MATH 2551-K/L FINAL EXAM
VERSION A
FALL 2022
SECTIONS 12.1-6, 13.1-13.4, 14.1-14.8, 15.1-15.8, 16.1-16.8

Full name: ______________________ **GT ID:** ______________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. **I will not discuss the exam with anyone until Wednesday December 14.**

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- You have 170 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	2
4	2
5	2
6	5
7	5
8	5
9	5

Question	Points
10	5
11	5
12	10
13	10
14	10
15	10
16	10
17	10
Total:	100

Choose whether the following statements are true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false.

1. (2 points) Every vector field $\mathbf{F}(x, y)$ is a gradient vector field, i.e. there is always some $f(x, y)$ so that $\mathbf{F} = \nabla f$.

   ○ **TRUE** $\surd$ **FALSE**

2. (2 points) If $\mathbf{F}(x, y, z)$ is a velocity field for a fluid flowing in space, then $\nabla \cdot \mathbf{F}$ is a vector whose direction is the right-hand rule direction of the axis of rotation of the fluid at each point.

   ○ **TRUE** $\surd$ **FALSE**

3. (2 points) Suppose $f(x, y)$ is a differentiable function defined on all of $\mathbb{R}^2$, $f_{xx}(0, -1) = -3$, $f_{xy}(0, -1) = f_{yx}(0, -1) = 4$, and $f_{yy}(0, -1) = -6$. Then $(0, -1)$ is the location of a local maximum of f.

   ○ **TRUE** $\surd$ **FALSE**

4. (2 points) For any continuously differentiable function $f(x, y, z)$, $\nabla \times (\nabla f) = \mathbf{0}$.

   $\surd$ **TRUE** ○ **FALSE**

5. (2 points) The domain of the function $f(x, y) = \ln(x - 2) + y^2$ is the interval $(2, \infty)$.

   ○ **TRUE** $\surd$ **FALSE**

6. (5 points) Let S be a surface with normal vector $\mathbf{n}$ and compatibly oriented boundary curve C. Which surface integral below is equivalent to the line integral

$$\int_C y\ dx + z^2\ dy + 2x\ dz?$$

○ **A)** $\iint_S \langle y, z^2, 2x\rangle \cdot \mathbf{n}\ d\sigma$

○ **B)** $\iint_S \langle 2z, 2, 1\rangle \cdot \mathbf{n}\ d\sigma$

√ **C)** $\iint_S \langle -2z, -2, -1\rangle \cdot \mathbf{n}\ d\sigma$

○ **D)** $\iint_S -1\ d\sigma$

○ **E)** $\iint_S \langle 0, 0, 0\rangle \cdot \mathbf{n}\ d\sigma$

7. (5 points) Which of the following is a parameterization of the surface S which is the portion of the cylinder $y^2 + z^2 = 9$ between the planes $x = 0$ and $x = 3$?

○ **A)** $\mathbf{r}(x, y) = \langle x, y, \sqrt{9 - z^2}\rangle \qquad 0 \le x \le 3, 0 \le y \le 3$

○ **B)** $\mathbf{r}(z, \theta) = \langle 3\cos(\theta), 3\sin(\theta), z\rangle \qquad 0 \le z \le 3, 0 \le \theta \le 2\pi$

√ **C)** $\mathbf{r}(x, \theta) = \langle x, 3\cos(\theta), 3\sin(\theta)\rangle \qquad 0 \le x \le 3, 0 \le \theta \le 2\pi$

○ **D)** $\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), 3\rangle \qquad 0 \le r \le 3, 0 \le \theta \le 2\pi$

○ **E)** $\mathbf{r}(y, z) = \langle 3, y, z\rangle \qquad 0 \le y \le \sqrt{9 - z^2}, 0 \le z \le 3$

8. (5 points) A vector field $\mathbf{F}$ and several curves C_1, C_2, C_3, C_4 are shown to the right. For which curve is the line integral $\int_{C_i} \mathbf{F} \cdot \mathbf{T}\ ds$ positive?

