MATH 2551-D MIDTERM 2
VERSION A
SPRING 2023
COVERS SECTIONS 14.1-14.8

Full name: ______________________ **GT ID:**______________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	2
4	3
5	9
6	8
7	7
8	7
9	10
Total:	50

For problems 1-3 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) The domain of the function $f(x, y, z) = \frac{1}{x-2} + yz^2$ is $(-\infty, 2) \cup (2, \infty)$.

   ○ **TRUE** ○ **FALSE**

2. (2 points) There does not exist a function $f(x, y)$ which is continuous and has continuous partial derivatives such that $f_x(x, y) = x^2 + 2xy$ and $f_y(x, y) = 3xy + y^2$.

   ○ **TRUE** ○ **FALSE**

3. (2 points) A function of two variables is differentiable at the point (a, b) if the surface $z = f(x, y)$ has a unique tangent plane at the point $(a, b, f(a, b))$.

   ○ **TRUE** ○ **FALSE**

4. (3 points) Which of the following is NOT a valid approach to finding the extreme values of the function $f(x, y) = 3x^2 - y^2 + 4$ on the unit disk $x^2 + y^2 \leq 1$?

   ○ **A)** Find all critical points of f inside the disk, use the formula $x^2 + y^2 = 1$ to rewrite f as function of only x and use this to find the critical points of f on the boundary, check for endpoints, and then evaluate f at all the points you found.

   ○ **B)** Find all critical points of f inside the disk, use the formula $x^2 + y^2 = 1$ to rewrite f as function of only y and use this to find the critical points of f on the boundary, check for endpoints, and then evaluate f at all the points you found.

   ○ **C)** Find all critical points of f inside the disk, parameterize the unit circle to rewrite f as function of only t and use this to find the critical points of f on the boundary, check for endpoints, and then evaluate f at all the points you found.

   ○ **D)** Find all critical points of f inside the disk, use Lagrange multipliers with the constraint $g(x, y) = 1 - x^2 - y^2 = 0$ to find possible extreme points on the boundary, check for endpoints, and then evaluate f at all the points you found.

   ○ **E)** None of the above.

5. (a) (4 points) By considering the paths $x = 0$ and $x = y^4$, show that the following limit does not exist. Your final answer should be a sentence that includes the test you are using.

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8}$$

(b) (5 points) Let $w = xz + y^2$, with $x = 3s - t, y = s^2$, and $z = 4st$. Use the Chain Rule, either via total derivatives or a tree diagram, to compute the partial derivatives $\frac{\partial w}{\partial s}(1,1)$ and $\frac{\partial w}{\partial t}(1,1)$.

6. Suppose you are on a landscape whose elevation can be modeled by the function $f(x, y) = e^{xy} - xy^2$, and you are standing at the point where $(x, y) = (1, 2)$.

   (a) (3 points) Find the rate of change of your elevation if you were to walk north (positive y-direction) or east (positive x-direction).

   (b) (3 points) Find the rate of change of your elevation if you were to walk in the direction 3 units east and 4 unit south.

   (c) (2 points) What is the relationship between the direction of maximum decrease of elevation from $(1, 2)$ and the contour line of f passing through $(1, 2)$?

7. (7 points) Use Lagrange multipliers to find the point on the plane $x + y + z = 2$ closest to the point $(0, 5, -1)$.

8. You need to approximate the value $\sqrt{4.9^2 - 3.1^2}$.

   (a) (1 point) Define a function $f(x, y)$ that you will linearize to approximate this square root.

   (b) (2 points) Choose a point (a, b) to base the approximation at and a nearby point (c, d) such that

   $$f(c, d) = \sqrt{4.9^2 - 3.1^2}$$

   (c) (4 points) Compute the linearization of your chosen f at (a, b) and use it to approximate the square root.

9. (10 points) Find and classify all of the critical points of the function $f(x, y) = xy(1-x-y)$.

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FORMULA SHEET

- For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n) \rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Near $\mathbf{a}$, $L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$
- If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$
- If the equation $F(x, y, z) = c$ implicitly defines z as a function of x and y, then $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.
- If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$
- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b \rangle$
- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle.$$

- For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$
- If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

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