MATH 2551-K/L EXAM 1
VERSION A
FALL 2022
COVERS SECTIONS 12.1-13.4

Full name: Solution Key **GT ID:**

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam.

( ) I attest to my integrity.

Read all instructions carefully before beginning.

- You have 50 minutes to take the exam.
- You may not use aids of any kind.
- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	10
2	10
3	10
4	10
5	10
Total:	50

FORMULA SHEET

- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta)$
- $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin(\theta)|$
- $L = \displaystyle\int_a^b |\mathbf{r}'(t)| \, dt$
- $s(t) = \displaystyle\int_{t_0}^t |\mathbf{r}'(T)| \, dT$
- $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{d\mathbf{r}}{ds}$
- $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$
- $\mathbf{N} = \dfrac{1}{\kappa}\dfrac{d\mathbf{T}}{ds} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

1. Choose whether the following statements are true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false.

   (a) (2 points) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, then $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.

   ○ **TRUE** ● **FALSE**

   (b) (2 points) If $\mathbf{T}(t)$ is the unit tangent to $\mathbf{r}(t)$ and $\mathbf{N}(t)$ is the principal normal vector, then $\mathbf{T}(t) \cdot \mathbf{N}(t) = 0$.

   ● **TRUE** ○ **FALSE**

   (c) (2 points) A smooth curve in the plane that never crosses itself can have two distinct tangent lines at a given point.

   ○ **TRUE** ● **FALSE**

   (d) (2 points) If a spaceship is in orbit around the Moon with a constant speed of 3750 miles per hour, then it is the case that the acceleration of the spaceship is zero.

   ○ **TRUE** ● **FALSE**

   (e) (2 points) The sphere $x^2 + (y-2)^2 + (z+1)^2 = 6$ has center $(0, -2, 1)$.

   ○ **TRUE** ● **FALSE**

2. (10 points) Find the plane containing the lines

$$L_1 : x(t) = -3 + t, y(t) = 8 - 3t, z(t) = 3 - t, \qquad -\infty < t < \infty$$

and

$$L_2 : \mathbf{r}(s) = \langle -1, 2, 1\rangle + s\langle 1, 1, 1\rangle, \qquad -\infty < s < \infty.$$

Give your answer in the form $Ax + By + Cz = D$.

Identify direction vectors:

$L_1: \langle 1, -3, -1\rangle$

$L_2: \langle 1, 1, 1\rangle$

Compute normal to plane:

$\vec{n} = \vec{v}_1 \times \vec{v}_2$

$= \begin{vmatrix} i & j & k \\ 1 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \langle -2, -2, 4\rangle$

Plug in to std eqn:

$-2x - 2y + 4z = D$ or $-2(x - x_0) - 2(y - y_0) + 4(z - z_0) = 0$

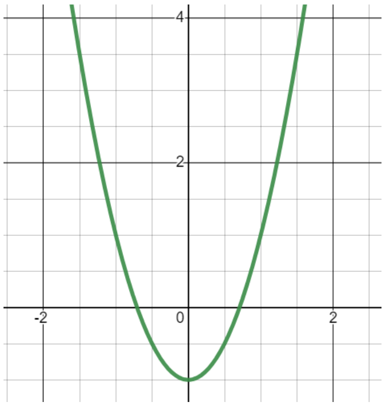
Determine D:

$D = \vec{OP} \cdot \vec{n} = (-1)(-2) + (2)(-2) + 1(4)$

$= 2$

Answer: $\boxed{-2x - 2y + 4z = 2}$

3. (a) (4 points) Suppose $\mathbf{r}(t) = e^t\mathbf{i} + (2e^{2t} - 1)\mathbf{j}$ for $-\infty < t < \infty$. Below is the graph of the curve with equation $y = 2x^2 - 1$. Does $\mathbf{r}(t)$ parameterize this curve? Explain why or why not.



· $\vec{r}(t)$ does not parameterize the curve. Although $2(x(t))^2-1 = y(t)$ we have $x(t) > 0$, so $\vec{r}(t)$ parameterizes only the right half of the curve.

