Math 2551 Learning Target G3 Practice

G3: Geometry of Curves. I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.

Concept Checks

1. **T/F:** If $\mathbf{T}(t)$ is the unit tangent to $\mathbf{r}(t)$ and $\mathbf{N}(t)$ is the principal normal vector, then $\mathbf{T}(t) \cdot \mathbf{N}(t) = 0$.

2. Order the following curves in increasing order of curvature.

   A) A helix of curvature 2
   B) A line segment of length 10
   C) A circle of radius 20

3. Choose the curve below whose curvature at the indicated point is larger than all of the other curves.

   (a) **A)** The line $\ell(t) = \langle t+1, 3t+3, 5t-5\rangle$ at the point $(2, 6, 0)$.
   (b) **B)** The circle $x^2 + y^2 = 1$ at the point $(1, 0)$
   (c) **C)** The circle $x^2 + y^2 = 4$ at the point $(\sqrt{2}, \sqrt{2})$
   (d) **D)** A curve with constant curvature $\kappa = \frac{1}{2}$ at any point on the curve.
   (e) **E)** No single curve above has the greatest curvature at the indicated point.

4. Which statement below about arc length is **not** true?

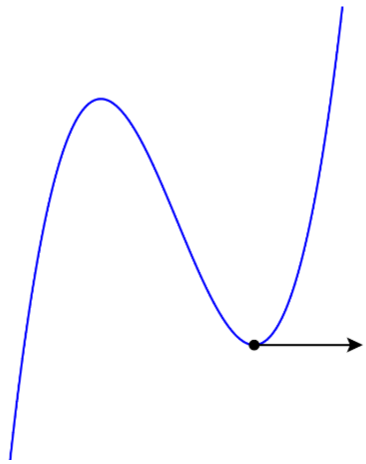
   (a) The length of a curve C is always positive.
   (b) The length of a curve C may be equal to the distance between its endpoints.
   (c) The length of a curve C depends on the parameterization we use to compute it.
   (d) The length of a curve C parameterized by arc length can be greater than 1.
   (e) Every smooth curve can be parameterized by arc length.

5. **T/F:** The curvature of a line is 0 at every point on the line.

Open Response

6. Find the coordinates of the point which lies a distance $\sqrt{5}\pi/3$ along the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t\rangle$ in the direction of increasing parameter t from $(2, 0, 0)$.

7. (a) A plane curve is pictured below. The given vector is the unit tangent at the marked point. Draw the principal unit normal vector at that point.



   (b) You are driving along a winding mountain road. Two miles along the road, the curvature of the road is $\kappa = 1$. In another two miles, the curvature of the road is $\kappa = \frac{1}{5}$. Write a few sentences explaining which curve in the road you would rather drive faster around and why.

   (c) Compute the unit tangent vector $\mathbf{T}(t)$ at $t = \pi$ for the helix $\mathbf{r}(t) = \langle 3t, -4\sin(t), -4\cos(t)\rangle$.

8. Find the length of the portion of the helix $\mathbf{r}(t) = \langle 2\sin(t), 5t, 2\cos(t)\rangle$ between $(0, 0, 2)$ and $(0, 5\pi, -2)$.

9. Consider the curve parameterized by $\mathbf{r}(t) = \langle \sqrt{7}e^t, e^t\cos(t), e^t\sin(t)\rangle$ for $t \in \mathbb{R}$.

   (a) Compute the unit tangent vector $\mathbf{T}(t)$.

   (b) Compute the curvature $\kappa(t)$.

10. Which of the following vectors could be the principal unit normal vector at time $t = 2$ to a curve whose tangent line at $t = 2$ is given by

$$\ell(p) = \langle 1, 0, -1\rangle + p\langle 1, 2, 1\rangle.$$

You must justify your answer to receive full credit.

A) $\langle 1, 1, 1\rangle$
B) $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$
C) $\langle -2, 1, 0\rangle$
D) $\left\langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right\rangle$

11. Tobert, a hard-working documentarian, is hiking through the African savannah for two days and his path is given by the curve $\mathbf{r}(t) = \langle t^2, \frac{1}{3}t^3, 2\rangle$ for $0 \leq t < 48$, where t is measured in hours and $\mathbf{r}(t)$ in meters.

