# MATH 2551-D MIDTERM 1
# VERSION A
# SPRING 2023
# COVERS SECTIONS 12.1-12.6, 13.1-13.4

**Full name:** ______________________________ **GT ID:** ____________________

**Honor code statement:** I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

(    ) I attest to my integrity.

**Read all instructions carefully** before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

| Question | Points |
|---|---|
| 1 | 2 |
| 2 | 2 |
| 3 | 2 |
| 4 | 4 |
| 5 | 10 |
| 6 | 10 |
| 7 | 10 |
| 8 | 10 |
| Total: | 50 |

For problems 1-3 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) The curvature of a line is 0 at every point on the line.

   $\surd$ **TRUE** ○ **FALSE**

2. (2 points) If $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^3$, then $\mathbf{u} \cdot \mathbf{v}$ is a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

   ○ **TRUE** $\surd$ **FALSE**

3. (2 points) There is exactly one possible parameterization for any line in space.

   ○ **TRUE** $\surd$ **FALSE**

4. (4 points) Which of the following is true of the quadric surface $z = x^2 + 2y^2$?

   ○ **A)** It is a plane, because all of its cross-sections in the $x = k, y = k$, and $z = k$ planes are straight lines.

   ○ **B)** It is a sphere, because all of its cross-sections in the $x = k, y = k$, and $z = k$ planes are circles.

   $\surd$ **C) It is an elliptical paraboloid, because its cross-sections are ellipses in the $z = k$ planes and parabolas in the $x = k$ and $y = k$ planes.**

   ○ **D)** It is a hyperbolic paraboloid, because its cross-sections are hyperbolas in the $z = k$ planes and parabolas in the $x = k$ and $y = k$ planes.

   ○ **E)** It is a cone, because its cross-sections are circles in the $z = k$ planes and straight lines in the $x = k$ and $y = k$ planes.

5. Let $C$ be the curve which is the graph of the vector-valued function $\mathbf{r}(t) = \left\langle \frac{t^2}{2}, \frac{4}{3}t^{3/2}, 2t \right\rangle$, $0 \le t < \infty$.

(a) (2 points) Compute $\mathbf{r}(0)$ and $\mathbf{r}(4)$.

**Solution:** $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$
$\mathbf{r}(4) = \langle 4^2/2, 4/3(4)^{(3/2)}, 2(4) \rangle = \langle 8, \frac{32}{3}, 8 \rangle$.

(b) (6 points) Compute the arc length function $s(t)$ for this parameterization, taking $t_0 = 0$ as your reference point.

**Solution:**

$$\begin{aligned} s(t) &= \int_{t_0}^{t} |\mathbf{r}'(T)| \, dT \\ &= \int_0^t |\langle T, 2\sqrt{T}, 2 \rangle| \, dT \\ &= \int_0^t \sqrt{T^2 + 4T + 4} \, dT \\ &= \int_0^t T + 2 \, dT \\ &= \frac{1}{2}t^2 + 2t \end{aligned}$$

(c) (2 points) Find the distance along $C$ between the points $(0, 0, 0)$ and $(8, \frac{32}{3}, 8)$.

**Solution:** The distance is $s(4) - s(0) = 1/2 \cdot 4^2 + 2(4) - (1/2 \cdot 0^2 + 2(0)) = 16$ units.

6. Suppose that a pegasus (a flying horse) is flying through the sky. Two hours after it left its stable, its velocity vector is $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ miles per hour and its position is $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ miles. The stable is not located at the origin.

   (a) (3 points) Find an equation of the tangent line to the flight path of the pegasus two hours after it left its stable.

   **Solution:** There are two options here, depending on how we define $t$. If $t$ is hours after leaving the stable, we have

   $$\ell_1(t) = \langle 3, -2, 1\rangle(t-2) + \langle 2, -3, 4\rangle.$$

   If $t$ is hours after this time, we have

   $$\ell_2(t) = \langle 3, -2, 1\rangle t + \langle 2, -3, 4\rangle.$$

   (b) (5 points) One hour after it left its stable, the position of the pegasus was $-\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ miles. Use this fact and your answer to part (a) to show that the pegasus cannot be flying in a straight line. Be sure to clearly explain your answer.

