# MATH 2551-K/L EXAM 3
# VERSION A
# FALL 2022
# COVERS SECTIONS 15.1-15.8

[redacted]

**Full name:** ______________________________ **GT ID:**________________

**Honor code statement:** I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam.

(    ) I attest to my integrity.

**Read all instructions carefully** before beginning.

- You have 50 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

| Question | Points |
|---|---|
| 1 | 10 |
| 2 | 9 |
| 3 | 7 |
| 4 | 5 |
| 5 | 10 |
| 6 | 9 |
| Total: | 50 |

## FORMULA SHEET

- $\sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- $f_{avg} = \dfrac{\iint_R f(x,y)dA}{\iint_R dA}, \qquad f_{avg} = \dfrac{\iiint_D f(x,y,z)dV}{\iiint_D dV}$
- $x = r\cos(\theta), \quad y = r\sin(\theta)$
- $r^2 = x^2 + y^2, \quad \tan(\theta) = \dfrac{y}{x}$
- $dA = dx\ dy = r\ dr\ d\theta$
- $M = \iiint_D \rho(x,y,z)\ dV$
- $M_{yz} = \iiint_D x\rho(x,y,z)\ dV, \quad M_{xz} = \iiint_D y\rho(x,y,z)\ dV, \quad M_{xy} = \iiint_D z\rho(x,y,z)\ dV$
- $I_x = \iiint_D (y^2 + z^2)\rho(x,y,z)\ dV$
- $I_y = \iiint_D (x^2 + z^2)\rho(x,y,z)\ dV$
- $I_z = \iiint_D (x^2 + y^2)\rho(x,y,z)\ dV$
- $(\bar{x}, \bar{y}, \bar{z}) = \left(\dfrac{M_{yz}}{M}, \dfrac{M_{xz}}{M}, \dfrac{M_{xy}}{M}\right)$
- $x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$
- $r^2 = x^2 + y^2, \quad \tan(\theta) = \dfrac{y}{x}, \quad z = z$
- $x = \rho\sin(\phi)\cos(\theta), \quad y = \rho\sin(\phi)\sin(\theta), \quad z = \rho\cos(\phi)$
- $\rho^2 = x^2 + y^2 + z^2, \quad \tan(\phi) = \dfrac{\sqrt{x^2 + y^2}}{z}, \quad \tan(\theta) = \dfrac{y}{x}$
- $r = \rho\sin(\phi), \quad \theta = \theta, \quad z = \rho\cos(\phi)$
- $\rho^2 = r^2 + z^2, \quad \tan(\phi) = \dfrac{r}{z}, \quad \theta = \theta$
- $dV = dx\ dy\ dz = r\ dz\ dr\ d\theta = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$
- If $\mathbf{T} : G \to R$ is a 1-to-1 differentiable function with $\mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle$ and $f(x,y)$ is continuous on $R$ then

$$\iint_R f(x,y)\ dx\ dy = \iint_G f(\mathbf{T}(u,v))|D\mathbf{T}(u,v)|\ du\ dv.$$

1. Choose whether the following statements are true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false.

(a) (2 points) The cylindrical coordinates of the point with Cartesian coordinates $(4, \pi, 0)$ are $(-4, 0, 0)$.

○ **TRUE** ● **FALSE**

(b) (2 points) The equation $r = 2\sin(\theta)$ describes a circle.

● **TRUE** ○ **FALSE**

$r^2 = r\sin(\theta)$

$x^2 + y^2 = y$

$x^2 + y^2 - y + \frac{1}{4} = 1$

(c) (2 points) The mass of a region in the plane might be given by $\int_0^x \int_0^y 3 \, dy \, dx$.

○ **TRUE** ● **FALSE**

(d) (2 points) If $f(x, y)$ is a positive continuous function, then

$$\int_0^1 \int_0^1 f(x,y) \, dy \, dx > \int_0^1 \int_x^1 f(x,y) \, dy \, dx.$$

● **TRUE** ○ **FALSE**

(e) (2 points) The region $0 \leq \rho \leq 4, 0 \leq \phi \leq \pi, 0 \leq \theta \leq \pi$ describes the bottom half of a sphere of radius 4 centered on the origin.

