane $x + y + z = 2$ closest to the point $(0, 5, -1)$.

**Solution:** Let $d(x, y, z) = \sqrt{x^2 + (y-5)^2 + (z+1)^2}$ be the distance from any point $(x, y, z)$ to the point $(0, 5, -1)$. Since any minimum of this function occurs when the quantity under the square root has a minimum, we will seek to minimize $f(x, y, z) = d^2 = x^2 + (y-5)^2 + (z+1)^2$. The constraint in this problem is that the point must lie on the plane, so we have $g(x, y, z) = x + y + z = 2$.

By the method of Lagrange multipliers, any solution satisfies $\nabla f = \lambda \nabla g$ and $g = 2$. This gives us the system

$$\langle 2x, 2(y-5), 2(z+1)\rangle = \lambda\langle 1, 1, 1\rangle, \qquad x + y + z = 2.$$

We therefore have $\lambda = 2x = 2(y-5) = 2(z+1)$, so $x = (y-5) = (z+1)$. Rearranging, we have $y = x + 5$ and $z = x - 1$, so the constraint equation yields

$$x + x + 5 + x - 1 = 2 \Rightarrow x = -\frac{2}{3}.$$

Therefore the unique solution of the system is $\left(-\frac{2}{3}, \frac{13}{3}, -\frac{5}{3}\right)$. We know that this is the location of a minimum of the distance function due to geometry - the line through $(0, 5, -1)$ normal to the plane contains the closest point on the plane, and there is no furthest point.

8. You need to approximate the value $\sqrt{4.9^2 - 3.1^2}$.

 (a) (1 point) Define a function $f(x, y)$ that you will linearize to approximate this square root.

 **Solution:** Many choices of $f$ will work. One of the simplest is $f(x, y) = \sqrt{x^2 - y^2}$.

 (b) (2 points) Choose a point $(a, b)$ to base the approximation at and a nearby point $(c, d)$ such that

$$f(c, d) = \sqrt{4.9^2 - 3.1^2}$$

 **Solution:** Answers will depend on choice in (a). For the choice of $f$ above, the point $(c, d) = (4.9, 3.1)$ and a good nearby point to base the approximation at is $(a, b) = (5, 3)$.

 (c) (4 points) Compute the linearization of your chosen $f$ at $(a, b)$ and use it to approximate the square root.

 **Solution:** Since $L(x, y) = f(a, b) + Df(a, b)\langle x - a, y - b\rangle$, we compute $f(5, 3) = \sqrt{5^2 - 3^2} = 4$ and

$$Df(5,3) = \begin{bmatrix} \frac{x}{\sqrt{x^2-y^2}} & \frac{-y}{\sqrt{x^2-y^2}}, \end{bmatrix} |_{(5,3)} = \begin{bmatrix} \frac{5}{4} & \frac{-3}{4} \end{bmatrix}.$$

 Therefore $L(x, y) = 4 + \frac{5}{4}(x - 5) - \frac{3}{4}(y - 3)$ and we have

$$\sqrt{4.9^2 - 3.1^2} = f(4.9, 3.1) \approxeq L(4.9, 3.1) = 4 + \frac{5}{4}(-.1) - \frac{3}{4}(.1) = 4 - .125 - .075 = 3.8.$$

9. (10 points) Find and classify all of the critical points of the function $f(x, y) = xy(1-x-y)$.

**Solution:** To find the critical points, we solve the equation $Df(x, y) = [0\ 0]$. Here

$$\begin{aligned} Df(x,y) &= \begin{bmatrix} y(1-x-y)+xy(-1) & x(1-x-y)+xy(-1) \end{bmatrix} \\ &= \begin{bmatrix} y(1-2x-y) & x(1-x-2y) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \end{bmatrix}. \end{aligned}$$

From the first component, we have either $y = 0$ or $1 - 2x - y = 0$. If $y = 0$, then the second component gives $x(1 - x) = 0$, so $x = 0$ or $x = 1$. Thus $(0, 0)$ and $(1, 0)$ are critical points.

If $1 - 2x - y = 0$, then $y = 1 - 2x$ and the second component gives

$$\begin{aligned} 0 &= x(1-x-2(1-2x)) \\ &= x(-1+3x). \end{aligned}$$

So either $x = 0$ and $y = 1 - 0 = 1$ or $x = \frac{1}{3}$ and $y = 1 - \frac{2}{3} = \frac{1}{3}$. Thus $(0, 1)$ and $(1/3, 1/3)$ are also critical points.

To classify the critical points, we compute the Hessian and use the second derivative test.

$$Hf(x,y) = \begin{bmatrix} -2y & 1-2x-2y \\ 1-2x-2y & -2x \end{bmatrix}$$

At $(0, 0)$, $\det(Hf(0, 0)) = (0)(0) - (1)(1) = -1 < 0$, so $f$ has a saddle point at $(0, 0)$.

At $(1, 0)$, $\det(Hf(1, 0)) = (0)(-2) - (-1)(-1) = -1 < 0$, so $f$ has a saddle point at $(1, 0)$.

At $(0, 1)$, $\det(Hf(0, 1)) = (-2)(0) - (-1)(-1) = -1 < 0$, so $f$ has a saddle point at $(0, 1)$.

At $(\frac{1}{3}, \frac{1}{3})$, $\det(Hf(\frac{1}{3}, \frac{1}{3})) = (\frac{-2}{3})(\frac{-2}{3}) - (\frac{-1}{3})(\frac{-1}{3}) = \frac{1}{3} > 0$ and $f_{xx}(\frac{1}{3}, \frac{1}{3}) = -\frac{2}{3} < 0$, so $f$ has a local max at $(\frac{1}{3}, \frac{1}{3})$.

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## FORMULA SHEET

- For $\mathbf{f}(x_1,\ldots,x_n) = \langle f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n)\rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Near $\mathbf{a}$, $L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x}-\mathbf{a})$
- If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$
- If the equation $F(x,y,z) = c$ implicitly defines $z$ as a function of $x$ and $y$, then $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.
- If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P)\cdot\mathbf{u}$
- The tangent line to a level curve of $f(x,y)$ at $(a,b)$ is $0 = \nabla f(a,b)\cdot\langle x-a, y-b\rangle$
- The tangent plane to a level surface of $f(x,y,z)$ at $(a,b,c)$ is

$$0 = \nabla f(a,b,c)\cdot\langle x-a, y-b, z-c\rangle.$$

- For $f(x,y)$, $Hf(x,y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$
- If $(a,b)$ is a critical point of $f(x,y)$ then
  1. If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) < 0$ then $f$ has a local maximum at $(a,b)$
  2. If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) > 0$ then $f$ has a local minimum at $(a,b)$
  3. If $\det(Hf(a,b)) < 0$ then $f$ has a saddle point at $(a,b)$
  4. If $\det(Hf(a,b)) = 0$ the test is inconclusive

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