Math 2551 Learning Target I3 Practice

I3: Change of Variables. I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.

Concept Checks

1. **T/F:** The cylindrical coordinates of the point with Cartesian coordinates $(4, \pi, 0)$ are $(-4, 0, 0)$.

2. **T/F:** The equation $r = 2\sin(\theta)$ describes a circle.

3. **T/F:** The region $0 \leq \rho \leq 4, 0 \leq \phi \leq \pi, 0 \leq \theta \leq \pi$ describes the bottom half of a sphere of radius 4 centered on the origin.

4. Which transformation can be used to simplify the integral below?

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\, dx\, dy$$

Hint: the domain of integration is the region bounded by $y = 2x, y = 2x - 4, y = 0, y = 2$

   (a) $u = x, v = y$
   (b) $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$
   (c) $u = 2x - y, v = y^7$
   (d) $u = 2x - y, v = y$
   (e) None of the above.

5. Consider the double integral

$$\iint_R \frac{2x-y}{3y-x}\, dA,$$

where R is the region bounded by the lines $2x-y=0$, $2x-y=4$, $3y-x=0$, $3y-x=2$. Which of the following coordinate transformations would be appropriate to use for change of variables on this integral?

(a) $u = \dfrac{y}{x}$
$v = \dfrac{x}{3y}$

(b) $x = u$
$y = v$

(c) $u = 2x - y$
$v = 3y - x$

(d) $x = 2u - v$
$y = 3v - y$

6. Suppose we wish to evaluate the integral $\iint_R f(x, y)\, dA$ using the change of variables

$$u = 3x \qquad v = y^5.$$

Which of the following is the new area element dA in terms of du and dv?

(a) $dA = 15v^{4/5}\ du\ dv$

(b) $dA = 15v^4\ du\ dv$

(c) $dA = \dfrac{1}{15}v^{-4/5}\ du\ dv$

(d) $dA = \ du\ dv$

(e) $dA = \dfrac{5}{3}v^{-4/5}\ du\ dv$

7. Which integral below computes the volume of the region inside the sphere $x^2+y^2+z^2 = 4$ and outside the cylinder $x^2 + y^2 = 2$?

(a) $\displaystyle\int_0^{2\pi} \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-r^2}} \ dz\ dr\ d\theta$

(b) $\displaystyle\int_0^{\pi} \int_0^{2} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\ dz\ dr\ d\theta$

(c) $\displaystyle\int_0^{2\pi} \int_0^{\pi/4} \int_{\sqrt{2}\csc(\phi)}^{2} \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta$

(d) $\displaystyle\int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_0^{2} \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta$

(e) $\displaystyle\int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_{\sqrt{2}\csc(\phi)}^{2} \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta$

8. What is the area of the region enclosed by one loop of the rose $r = \cos(3\theta)$?

(a) $\displaystyle\int_{\pi/6}^{\pi/2}\int_0^{\cos(3\theta)} dr\, d\theta$

(b) $\displaystyle\int_{\pi/2}^{3\pi/2}\int_0^{\cos(3\theta)} r\, dr\, d\theta$

(c) $\displaystyle\int_{0}^{\pi/6}\int_0^{\cos(3\theta)} r\, dr\, d\theta$

(d) $\displaystyle\int_{0}^{2\pi}\int_0^{\cos(3\theta)} r\, dr\, d\theta$

(e) $\displaystyle\int_{\pi/6}^{\pi/2}\int_0^{\cos(3\theta)} r\, dr\, d\theta$

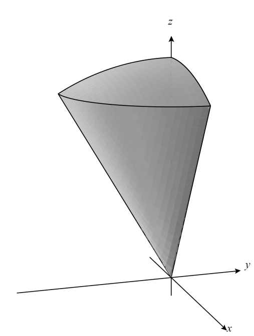
9. **T/F:** The Jacobian determinant for a change of variables is always constant.

Open Response

1. Compute the average distance to the origin among points in a disk D of radius 3 centered at the origin in $\mathbb{R}^2$, $D = \{(x, y) \mid x^2 + y^2 \leq 9\}$ given that

$$d_{avg} = \frac{1}{9\pi}\iint_D \sqrt{x^2+y^2}\, dA.$$

2. Let D be the solid in $\mathbb{R}^3$ which is bounded below by the cone $z = \sqrt{3x^2+3y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 16$ and lies in the fourth octant ($x \geq 0, y \leq 0, z \geq 0$). The projection of D to the xy-plane is the portion of the disk $x^2 + y^2 \leq 4$ in the fourth quadrant.



(a) Write an integral in cylindrical coordinates for the volume of D. Do not evaluate your integral.

(b) Write an integral in spherical coordinates for the volume of D. Do not evaluate your integral.