(a) What is Tobert's position two hours into his hike?
(b) Find a function $s(t)$ that gives the total distance Tobert has hiked for any time t since he started at $t = 0$.
(c) How far has Tobert hiked when he is at the point $(4, 8/3, 2)$?

12. Consider the curve C which is the part of a helix between $(2, 0, 0)$ and $(2, 0, 4\pi)$ parameterized by

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t\rangle, \quad 0 \leq t \leq 4\pi.$$

(a) Compute the arc length function for C based at $(0, 2, \pi/2)$.
(b) Use your arc length function to compute the length of the section of the curve between $(-2, 0, \pi)$ and $(0, 2, 5\pi/2)$. Fully simplify your answer.

13. Find the curvature $\kappa(t)$ of the plane curve

$$\mathbf{r}(t) = \langle \cos^3(t), \sin^3(t)\rangle, 0 \leq t \leq \pi/2.$$

14. Let C be the curve which is the graph of the vector-valued function $\mathbf{r}(t) = \left\langle \frac{t^2}{2}, \frac{4}{3}t^{3/2}, 2t \right\rangle$, $0 \leq t < \infty$.

    (a) Compute $\mathbf{r}(0)$ and $\mathbf{r}(4)$.

    (b) Compute the arc length function $s(t)$ for this parameterization, taking $t_0 = 0$ as your reference point.

    (c) Find the distance along C between the points $(0, 0, 0)$ and $(8, \frac{32}{3}, 8)$.

15. In this problem, you will work with normal vectors and curvature.

    (a) Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ be any parameterized curve in the plane. Show that the vector $\mathbf{n}(t) = -g'(t)\mathbf{i} + f'(t)\mathbf{j}$ is orthogonal to $\mathbf{r}'(t)$ for every value of t.

    (b) Use your result from part (a) to produce a **unit** normal vector to the curve $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle, 0 \leq t < 2\pi$.

    (c) Is the vector that you produced in part (b) the principal unit normal vector? Explain your answer; a picture might be useful.
    Hint: Consider whether your vector is pointing in the direction of curvature of this curve.

16. In this problem, you will work with the vector-valued function $\mathbf{r}(t) = \langle \sqrt{2}t, e^t, e^{-t}\rangle$, defined for $t \in \mathbb{R}$.

    (a) Find the two values of t where $\mathbf{r}(t)$ is $\langle 0, 1, 1\rangle$ and $2\sqrt{2}\mathbf{i} + e^2\mathbf{j} + \frac{1}{e^2}\mathbf{k}$, respectively.

    (b) Find the length of the curve $\mathbf{r}(t)$ between $(0, 1, 1)$ and $(2\sqrt{2}, e^2, \frac{1}{e^2})$. Show all of your work and fully simplify your final answer.

17. In this problem, you will work with the spiral plane curve

$$\mathbf{r}(t) = \langle \cos(t) + t\sin(t), \sin(t) - t\cos(t)\rangle$$

    for $t > 0$.

    (a) Compute the unit tangent vector $\mathbf{T}(t)$.

    (b) Compute the principal unit normal vector $\mathbf{N}(t)$.

    (c) Compute the curvature $\kappa(t)$.

18. Find the arc length of the portion of the circle

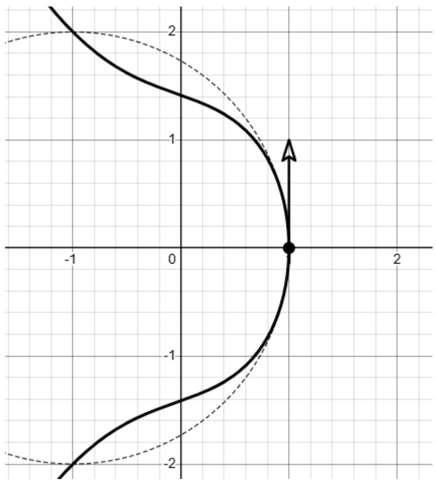
$$\langle \sqrt{2}\cos(t), \sqrt{2}\sin(t), 2\rangle, \qquad 0 \leq t \leq 2\pi$$

    that lies between $(1, 1, 2)$ and $(-1, 1, 2)$.