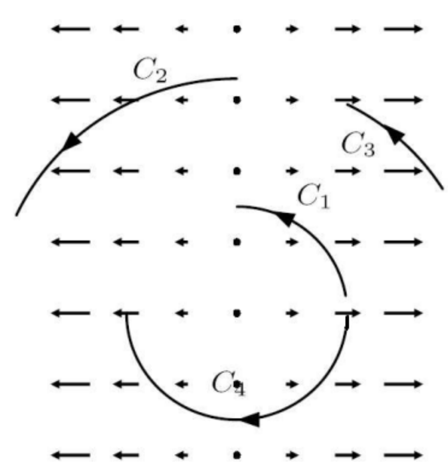
○ **A)** C_4

○ **B)** C_1

○ **C)** C_3

√ **D)** C_2

○ **E)** None are positive.



9. (5 points) The integral below could describe the mass of:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (x^2 + y^2 + z^2)\, dz\, dy\, dx$$

○ **A)** a cone that gets heavier towards the outside

√ **B) a ball that gets heavier towards the outside**

○ **C)** a cone that gets lighter towards the outside

○ **D)** a ball that is equally heavy at all points

○ **E)** a ball that gets lighter towards the outside

10. (5 points) A vector equation for the tangent line to the curve parameterized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $(2, 4, 8)$ is:

○ **A)** $\ell(s) = \langle 2, 2, 2 \rangle s + \langle 1, 4, 12 \rangle$

○ **B)** $\ell(s) = \langle 1, 2t, 3t^2 \rangle s + \langle 2, 4, 8 \rangle$

○ **C)** $\ell(s) = \langle t, t^2, t^3 \rangle s + \langle 2, 4, 8 \rangle$

√ **D)** $\ell(s) = \langle 1, 4, 12 \rangle s + \langle 2, 4, 8 \rangle$

○ **E)** $\ell(s) = \langle 2, 4, 8 \rangle s + \langle 1, 4, 12 \rangle$

11. (5 points) The point $(2, 1, 3)$ is closest to:

○ **A)** the xy-plane

○ **B)** the plane $z = 55$

○ **C)** the plane $x = -6$

√ **D) the xz-plane**

○ **E)** the yz-plane

12. (10 points) Show that the vector field $\mathbf{F}(x, y) = \langle ye^{yx} + 2x, xe^{yx} + 4\rangle$ is a conservative vector field, find its potential function $f(x, y)$, and use that to compute the line integral

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r},$$

where C is any path from $(0, 0)$ to $(1, -1)$.

Solution: $Q_x = e^{yx}(1 + xy)$ and $P_y = e^{yx}(1 + xy)$, so this is a conservative vector field. Integrating Q with respect to y, we have

$$f(x, y) = \int xe^{yx} + 4 \; dy = e^{yx} + 4y + g(x).$$

Computing f_x and comparing to P gives $g'(x) = 2x$, so $g(x) = x^2 + C$. Thus $f(x, y) = e^{xy} + 4y + x^2 + C$.

By the fundamental theorem of line integrals,

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = f(x, y)\big|_{(0,0)}^{(1,-1)} = (e^{-1} - 4 + 1) - (e^0 + 0 + 0) = e^{-1} - 4.$$

13. (10 points) Let C be the boundary curve of the triangle with vertices $(0,0),(-1,1)$, and $(1,1)$, oriented counter-clockwise. Let $\mathbf{F}(x,y) = \langle 2xy, x^2+y^2\rangle$. Compute the circulation of $\mathbf{F}$ around C using any method. Write a sentence explaining why you chose the method you used.

Solution: There are many correct solutions to this problem. One of the simplest is to apply Green's theorem, since the curve C is closed and oriented counter-clockwise.

If R is the region enclosed by C, we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R Q_x - P_y \; dA = \iint_R (2x) - (2x) \; dA = \iint_R 0 \; dA = 0.$$

14. (10 points) Find and classify the critical points of the function

$$f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1.$$

Include both the function value and location for each point you classify.

Solution: To find the critical points, we identify all points where $\nabla f = \mathbf{0}$. Since $\nabla f = \langle 2x - 2y - 2, -2x + 4y + 2\rangle$, we have the system

$$\begin{aligned} 2x - 2y - 2 &= 0 \\ -2x + 4y + 2 &= 0. \end{aligned}$$

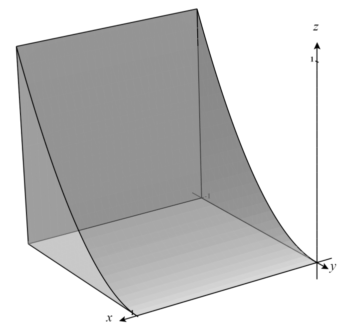
Adding these equations gives $2y = 0$,so $y = 0$ and substitution yields $x = 1$. Therefore the unique critical point of f is $(1, 0)$.

