

**MATH 2551-K/L FINAL EXAM**  
**VERSION A**  
**FALL 2022**  
**SECTIONS 12.1-6, 13.1-13.4, 14.1-14.8, 15.1-15.8, 16.1-16.8**

Full name: \_\_\_\_\_

GT ID: \_\_\_\_\_

**Honor code statement:** I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. **I will not discuss the exam with anyone until Wednesday December 14.**

(     ) I attest to my integrity.

**Read all instructions carefully** before beginning.

- You have 170 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	2
4	2
5	2
6	5
7	5
8	5
9	5

Question	Points
10	5
11	5
12	10
13	10
14	10
15	10
16	10
17	10
Total:	100

Choose whether the following statements are true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false.

1. (2 points) Every vector field  $\mathbf{F}(x, y)$  is a gradient vector field, i.e. there is always some  $f(x, y)$  so that  $\mathbf{F} = \nabla f$ .

TRUE

FALSE

2. (2 points) If  $\mathbf{F}(x, y, z)$  is a velocity field for a fluid flowing in space, then  $\nabla \cdot \mathbf{F}$  is a vector whose direction is the right-hand rule direction of the axis of rotation of the fluid at each point.

TRUE

FALSE

3. (2 points) Suppose  $f(x, y)$  is a differentiable function defined on all of  $\mathbb{R}^2$ ,  $f_{xx}(0, -1) = -3$ ,  $f_{xy}(0, -1) = f_{yx}(0, -1) = 4$ , and  $f_{yy}(0, -1) = -6$ . Then  $(0, -1)$  is the location of a local maximum of  $f$ .

TRUE

FALSE

4. (2 points) For any continuously differentiable function  $f(x, y, z)$ ,  $\nabla \times (\nabla f) = \mathbf{0}$ .

TRUE

FALSE

5. (2 points) The domain of the function  $f(x, y) = \ln(x - 2) + y^2$  is the interval  $(2, \infty)$ .

TRUE

FALSE

6. (5 points) Let  $S$  be a surface with normal vector  $\mathbf{n}$  and compatibly oriented boundary curve  $C$ . Which surface integral below is equivalent to the line integral

$$\int_C y \, dx + z^2 \, dy + 2x \, dz?$$

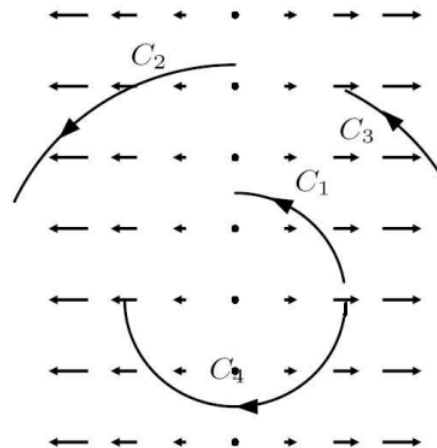
- A)  $\iint_S \langle y, z^2, 2x \rangle \cdot \mathbf{n} \, d\sigma$
- B)  $\iint_S \langle 2z, 2, 1 \rangle \cdot \mathbf{n} \, d\sigma$
- C)  $\iint_S \langle -2z, -2, -1 \rangle \cdot \mathbf{n} \, d\sigma$
- D)  $\iint_S -1 \, d\sigma$
- E)  $\iint_S \langle 0, 0, 0 \rangle \cdot \mathbf{n} \, d\sigma$

7. (5 points) Which of the following is a parameterization of the surface  $S$  which is the portion of the cylinder  $y^2 + z^2 = 9$  between the planes  $x = 0$  and  $x = 3$ ?

- A)  $\mathbf{r}(x, y) = \langle x, y, \sqrt{9 - z^2} \rangle \quad 0 \leq x \leq 3, 0 \leq y \leq 3$
- B)  $\mathbf{r}(z, \theta) = \langle 3 \cos(\theta), 3 \sin(\theta), z \rangle \quad 0 \leq z \leq 3, 0 \leq \theta \leq 2\pi$
- C)  $\mathbf{r}(x, \theta) = \langle x, 3 \cos(\theta), 3 \sin(\theta) \rangle \quad 0 \leq x \leq 3, 0 \leq \theta \leq 2\pi$
- D)  $\mathbf{r}(r, \theta) = \langle r \cos(\theta), r \sin(\theta), 3 \rangle \quad 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi$
- E)  $\mathbf{r}(y, z) = \langle 3, y, z \rangle \quad 0 \leq y \leq \sqrt{9 - z^2}, 0 \leq z \leq 3$

8. (5 points) A vector field  $\mathbf{F}$  and several curves  $C_1, C_2, C_3, C_4$  are shown to the right. For which curve is the line integral  $\int_{C_i} \mathbf{F} \cdot \mathbf{T} \, ds$  positive?