3. Convert the integral $\displaystyle\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} xy\, dy\, dx$ to an equivalent integral in polar coordinates.

4. In this problem you will compute the integral

$$\iint_R (2x+y)(x-y)\ dx\ dy$$

for the region R in the first quadrant bounded by the lines $2x+y=4, 2x+y=7, x-y=2$, and $x-y=-1$ using the transformation $u=x-y, v=2x+y$.

   (a) Transform the given bounds into equations in u and v. Use these to sketch the region G in the uv-plane that is the image of R under this transformation.

   (b) Solve the transformation equations for x and y in terms of u and v and compute the Jacobian determinant, i.e. find $\mathbf{T}(u,v)$ and $|D\mathbf{T}(u,v)|$.

   (c) Use your results from (a) and (b) to use change of variables with $u=x-y, v=2x+y$ to rewrite the integral

$$\iint_R (2x+y)(x-y)\ dx\ dy$$

   as an integral with respect to u and v over the region G. Do not evaluate your integral.

5. Consider the volume D in the first octant of $\mathbb{R}^3$ which is bounded above by the sphere $x^2+y^2+z^2=12$ and below by the paraboloid $z=x^2+y^2$. If D has a constant density function $\delta=16$, compute $M_{xy}=\iiint_D z\delta\ dV$, the first moment of D about the xy-plane. Fully simplify your answer.

   Hint: Be careful with your choice of coordinate system.

6. Consider the volume D in the second octant of $\mathbb{R}^3$ ($x\le 0, y\ge 0, z\ge 0$) which is bounded above by the sphere $x^2+y^2+z^2=2$ and below by the paraboloid $z=x^2+y^2$.

   (a) Write an integral for the volume of D using Cartesian coordinates in the order $dz\ dy\ dx$. Show your work. **Do not evaluate your integral**.

   (b) Write an integral for the volume of D using cylindrical coordinates. Show your work. **Do not evaluate your integral**.

   (c) Sketch the shadow of D in the yz-plane.

   (d) Are spherical coordinates a good choice to use to compute this volume? Justify your answer.

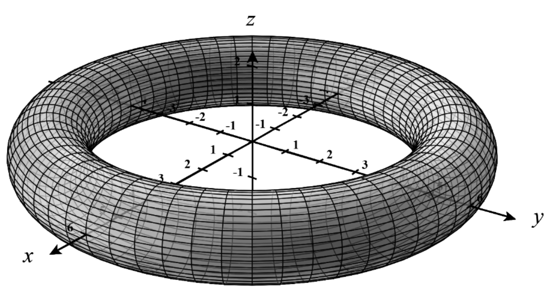
7. Use polar coordinates to find

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{-6}{x^2+y^2+1}\ dy\ dx.$$

8. Use the transformation
$$\mathbf{T}(u,v) = \begin{bmatrix} \frac{1}{4}(u+v) \\ \frac{1}{4}(v-3u) \end{bmatrix}$$
to change variables in the integral
$$\iint_R 4x + 8y \, dA$$
where R is the parallelogram bounded by the lines
$$x - y = -4, \quad x - y = 4, \quad y + 3x = 0, \quad y + 3x = 8.$$
A complete solution will include a sketch of the new region of integration in the uv-plane. **Do not evaluate the resulting integral in terms of u and v.**

9. Set up an integral using spherical or cylindrical coordinates to compute the volume of the **torus** centered on the z-axis with inner radius 4 and cross-sectional radius 1. A picture of this solid is below.

   Hint: It may help to sketch a vertical cross-section of the shape and exploit symmetry.



10. Convert the integral $\displaystyle\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x^2 + y^2 \; dy \; dx$ to an equivalent integral in polar coordinates.

11. Consider the region D in $\mathbb{R}^3$ given by $\sqrt{x^2+y^2} \leq z \leq \sqrt{4-x^2-y^2}$ and suppose a solid occupying this region has mass density $\delta(x,y,z) = x^2+y^2+z^2$.

    (a) Write an integral in cylindrical coordinates for the mass $\iiint_D \delta \; dV$ of the solid. **Do not evaluate your integral.**

    (b) Write an integral in spherical coordinates for the mass $\iiint_D \delta \; dV$ of the solid. **Do not evaluate your integral.**

Math 2551 LT I3 Practice Answers

Concept Checks

1. False
2. True
3. False
4. d) $u = 2x - y, v = y$
5. Choice three is correct, $u = 2x - y, v = 3y - x$. Applying this transformation results in a region of integration $0 \leq u \leq 4, 0 \leq v \leq 2$ and an integrand of $\frac{u}{v}$ multiplied by the Jacobian (a constant). Choice one does not map either the integrand or the region to a nice object, choice two does nothing but rename the variables, and choice four has x, y swapped with u, v, so also works poorly.
6. (c) $dA = \dfrac{1}{15} v^{-4/5}\ du\ dv$
7. (e)
8. (e)
9. False