   **Solution:** One hour after leaving the stable corresponds to $t = 1$ or $t = -1$ in the solutions above, respectively. If the pegasus was flying in a straight line, then its path would be exactly the same as its tangent line! So we compute its position on the tangent line and compare with the given position. Using the first equation above, this gives:

   $$\mathbf{r}(1) = \langle -1, -1, 5\rangle,$$

   but

   $$\ell_1(1) = \langle 3, -2, 1\rangle(1-2) + \langle 2, -3, 4\rangle = \langle -1, -1, 3\rangle.$$

   Since these are not equal, the pegasus must not be flying in a straight line.

   (c) (2 points) If the pegasus maintains its flight speed at the two hour mark of $\sqrt{14}$ miles per hour for the rest of its flight, is it possible for it to return to its stable? Be sure to clearly explain your answer.

   **Solution:** Yes, the pegasus can still return to its stable without changing its speed, because it can change the direction of its flight while maintaining the same speed.

7. Let $P_1$ be the plane $2x + 2y - z = 1$ and $P_2$ be the plane $2x - y + 2z = 4$.

(a) (2 points) Show that $P_1$ and $P_2$ are intersecting planes by showing that the point $(-1, 4, 5)$ lies on both planes.

**Solution:** Plug $(-1, 4, 5)$ into each plane's equation:

$$2(-1) + 2(4) - 5 = -2 + 8 - 5 = 1$$

and

$$2(-1) - 4 + 2(5) = -2 - 4 + 10 = 4.$$

(b) (2 points) Find normal vectors to both planes.

**Solution:** We can read these vectors off the given equations: $\mathbf{n}_1 = \langle 2, 2, -1 \rangle$ and $\mathbf{n}_2 = \langle 2, -1, 2 \rangle$.

(c) (6 points) Find a parameterization of the line of intersection of the two planes.

**Solution:** A direction vector for this line is $\mathbf{n}_1 \times \mathbf{n}_2 = \langle 3, -6, -6 \rangle$. Since we found previously that the point $(-1, 4, 5)$ is on both planes, it lies on the line of intersection. So one possible equation is

$$\ell(t) = \langle 3, -6, -6 \rangle t + \langle -1, 4, 5 \rangle.$$

8. In this problem, you will work with normal vectors and curvature.

(a) (4 points) Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ be any parameterized curve in the plane. Show that the vector $\mathbf{n}(t) = -g'(t)\mathbf{i} + f'(t)\mathbf{j}$ is orthogonal to $\mathbf{r}'(t)$ for every value of $t$.

**Solution:** $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$. Therefore we have

$$\mathbf{r}'(t) \cdot \mathbf{n}(t) = (f'(t)\mathbf{i} + g'(t)\mathbf{j}) \cdot (-g'(t)\mathbf{i} + f'(t)\mathbf{j}) = -f'(t)g'(t) + f'(t)g'(t) = 0.$$

Since this dot product is 0 for every value of $t$, $\mathbf{n}(t)$ and $\mathbf{r}'(t)$ are orthogonal for every value of $t$.

(b) (4 points) Use your result from part (a) to produce a **unit** normal vector to the curve $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle, 0 \leq t < 2\pi$.

**Solution:** We showed in (a) that $\mathbf{n}(t)$ is normal to the curve $\mathbf{r}(t)$. So $\mathbf{n}(t) = \langle -2\cos(t), -2\sin(t)$ is a normal vector to this curve. To get a unit normal vector, we divide by $|\mathbf{n}(t)| = 2$ to get the vector

$$\hat{\mathbf{n}}(t) = \langle -\cos(t), -\sin(t)\rangle.$$

(c) (2 points) Is the vector that you produced in part (b) the principal unit normal vector? Explain your answer; a picture might be useful.
*Hint: Consider whether your vector is pointing in the direction of curvature of this curve.*

**Solution:** The principal unit normal vector to this curve (a circle of radius 2) points inward toward the origin, which this vector does do, so $\hat{\mathbf{n}}(t)$ is the principal unit normal vector.

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## FORMULA SHEET

- $\langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$
- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta)$
- $\langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|| \sin(\theta)|$
- $L = \displaystyle\int_a^b |\mathbf{r}'(t)| \; dt$
- $s(t) = \displaystyle\int_{t_0}^t |\mathbf{r}'(T)| \; dT$
- $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{d\mathbf{r}}{ds}$
- $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$
- $\mathbf{N} = \dfrac{1}{\kappa}\dfrac{d\mathbf{T}}{ds} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

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