To classify this point, we find the Hessian determinant at $(1, 0)$:

$$Hf(x, y) = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}, \qquad \text{so } \det(Hf(1, 0)) = 2(4) - (-2)^2 = 4.$$

Since $\det(Hf(1, 0)) > 0$ and $f_{xx}(1, 0) > 0$, the second derivative test tells us that f has a local minimum of $f(1, 0) = 1 - 0 + 0 - 2 + 0 + 1 = 0$ at $(1, 0)$.

15. Let D be the solid in $\mathbb{R}^3$ which is bounded by the surface $z = y^2$ and the planes $z = 0, x = 0$, $x = 1, y = -1$, and $y = 0$, pictured to the right.



(a) (5 points) Write a triple integral in Cartesian coordinates for the volume of D using the order $dz\ dy\ dx$. Do not evaluate your integral.

Solution:

$$V = \int_0^1 \int_{-1}^1 \int_0^{y^2} dz\ dy\ dx$$

(b) (5 points) Write a triple integral in Cartesian coordinates for the volume of D using the order $dx\ dy\ dz$. Do not evaluate your integral.

Solution:

$$V = \int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx\ dy\ dz$$

16. Let $f(x,y) = \ln(x^2+y^2)$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$, and $P = (1,0)$.

(a) (4 points) Find the rate of change of f at P in the direction of the vector $\mathbf{v}$.

Solution: First, $\nabla f = \langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \rangle$. We need a unit vector for our direction, so let $\mathbf{u} = \mathbf{v}/|\mathbf{v}| = \langle \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}} \rangle$. Then the rate of change of f at P in the direction of $\mathbf{v}$ is $D_{\mathbf{u}} f(P)$.

$$D_{\mathbf{u}} f(P) = \nabla f(P) \cdot \mathbf{u} = \langle \frac{2}{1+0}, \frac{0}{1+0} \rangle \cdot \langle \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}} \rangle = \frac{6}{\sqrt{13}}.$$

(b) (3 points) Find the linear approximation of f at P.

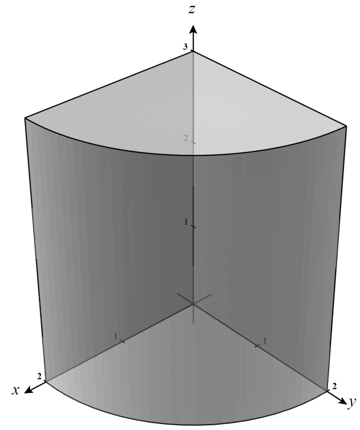
Solution: Note $f(P) = \ln(1+0) = 0$, so we have the linear approximation

$$L(x,y) = f(P) + \nabla f(P) \cdot (\mathbf{x} - \vec{OP}) = 0 + \langle 2,0 \rangle \cdot \langle x-1, y-0 \rangle = 2x - 2.$$

(c) (3 points) Is there a direction in which the rate of change of f at P is at least 4? If so, find the direction. If not, explain why not.

Solution: There is no such direction. The maximum rate of change of f at P is $|\nabla f(P)| = \sqrt{4+0} = 2$.

17. (10 points) Compute the outward flux of the field $\mathbf{F}(x, y, z) = (6x^2 + 2xy)\mathbf{i} + (2y + x^2 z^x)\mathbf{j} + 4x^{-57} y^{y^y}\mathbf{k}$ through the closed surface S surrounding the region D cut from the first octant by the cylinder $x^2 + y^2 = 4$ and the plane $z = 3$.



