MATH 2551-K/L EXAM 2
VERSION A
FALL 2022
COVERS SECTIONS 14.1-14.8

Full name: ______________________________ **GT ID:**____________________

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam.

(   ) I attest to my integrity.

Read all instructions carefully before beginning.

- You have 50 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	10
2	10
3	10
4	10
5	10
Total:	50

FORMULA SHEET

- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$
- $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin(\theta)|$
- For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n) \rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- At $\mathbf{a}$, $L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$
- If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$
- If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$
- For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$
- If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive
- If (a, b) is the location of a min or max of $f(x, y)$ subject to the constraint $g(x, y) = c$, then $\nabla f(a, b) = \lambda \nabla g(a, b)$ for some λ.

1. Choose whether the following statements are true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false.

   (a) (2 points) Suppose we have a function $f(x, y)$ and $\mathbf{u} \in \mathbb{R}^2$ is a unit vector. Then $D_{\mathbf{u}} f(a, b)$ is a vector.

   ○ **TRUE** √ **FALSE**

   (b) (2 points) If $f(x, y) = (x + y)^{50}$, then all of the 100^{th} order partial derivatives of f will be 0.

   √ **TRUE** ○ **FALSE**

   (c) (2 points) The total derivative Df of the function $f(x, y, z, w) = 2xyz + 3\ln(w) + \sin(e^{xw+z})$ is represented by a 1×4 matrix.

   √ **TRUE** ○ **FALSE**

   (d) (2 points) Suppose $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 1$ along the y-axis, the x-axis, and the line $y = x$ and $f(0, 0) = 1$. Then f must be continuous at $(0, 0)$.

   ○ **TRUE** √ **FALSE**

   (e) (2 points) The domain of the function $f(x, y) = \sqrt{x - 2} + y$ is the interval $[2, \infty)$.

   ○ **TRUE** √ **FALSE**

2. Let $f(x,y,z) = x^2 + z^2 e^{y-x}$.

(a) (5 points) Find an equation of the tangent plane to the level surface $f = 13$ at the point $P = (2, 3, \frac{3}{\sqrt{e}})$.

Solution: The equation of a tangent plane to a level surface of a function of three variables at a point $P = (x_0, y_0, z_0)$ is $\nabla f(P) \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$.
$\nabla f = \langle 2x - z^2 e^{y-x}, z^2 e^{y-x}, 2ze^{y-x}\rangle$, so $\nabla f(P) = \langle -5, 9, 6\sqrt{e}\rangle$.
Thus an equation of the tangent plane is

$$-5(x-2) + 9(y-3) + 6\sqrt{e}(z - \frac{3}{\sqrt{e}}) = 0.$$

(b) (3 points) Find the linearization $L(x,y,z)$ of f at P.

Solution: The linearization $L(x,y,z)$ at P is $L(x,y,z) = f(P) + \nabla f(P) \cdot \langle x - x_0, y - y_0, z - z_0\rangle$. $f(P) = 13$ and we computed the rest of this in part (a).
So the linearization is

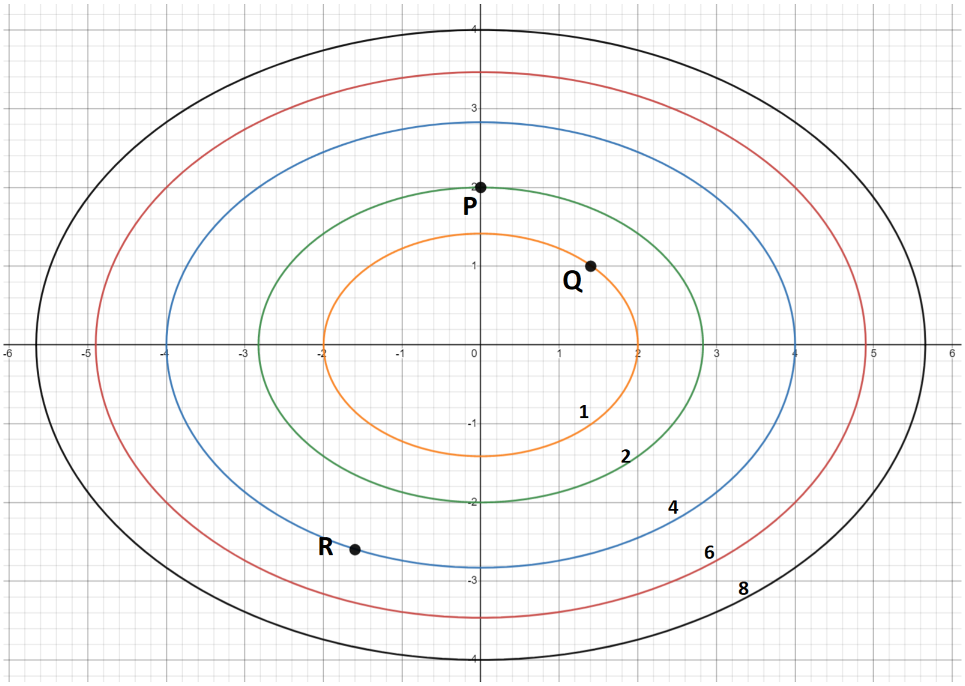
$$L(x,y,z) = 13 - 5(x-2) + 9(y-3) + 6\sqrt{e}(z - \frac{3}{\sqrt{e}}).$$