19. Consider the curve parameterized by $\mathbf{r}(t) = \langle e^t + e^{-t}, 2t \rangle$ for $t \in \mathbb{R}$.

    (a) Compute the unit tangent vector $\mathbf{T}(t)$. Fully simplify your answer.

    (b) Sketch the principal unit normal vector to the solid curve drawn below at the marked point. The drawn arrow is the tangent vector at that point.



    (c) The dashed circle drawn in the graph above has radius 2 and exactly matches the curve at the marked point. What can you conclude about the curvature of the curve at the marked point?

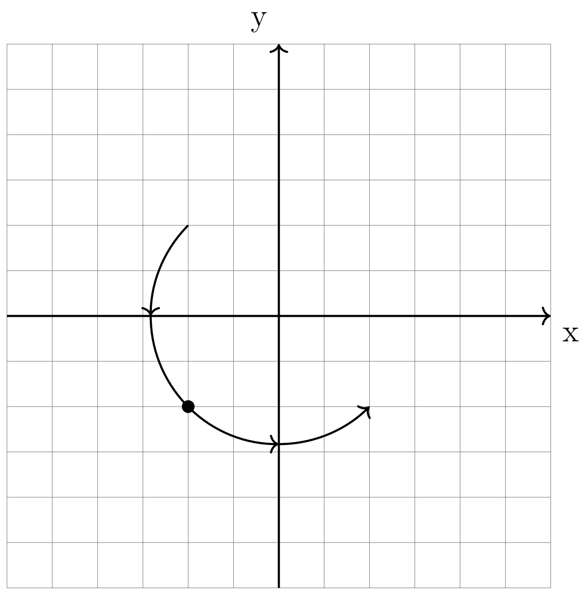
20. Pictured below is a curve C. The marked point on the curve C is the point P referenced below. Each grid line is spaced 0.5 units apart. Sketch the following on this graph. Clearly indicate which vector is which.

    (a) The unit tangent vector to C at P.

    (b) The principal unit normal vector to C at P.

    (c) Another curve passing through P which has the same principal unit normal vector as C at P, but has **higher** curvature at P and passes through P with the **opposite** orientation as C.

    (d) Explain in one or two sentences why your curve in (c) does or does not have the same **tangent line** as C at P.



21. Compute the length of the curve

$$\mathbf{r}(t) = \langle t^2, \frac{1}{3}t^3, 2\rangle, \qquad 0 \leq t < 4.$$

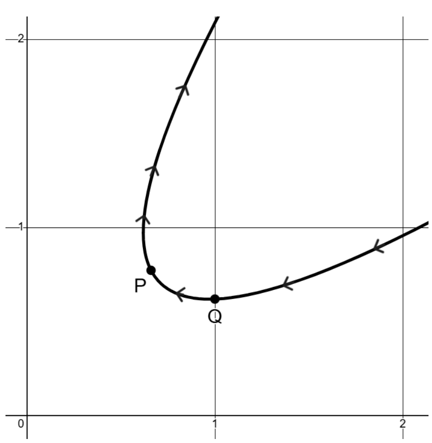
22. Pictured below is a curve C. Sketch the following on this graph and answer the True/False question. Clearly indicate which vector is which.

    (a) The unit tangent vector to C at P.

    (b) The principal unit normal vector to C at P.

    (c) Another curve passing through P which has the **same** unit tangent vector as C at P, but has **lower** curvature at P and has a **different** principal unit normal vector at C.

    (d) **True/False:** The length of the curve between points P and Q is the distance between P and Q.