- A)  $C_4$
- B)  $C_1$
- C)  $C_3$
- D)  $C_2$
- E) None are positive.



9. (5 points) The integral below could describe the mass of:

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (x^2 + y^2 + z^2) dz dy dx$$

- A) a cone that gets heavier towards the outside
- B) a ball that gets heavier towards the outside
- C) a cone that gets lighter towards the outside
- D) a ball that is equally heavy at all points
- E) a ball that gets lighter towards the outside

10. (5 points) A vector equation for the tangent line to the curve parameterized by  $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$  at the point  $(2, 4, 8)$  is:

- A)  $\ell(s) = \langle 2, 2, 2 \rangle s + \langle 1, 4, 12 \rangle$
- B)  $\ell(s) = \langle 1, 2t, 3t^2 \rangle s + \langle 2, 4, 8 \rangle$
- C)  $\ell(s) = \langle t, t^2, t^3 \rangle s + \langle 2, 4, 8 \rangle$
- D)  $\ell(s) = \langle 1, 4, 12 \rangle s + \langle 2, 4, 8 \rangle$
- E)  $\ell(s) = \langle 2, 4, 8 \rangle s + \langle 1, 4, 12 \rangle$

11. (5 points) The point  $(2, 1, 3)$  is closest to:

- A) the  $xy$ -plane
- B) the plane  $z = 55$
- C) the plane  $x = -6$
- D) the  $xz$ -plane
- E) the  $yz$ -plane

12. (10 points) Show that the vector field  $\mathbf{F}(x, y) = \langle ye^{yx} + 2x, xe^{yx} + 4 \rangle$  is a conservative vector field, find its potential function  $f(x, y)$ , and use that to compute the line integral

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r},$$

where  $C$  is any path from  $(0, 0)$  to  $(1, -1)$ .

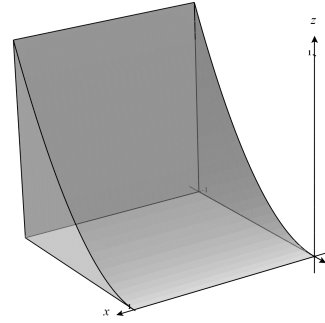
13. (10 points) Let  $C$  be the boundary curve of the triangle with vertices  $(0, 0)$ ,  $(-1, 1)$ , and  $(1, 1)$ , oriented counter-clockwise. Let  $\mathbf{F}(x, y) = \langle 2xy, x^2 + y^2 \rangle$ . Compute the circulation of  $\mathbf{F}$  around  $C$  using any method. Write a sentence explaining why you chose the method you used.

14. (10 points) Find and classify the critical points of the function

$$f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1.$$

Include both the function value and location for each point you classify.

15. Let  $D$  be the solid in  $\mathbb{R}^3$  which is bounded by the surface  $z = y^2$  and the planes  $z = 0$ ,  $x = 0$ ,  $x = 1$ ,  $y = -1$ , and  $y = 0$ , pictured to the right.



- (a) (5 points) Write a triple integral in Cartesian coordinates for the volume of  $D$  using the order  $dz \, dy \, dx$ . Do not evaluate your integral.
- (b) (5 points) Write a triple integral in Cartesian coordinates for the volume of  $D$  using the order  $dx \, dy \, dz$ . Do not evaluate your integral.



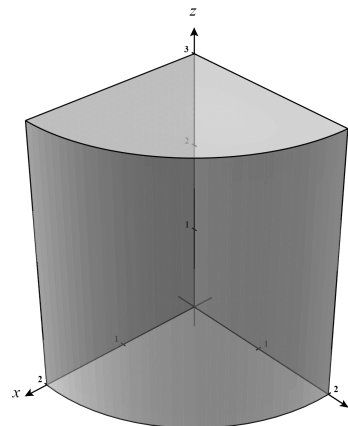
16. Let  $f(x, y) = \ln(x^2 + y^2)$ ,  $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ , and  $P = (1, 0)$ .

(a) (4 points) Find the rate of change of  $f$  at  $P$  in the direction of the vector  $\mathbf{v}$ .

(b) (3 points) Find the linear approximation of  $f$  at  $P$ .

(c) (3 points) Is there a direction in which the rate of change of  $f$  at  $P$  is at least 4? If so, find the direction. If not, explain why not.