(b) (6 points) Find an equation of the tangent line to the space curve

$$\mathbf{r}(t) = \langle \ln(t), t - 1, t\ln(t)\rangle$$

at $t = 1$.

Tangent line is $L(s) = \vec{r}(t_0) + s\,\vec{r}\,'(t_0)$

-Identify $t_0 = 1$

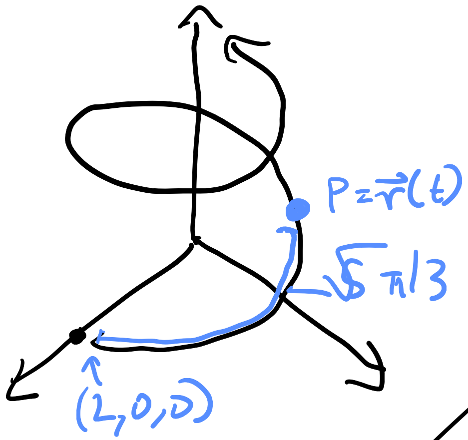
$\vec{r}(1) = \langle \ln(1), 1-1, 1\cdot\ln(1)\rangle = \langle 0, 0, 0\rangle$

$\vec{r}\,'(t) = \langle \frac{1}{t}, 1, 1\cdot\ln(t) + t\cdot\frac{1}{t}\rangle$

$\vec{r}\,'(1) = \langle 1, 1, 0+1\rangle = \langle 1, 1, 1\rangle$

Answer: $\boxed{L(s) = s\langle 1,1,1\rangle}$

4. (10 points) Find the coordinates of the point which lies a distance $\sqrt{5}\pi/3$ along the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t\rangle$ in the direction of increasing parameter t from $(2, 0, 0)$.



Recognize we need arc length with unknown upper bd

$$\int_0^t |\vec{r}\,'(T)|\, dT$$

$$\vec{r}\,'(T) = \langle -2\sin(t), 2\cos(t), 1\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{5}$$

$$= \int_0^t \sqrt{5}\, dT = \sqrt{5}\, t$$

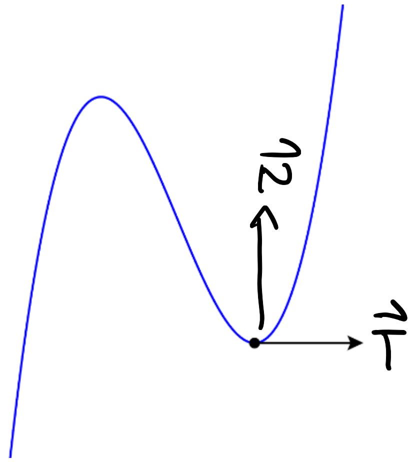
set equal to $\sqrt{5}\pi/3$

solve: $t = \pi/3$

get point: $P = (1, \sqrt{3}, \pi/3)$

Answer: $(1, \sqrt{3}, \pi/3)$

5. (a) (3 points) A plane curve is pictured below. The given vector is the unit tangent at the marked point. Draw the principal unit normal vector at that point.



(b) (4 points) You are driving along a winding mountain road. Two miles along the road, the curvature of the road is $\kappa = 1$. In another two miles, the curvature of the road is $\kappa = \frac{1}{5}$. Write a few sentences explaining which curve in the road you would rather drive faster around and why.

It is safer to drive quickly around the curve with $\kappa = \frac{1}{5}$ b/c that represents a shallower curve than $\kappa = 1$ and so we will not have to turn as fast.

(c) (3 points) Compute the unit tangent vector $\mathbf{T}(t)$ at $t = \pi$ for the helix $\mathbf{r}(t) = \langle 3t, -4\sin(t), -4\cos(t)\rangle$.

$$\vec{r}\,'(t) = \langle 3, -4\cos(t), 4\sin(t)\rangle$$

$$\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \left\langle 3/5, -4/5\cos(t), 4/5\sin(t)\right\rangle$$

Answer: $\boxed{\vec{T}(\pi) = \langle \frac{3}{5}, \frac{4}{5}, 0\rangle}$