Solution: We apply the Divergence Theorem and cylindrical coordinates:

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma &= \iiint_D \nabla \cdot \mathbf{F} \; dV \\
&= \iiint_D 12x + 2y + 2 \; dV \\
&= \int_0^{\pi/2} \int_0^2 \int_0^3 12r^2 \cos(\theta) + 2r^2 \sin(\theta) + 2r \; dz \; dr \; d\theta \\
&= 3 \int_0^{\pi/2} \int_0^2 12r^2 \cos(\theta) + 2r^2 \sin(\theta) + 2r \; dr \; d\theta \\
&= 3 \int_0^{\pi/2} 4r^3 \cos(\theta) + \frac{2}{3} r^3 \sin(\theta) + r^2 |_0^2 \; d\theta \\
&= 3 \int_0^{\pi/2} 32 \cos(\theta) + \frac{16}{3} \sin(\theta) + 4 \; d\theta \\
&= 96 \sin(\theta) - 16 \cos(\theta) + 4\theta |_0^{\pi/2} \\
&= (96 - 0 + 2\pi) - (0 - 16 + 0) \\
&= 112 + 2\pi.
\end{aligned}$$

18. (5 points (bonus)) Reflect on a problem that you struggled with at some point during this course but since figured out. What was difficult about the problem at first? How did you overcome your struggle with the problem? What did you learn from that struggle?

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FORMULA SHEET

- $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$
- $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin(\theta)|$
- $L = \int_a^b |\mathbf{r}'(t)|\ dt$
- $s(t) = \int_{t_0}^t |\mathbf{r}'(T)|\ dT$
- $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{r}}{ds}$
- $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$
- $\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$
- $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$
- $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$
- $x = r\cos(\theta)$
  $y = r\sin(\theta)$
- $dA = dx\ dy = r\ dr\ d\theta$
- $x = r\cos(\theta)$
  $y = r\sin(\theta)$
  $z = z$
- $x = \rho\sin(\phi)\cos(\theta)$
  $y = \rho\sin(\phi)\sin(\theta)$
  $z = \rho\cos(\phi)$
- $dV = dx\ dy\ dz$
  $= r\ dz\ dr\ d\theta$
  $= \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$
- For $\mathbf{f}(x_1,\ldots,x_n) = \langle f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n)\rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- At $\mathbf{a}$, $L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x}-\mathbf{a})$
- If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$
- If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P)\cdot\mathbf{u}$
- For $f(x,y)$, $Hf(x,y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$
- If (a,b) is a critical point of $f(x,y)$ then
  1. If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) < 0$ then f has a local maximum at (a,b)
  2. If $\det(Hf(a,b)) > 0$ and $f_{xx}(a,b) > 0$ then f has a local minimum at (a,b)
  3. If $\det(Hf(a,b)) < 0$ then f has a saddle point at (a,b)
  4. If $\det(Hf(a,b)) = 0$ the test is inconclusive
- $f_{avg} = \frac{\iint_R f(x,y)dA}{\iint_R dA}, \qquad f_{avg} = \frac{\iiint_D f(x,y,z)dV}{\iiint_D dV}$
- $M = \iiint_D \rho(x,y,z)\ dV$
- $M_{yz} = \iiint_D x\rho(x,y,z)\ dV$
  $M_{xz} = \iiint_D y\rho(x,y,z)\ dV$
  $M_{xy} = \iiint_D z\rho(x,y,z)\ dV$
- $(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$
- If $\mathbf{T}: G \to R$ is a 1-to-1 differentiable function with $\mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle$ and $f(x,y)$ is continuous on R then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(\mathbf{T}(u,v))|\det(D\mathbf{T}(u,v))|\,du\,dv.$$

FORMULA SHEET

- $\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- $\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt = \int_C P\ dx + Q\ dy + R\ dz$
- $\int_C \mathbf{F}(x,y)\cdot\mathbf{n}\ ds = \int_C P\ dy - Q\ dx = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\langle y'(t), -x'(t)\rangle\ dt.$
- $\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- $\mathbf{F} = \nabla f$ if $P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.
- div $\mathbf{F} = \nabla\cdot\mathbf{F}$
- curl $\mathbf{F} = \nabla\times\mathbf{F}$
- If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dA \qquad\qquad \int_C \mathbf{F}\cdot\mathbf{n}\ ds = \iint_R (\nabla\cdot\mathbf{F})\ dA.$$

- SA=$\iint_S 1\ d\sigma = \iint_R |\mathbf{r}_u\times\mathbf{r}_v|dA$
- $\iint_S f(x,y,z)\ d\sigma = \iint_R f(\mathbf{r}(u,v))\ |\mathbf{r}_u\times\mathbf{r}_v|dA$
- $\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iint_S \mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\ dA$
- If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ d\sigma.$$

- If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iiint_D \nabla\cdot\mathbf{F}\ dV.$$