17. (10 points) Compute the outward flux of the field  $\mathbf{F}(x, y, z) = (6x^2 + 2xy)\mathbf{i} + (2y + x^2z^x)\mathbf{j} + 4x^{-57}y^{y^y}\mathbf{k}$  through the closed surface  $S$  surrounding the region  $D$  cut from the first octant by the cylinder  $x^2 + y^2 = 4$  and the plane  $z = 3$ .



18. (5 points (bonus)) Reflect on a problem that you struggled with at some point during this course but since figured out. What was difficult about the problem at first? How did you overcome your struggle with the problem? What did you learn from that struggle?

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## FORMULA SHEET

- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta)$
- $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin(\theta)$
- $L = \int_a^b |\mathbf{r}'(t)| dt$
- $s(t) = \int_{t_0}^t |\mathbf{r}'(T)| dT$
- $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{r}}{ds}$
- $\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$
- $\mathbf{N} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$
- $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$
- $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- $x = r \cos(\theta)$   
 $y = r \sin(\theta)$
- $dA = dx dy = r dr d\theta$
- $x = r \cos(\theta)$   
 $y = r \sin(\theta)$   
 $z = z$
- $x = \rho \sin(\phi) \cos(\theta)$   
 $y = \rho \sin(\phi) \sin(\theta)$   
 $z = \rho \cos(\phi)$
- $dV = dx dy dz$   
 $= r dz dr d\theta$   
 $= \rho^2 \sin(\phi) d\rho d\phi d\theta$
- For  $\mathbf{f}(x_1, \dots, x_n) = \langle f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n) \rangle$
- $D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$
- At  $\mathbf{a}$ ,  $L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$
- If  $h = g(f(\mathbf{x}))$  then  $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$
- If  $\mathbf{u}$  is a unit vector,  $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$
- For  $f(x, y)$ ,  $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$
- If  $(a, b)$  is a critical point of  $f(x, y)$  then
  1. If  $\det(Hf(a, b)) > 0$  and  $f_{xx}(a, b) < 0$  then  $f$  has a local maximum at  $(a, b)$
  2. If  $\det(Hf(a, b)) > 0$  and  $f_{xx}(a, b) > 0$  then  $f$  has a local minimum at  $(a, b)$
  3. If  $\det(Hf(a, b)) < 0$  then  $f$  has a saddle point at  $(a, b)$
  4. If  $\det(Hf(a, b)) = 0$  the test is inconclusive
- $f_{avg} = \frac{\iint_R f(x, y) dA}{\iint_R dA}$ ,  $f_{avg} = \frac{\iiint_D f(x, y, z) dV}{\iiint_D dV}$
- $M = \iiint_D \rho(x, y, z) dV$
- $M_{yz} = \iiint_D x\rho(x, y, z) dV$   
 $M_{xz} = \iiint_D y\rho(x, y, z) dV$   
 $M_{xy} = \iiint_D z\rho(x, y, z) dV$
- $(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right)$
- If  $\mathbf{T} : G \rightarrow R$  is a 1-to-1 differentiable function with  $\mathbf{T}(u, v) = \langle x(u, v), y(u, v) \rangle$  and  $f(x, y)$  is continuous on  $R$  then
 
$$\iint_R f(x, y) dx dy = \iint_G f(\mathbf{T}(u, v)) |\det(D\mathbf{T}(u, v))| du dv.$$

### FORMULA SHEET

- $\int_C f(x, y, z) ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt$
- $\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C P dx + Q dy + R dz$
- $\int_C \mathbf{F}(x, y) \cdot \mathbf{n} ds = \int_C P dy - Q dx = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t) \rangle dt.$
- $\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$  if  $C$  is a path from  $A$  to  $B$
- $\mathbf{F} = \nabla f$  if  $P_z = R_x, Q_z = R_y,$  and  $Q_x = P_y.$
- $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$
- $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$
- If  $C$  is a simple closed curve with positive orientation and  $R$  is the simply-connected region it encloses, then

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA \qquad \int_C \mathbf{F} \cdot \mathbf{n} ds = \iint_R (\nabla \cdot \mathbf{F}) dA.$$

- $SA = \iint_S 1 d\sigma = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| dA$
- $\iint_S f(x, y, z) d\sigma = \iint_R f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA$
- $\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S \mathbf{F} \cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$
- If  $S$  is a piecewise smooth oriented surface bounded by a piecewise smooth curve  $C$  and  $\mathbf{F}$  is a vector field whose components have continuous partial derivatives on an open region containing  $S$ , then

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma.$$

- If  $S$  is a piecewise smooth closed oriented surface enclosing a volume  $D$  and  $\mathbf{F}$  is a vector field whose components have continuous partial derivatives on  $D$ , then

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \mathbf{F} dV.$$