○ **TRUE** ● **FALSE**

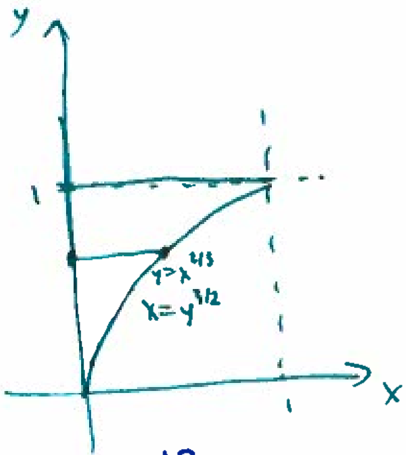
2. (9 points) Sketch the domain of integration corresponding to

$$\int_0^1 \int_{x^{2/3}}^1 x e^{y^4}\, dy\, dx.$$

Then change the order of integration and evaluate. Explain the simplification achieved by changing the order.

$x^{2/3} \le y \le 1$ +1

$0 \le x \le 1$




+2

$$\int_0^1 \int_0^{y^{3/2}} x e^{y^4}\, dx\, dy = \int_0^1 \frac{x^2}{2} e^{y^4} \Big|_0^{y^{3/2}} dy$$

+1 +1

$$= \int_0^1 \frac{y^3}{2} e^{y^4}\, dy$$ +1

$u = y^4 \quad du = 4y^3\, dy$

$\left.\begin{array}{l} y=0 \\ y=1 \end{array}\right\} \to \left.\begin{array}{l} u=0 \\ u=1 \end{array}\right\}$

$$= \int_0^1 \frac{1}{8} e^u\, du$$

$$= \frac{1}{8} e^u \Big|_0^1$$ +1

$$= \boxed{\frac{1}{8}(e-1)}$$

The change of order lets us evaluate the integral wrt x first so we can make a u-sub. +2

3. (7 points) Compute the average distance to the origin among points in a disk $D$ of radius 3 centered at the origin in $\mathbb{R}^2$, $D = \{(x, y) \mid x^2 + y^2 \leq 9\}$.

$d(x,y) = \sqrt{x^2+y^2}$ +1

$$f_{avg} \underset{+1}{=} \frac{\iint_D d(x,y)\, dA}{\text{area } D} = \frac{\overbrace{\int_0^{2\pi}}^{+1} \int_0^3 \overbrace{r\,(r\,dr\,d\theta)}^{+1}}{9\pi} \; +1$$

$$= \frac{1}{9\pi} \int_0^{2\pi} \frac{r^3}{3}\Big|_0^3 d\theta \; +1$$

$$= \frac{1}{9\pi} \cdot 9 \cdot \theta \Big|_0^{2\pi} \; +1$$

$$= 2$$

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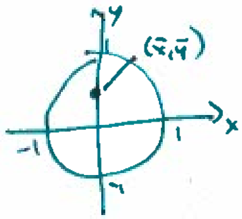
$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

4. (5 points) Suppose that the mass of a thin metal plate $D$ in the shape of a disk of radius 1 m centered at the origin is 12 kg and the center of mass of this disk is the point $(0, 1/2)$. Is it possible that the density $\rho(x, y)$ of the disk depends only on $x$ and not on $y$, i.e. $\rho = f(x)$? Give an explanation for your answer. It may help to consider $\bar{y}$.




$m = 12$ kg

It is not possible!

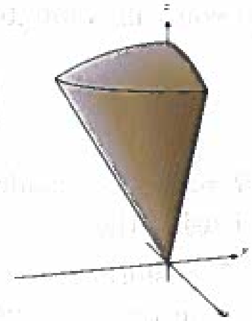
$$\bar{y} = \frac{\iint_D y\rho(x)\, dA}{12} = \frac{1}{12}\int_{-1}^{1} \rho(x) \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y \, dy \, dx$$

$$= \frac{1}{12}\int_{-1}^{1} \rho(x) \left. \frac{1}{2}y^2 \right|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{12}\int_{-1}^{1} \rho(x) \cdot 0 \, dx$$

$= 0$, not $\frac{1}{2}$!