Open Response

1. This region and function are nicely described in polar coordinates, so we have

$$\begin{aligned} d_{avg} &= \frac{1}{9\pi} \iint_D \sqrt{x^2 + y^2}\ dA \\ &= \frac{1}{9\pi} \int_0^{2\pi} \int_0^3 r^2\ dr\ d\theta \\ &= \frac{1}{9\pi} 18\pi \\ &= 2 \end{aligned}$$

2. (a)

$$\int_{-\pi/2}^{0} \int_0^2 \int_{\sqrt{3}r}^{\sqrt{16-r^2}} r\ dz\ dr\ d\theta$$

(b)

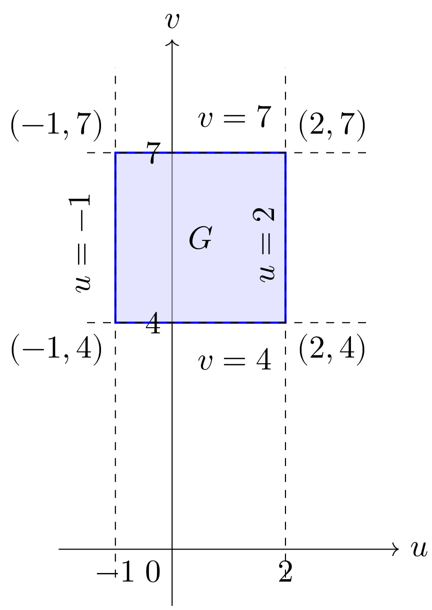
$$\int_{-\pi/2}^{0}\int_{0}^{\pi/6}\int_{0}^{4} \rho^2 \sin(\varphi)\ d\rho\ d\varphi\ d\theta$$

3. We have $-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$ and $0 \leq x \leq 2$, so this region of integration is the right half of a circle of radius 2 centered at the origin.

   Thus the integral becomes

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{2} (r\cos(\theta))(r\sin(\theta))r\ dr\ d\theta.$$

4. (a) $-1 \leq u \leq 2, 4 \leq v \leq 7$



(b)

$$\mathbf{T}(u,v) = \begin{bmatrix} \frac{1}{3}u + \frac{1}{3}v \\ -\frac{2}{3}u + \frac{1}{3}v \end{bmatrix} \qquad |D\mathbf{T}(u,v)| = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}$$

(c)

$$\iint_R (2x+y)(x-y)\ dx\ dy = \int_{-1}^{2}\int_{4}^{7} \frac{1}{3}uv\ dv\ du.$$

5. We have $M_{xy} = \iiint_D z\delta\ dV$, so we compute $\iiint_D 16z\ dV$. This is easiest in cylindrical coordinates, in which D is described by $r^2 \leq z \leq \sqrt{12-r^2}$, $0 \leq \theta \leq \pi/2$. To find bounds on r, we solve $r^2 = \sqrt{12-r^2}$ to get $r^4 = 12 - r^2$, which then gives $(r^2+4)(r^2-3) = 0$. Thus we have $0 \leq r \leq \sqrt{3}$.

Now we integrate:

$$\begin{aligned}
M_{xy} &= \iiint_D 16z\, dV = \int_0^{\pi/2} \int_0^{\sqrt{3}} \int_{r^2}^{\sqrt{12-r^2}} 16zr\, dz\, dr\, d\theta \\
&= \frac{\pi}{2} \int_0^{\sqrt{3}} (8rz^2)|_{r^2}^{\sqrt{12-r^2}}\, dr = 4\pi \int_0^{\sqrt{3}} r(12 - r^2 - r^4)\, dr \\
&= 4\pi \int_0^{\sqrt{3}} 12r - r^3 - r^5\, dr \\
&= 4\pi(6r^2 - \frac{1}{4}r^4 - \frac{1}{6}r^6)|_0^{\sqrt{3}} \\
&= 4\pi(18 - \frac{9}{4} - \frac{27}{6}) = \boxed{45\pi}
\end{aligned}$$

6. Consider the volume D in the second octant of $\mathbb{R}^3$ $(x \leq 0, y \geq 0, z \geq 0)$ which is bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

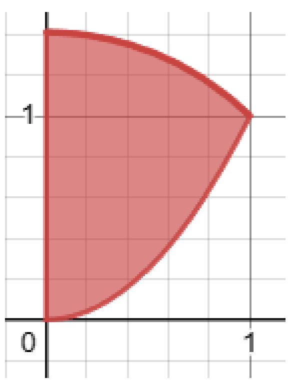
   (a) These two surfaces intersect when $z + z^2 = 2$ and $z \geq 0$, i.e. $z = 1$. When $z = 1$, they meet in the circle $x^2 + y^2 = 1$. Therefore since we are taking only the portion of this volume in the second octant we have

   $$V = \int_{-1}^{0} \int_0^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dz\, dy\, dx.$$

   (b) In cylindrical coordinates, the surfaces are $r^2 + z^2 = 2$ and $z = r^2$, meeting in the circle $r = 1$ in the plane $z = 1$. Now θ runs from $\pi/2$ to π so that we get the portion in the second octant.