(c) (2 points) Use the linearization you found to approximate the value of $f(2, 3, \frac{4}{\sqrt{e}})$

Solution:

$$f(2,3,\frac{4}{\sqrt{e}}) \approx L(2,3,\frac{4}{\sqrt{e}}) = 13 - 5(2-2) + 9(3-3) + 6\sqrt{e}(\frac{4}{\sqrt{e}} - \frac{3}{\sqrt{e}}) = 19$$

3. Below is a contour plot for a smooth function $f(x, y)$. Use this contour plot to answer parts (a)-(c) below.



(a) (2 points) Determine the sign $(+, -, 0)$ of f_x at the point Q.

Solution: $f_x(Q) > 0$ since the contours are increasing in the positive x-direction from Q

(b) (2 points) Determine the sign $(+, -, 0)$ of the directional derivative of f in the direction $\mathbf{i} + \mathbf{j}$ at the point R.

Solution: This derivative is negative since the contours are decreasing in this direction from R

(c) (2 points) Draw a vector in the direction of greatest increase of f at the point P.

Solution: The vector is in the $\mathbf{j}$ direction.

(d) (4 points) If $g(x, y) = 2x^2 - y^2$ and $\langle x(t), y(t)\rangle = \langle \cos(t), \sin(t)\rangle$, compute $\frac{dg}{dt}$ at $t = \pi/3$.

Solution: By the Chain Rule $\frac{dg}{dt}(\pi/3) = Dg(x(\pi/3), y(\pi/3))D\mathbf{r}(\pi/3)$ where $\mathbf{r}(t) = \langle x(t), y(t)\rangle$.

Thus we have

$$\frac{dg}{dt}\left(\frac{\pi}{3}\right) = \begin{bmatrix} 4x & -2y \end{bmatrix} |_{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)} \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} |_{\pi/3} = -2\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = -\frac{3\sqrt{3}}{2}$$

4. For all parts of this question be sure to justify your conclusion, using at least one complete sentence.

   (a) (4 points) Suppose $f(x, y)$ is a differentiable function defined on all of $\mathbb{R}^2$, $Df(1,3) = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $f_{xx}(1,3) = -2$, $f_{xy}(1,3) = f_{yx}(1,3) = 4$, and $f_{yy}(1,3) = -10$.

   What can you conclude about the behavior of f at $(1, 3)$?

   Solution: Since $Df(1,3) = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $(1,3)$ is a critical point of f. Further we have $Hf(1,3) = \begin{bmatrix} -2 & 4 \\ 4 & -10 \end{bmatrix}$ with $\det(Hf(1,3)) = 20 - 16 = 4 > 0$ and $f_{xx}(1,3) < 0$. Therefore by the 2nd derivative test f has a local maximum at $(1, 3)$.

   (b) (6 points) Let $g(x, y) = 2x^2 + y^2$ defined on the disk $R = \{(x, y) \mid x^2 + y^2 \leq 1\}$. Determine the absolute extreme values of g on R.

   Solution: The critical points of g inside the disk are (x, y) such that $Dg = \begin{bmatrix} 4x & 2y \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$. The only such point is $(0, 0)$ and $g(0, 0) = 0$.

   On the boundary $x^2 + y^2 = 1$ of the disk, $g(x, y) = x^2 + (x^2 + y^2) = x^2 + 1$, and $-1 \leq x \leq 1$. The function $h(x) = x^2 + 1$ has a critical number at $x = 0$ since its derivative $h'(x) = 2x$. If $x = 0$ on the boundary, $y = \pm 1$ and at both points $g = 1$.

   Finally we check the endpoints of the domain of h, when $x = \pm 1$ and $y = 0$. At both of these points $g = 2$, so the absolute maximum of g on R is 2, achieved at $(\pm 1, 0)$ and the absolute minimum of g on R is 0 achieved at $(0, 0)$.

5. (10 points) You are traveling on an airplane and would like to know what the largest carry-on suitcase you can bring with you is. The airline requires that any carry-on item has linear size (sum of the length, width, and height) at most 72 inches. Find the dimensions and maximum volume of the box-shaped suitcase with linear size 72 inches assuming that all of the dimensions must be non-negative. Be sure to explain how you know you have found a maximum.

Solution: We are optimizing the function $V = xyz$ subject to the constraint $x + y + z = 72$. This is simplest to do with Lagrange multipliers. $\nabla V = \langle yz, xz, xy\rangle$ and $\nabla g = \langle 1, 1, 1\rangle$, so we get the system

$$yz = \lambda, \qquad xz = \lambda, \qquad xy = \lambda.$$

Hence we have

$$yz = xz = xy,$$

so from the first equality we conclude either $z = 0$ and thus the volume is 0 or $x = y$. Substituting this into the second quality gives $xz = x^2$ so either $x = 0$ and the volume is 0 or $z = x$. Therefore we have $x = y = z = 72/3 = 24$ and $V(24, 24, 24) = 24^3$ cubic inches. The two extreme values are 0 and 24^3, so the larger one must be the maximum since our domain is closed and bounded.