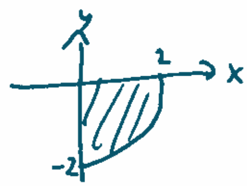
5. Let $D$ be the solid in $\mathbb{R}^3$ which is bounded below by the cone $z = \sqrt{3x^2 + 3y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 16$ and lies in the fourth octant $(x \geq 0, y \leq 0, z \geq 0)$. The projection of $D$ to the $xy$-plane is the portion of the disk $x^2 + y^2 \leq 4$ in the fourth quadrant.



(a) (5 points) Write an integral in cylindrical coordinates for the volume of $D$. Do not evaluate your integral.

$$V = \int_{-\pi/2}^{0} \int_{0}^{2} \int_{\sqrt{3}r}^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta$$

cone: $z = \sqrt{3r^2} = \sqrt{3}\, r$
sphere: $z^2 + r^2 = 16$
$z = \sqrt{16 - r^2}$




(b) (5 points) Write an integral in spherical coordinates for the volume of $D$. Do not evaluate your integral.

$$V = \int_{-\pi/2}^{0} \int_{0}^{\pi/6} \int_{0}^{4} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

cone: $z = \sqrt{3x^2 + 3y^2}$
$\rho\cos\varphi = \sqrt{3\rho^2\sin^2\varphi}$
$= \sqrt{3}\,\rho\sin\varphi$
$\tan\varphi = \frac{1}{\sqrt{3}}$
$\varphi = \pi/6$

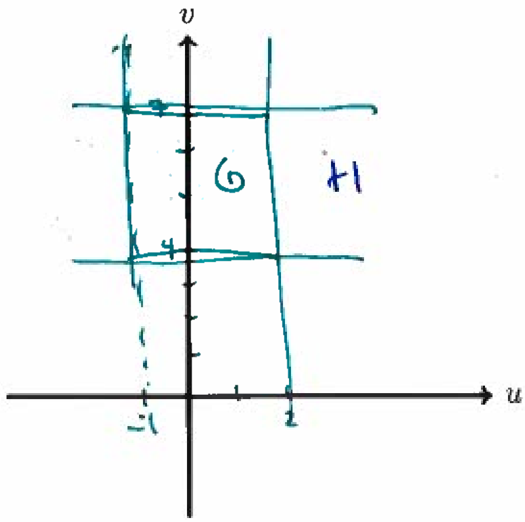
Sphere: $\rho = 4$

6. In this problem you will compute the integral

$$\iint_R (2x+y)(x-y)\; dx\; dy$$

for the region $R$ in the first quadrant bounded by the lines $2x+y=4$, $2x+y=7$, $x-y=2$, and $x-y=-1$ using the transformation $u=x-y$, $v=2x+y$.

(a) (3 points) Transform the given bounds into equations in $u$ and $v$. Use these to sketch the region $G$ in the $uv$-plane that is the image of $R$ under this transformation.




$2x+y=4 \quad v=4$
$2x+y=7 \quad v=7$ ] +1

$x-y=2 \quad u=2$
$x-y=-1 \quad u=-1$ ] +1

(b) (4 points) Solve the transformation equations for $x$ and $y$ in terms of $u$ and $v$ and compute the Jacobian determinant, i.e. find $\mathbf{T}(u,v)$ and $|D\mathbf{T}(u,V)|$. If you need more space, use the top of the next page.

$u=x-y$
$v=2x+y$

$u+v=3x \quad x=\frac{1}{3}u+\frac{1}{3}v \quad \vec{T}(u,v)=\langle \frac{1}{3}u+\frac{1}{3}v, -\frac{2}{3}u+\frac{1}{3}v\rangle$

$y=x-u=-\frac{2}{3}u+\frac{1}{3}v$

$$|D\vec{T}(u,v)| = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9}+\frac{2}{9}=\frac{1}{3}$$

(c) (2 points) (Problem 6 continued) Use your results from (a) and (b) to use change of variables with $u = x - y, v = 2x + y$ to rewrite the integral

$$\iint_R (2x + y)(x - y)\, dx\, dy$$

as an integral with respect to $u$ and $v$ over the region $G$. Do not evaluate your integral.

$$\iint_R (2x+y)(x-y)\,dx\,dy = \int_{-1}^{2} \int_{4}^{7} uv \cdot \frac{1}{3}\, dv\, du$$