   So we have

   $$V = \int_{\pi/2}^{\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r\, dz\, dr\, d\theta.$$

   (c) 

   (d) They are not, because this region is not simple in the ρ-direction. This can be seen in the picture in part (c); a ray outward from the origin cuts through the sphere for $0 \leq \phi \leq \pi/4$ and through the paraboloid for $\pi/4 \leq \phi \leq \pi/2$.

7. In polar coordinates, the region

$$0 \le x \le 2, \quad -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$$

becomes

$$0 \le r \le 2, -\pi/2 \le \theta \le \pi/2,$$

since this is the right half of the disk of radius 2 centered at the origin. So we have

$$\begin{aligned}
\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{-6}{x^2+y^2+1}\ dy\ dx &= \int_{-\pi/2}^{\pi/2} \int_0^2 \frac{-6}{r^2+1} r\ dr\ d\theta \\
&= \int_{-\pi/2}^{\pi/2} -3\ln(r^2+1)|_0^2\ d\theta \\
&= \int_{-\pi/2}^{\pi/2} -3\ln(5)\ d\theta \\
&= -3\ln(5)\theta|_{-\pi/2}^{\pi/2} \\
&= -3\pi\ln(5)
\end{aligned}$$

8. First, notice that we have

$$x - y = \frac{1}{4}u + \frac{1}{4}v - \frac{1}{4}v + \frac{3}{4}u = u \qquad \text{and} \qquad y + 3x = \frac{1}{4}v - \frac{3}{4}u + \frac{3}{4}u + \frac{3}{4}v = v.$$

So the new region of integration is the rectangle $-4 \le u \le 4, 0 \le v \le 8$ in the uv-plane. Next, we have

$$|\det(D\mathbf{T}(u,v)) = |\det \begin{bmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{bmatrix}| = \left|\frac{1}{16} + \frac{3}{16}\right| = \frac{1}{4}.$$

Finally, we have

$$4x + 8y = u + v - 6u + 2v = -5u + 3v.$$

Combining all of this, the Substitution Theorem gives

$$\iint_R 4x + 8y\ dA = \int_{-4}^4 \int_0^8 \frac{3v-5u}{4}\ dv\ du.$$

9. It is easiest to work in cylindrical coordinates. In any cross-sectional plane $\theta = c$ (for example the yz-plane) we have a circle of radius 1 centered at the point $r = 5, z = 0$. So each such cross-section has equation

$$(r-5)^2 + z^2 = 1$$

and therefore an integral for the volume of the torus is

$$\int_0^{2\pi} \int_4^6 \int_{-\sqrt{1-(r-5)^2}}^{\sqrt{1-(r-5)^2}} r\ dz\ dr\ d\theta.$$

10. We have $-\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2}$ and $0 \leq x \leq 3$, so this region of integration is the right half of a circle of radius 3 centered at the origin.

    Thus the integral becomes
$$\int_{-\pi/2}^{\pi/2} \int_0^3 (r^2)\ r\ dr\ d\theta.$$

11. Consider the region D in $\mathbb{R}^3$ given by $\sqrt{x^2+y^2} \leq z \leq \sqrt{4-x^2-y^2}$ and suppose a solid occupying this region has mass density $\delta(x,y,z) = x^2+y^2+z^2$.

    (a) The cone $z = r$ intersects the upper hemisphere $z = \sqrt{4-r^2}$ when $r^2 = 4-r^2$, i.e. $r = \sqrt{2}$. The density is $\delta = r^2+z^2$. Therefore we have

$$\text{mass} = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} (r^2+z^2) r\ dz\ dr\ d\theta.$$

    (b) In spherical coordinates, the hemisphere is the part of $\rho = 2$ with $0 \leq \varphi \leq \pi/2$ and the region above the cone $z \geq \sqrt{x^2+y^2}$ is the region with $\varphi \leq \pi/4$. The density is $\delta = \rho^2$. Therefore we have

$$\text{mass} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^4 \sin(\varphi)\ d\rho\ d\varphi\ d\theta.$$