Math 2551 LT G3 Practice Answers

Concept Checks

1. True.
2. $B < C < A$
3. (b)
4. (c)
5. True

Open Response

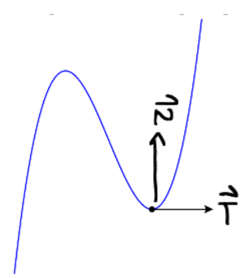
6. The distance from $(2, 0, 0)$ to the point on the helix at time t is

$$s(t) = \int_0^t \|\mathbf{r}'(T)\| \, dT.$$

Here we have $\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t), 1\rangle$ so $\|\mathbf{r}'(t)\| = \sqrt{5}$ and therefore

$$s(t) = \sqrt{5}t.$$

We set this equal to $\sqrt{5}\pi/3$ to find $t_1 = \pi/3$, so the point at this distance is $\mathbf{r}(\pi/3) = \langle(1, \sqrt{3}, \pi/3\rangle$.

7. (a) 

   (b) Many possible correct answers. One is below.

   It is safer to drive quickly around the curve with $\kappa = \frac{1}{5}$ because that is a shallower curve than the one with $\kappa = 1$ and so we will not have to turn as fast.

(c) Here $\mathbf{r}'(t) = \langle 3, -4\cos(t), 4\sin(t)\rangle$ and $\|\mathbf{r}'(t)\| = 5$, so

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \langle 3/5, (-4/5)\cos(t), 4/5\sin(t)\rangle$$

and so

$$\mathbf{T}(\pi) = \langle 3/5, 4/5, 0\rangle.$$

8. We need to compute $\|\mathbf{r}'(t)|$ to find this arc length. We have $\mathbf{r}'(t) = \langle 2\cos(t), 5, -2\sin(t)\rangle$, so $|\mathbf{r}'(t)| = \sqrt{(2\cos(t))^2 + 5^2 + (-2\sin(t))^2} = \sqrt{29}$. The given parameterization reaches $(0, 0, 2)$ when $t = 0$ and $(0, 5\pi, -2)$ when $t = \pi$.

   Thus we have

$$\begin{aligned} \text{length} &= \int_0^\pi |\mathbf{r}'(t)|\ dt \\ &= \int_0^\pi \sqrt{29}\ dt \\ &= \sqrt{29}\pi. \end{aligned}$$

(a) We have $\mathbf{r}(t) = e^t\langle \sqrt{7}, \cos(t), \sin(t)\rangle$, so by the product rule

$$\begin{aligned} \mathbf{r}'(t) &= e^t\langle \sqrt{7}, \cos(t), \sin(t)\rangle + e^t\langle 0, -\sin(t), \cos(t)\rangle \\ &= e^t\langle \sqrt{7}, \cos(t) - \sin(t), \cos(t) + \sin(t)\rangle. \end{aligned}$$

Therefore

$$\begin{aligned} |\mathbf{r}'(t)| &= e^t\sqrt{7 + (\cos(t) - \sin(t))^2 + (\cos(t) + \sin(t))^2} \\ &= e^t\sqrt{7 + \cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t) + \cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t)} \\ &= e^t\sqrt{7 + 2\cos^2(t) + 2\sin^2(t)} \\ &= e^t\sqrt{9} \\ &= 3e^t \end{aligned}$$

Combining these, we get that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{3}\langle \sqrt{7}, \cos(t) - \sin(t), \cos(t) + \sin(t)\rangle.$$

(b) First we compute $\mathbf{T}'(t)$:

$$\mathbf{T}'(t) = \frac{1}{3}\langle 0, -\sin(t) - \cos(t), -\sin(t) + \cos(t)\rangle$$

Then we have

$$\begin{aligned}
|\mathbf{T}'(t)| &= \frac{1}{3}\sqrt{0^2 + (-\sin(t) - \cos(t))^2 + (-\sin(t) + \cos(t))^2} \\
&= \frac{1}{3}\sqrt{\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t) + \cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t)} \\
&= \frac{1}{3}\sqrt{2\cos^2(t) + 2\sin^2(t)} \\
&= \frac{\sqrt{2}}{3}.
\end{aligned}$$

Therefore

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{2}/3}{3e^t} = \frac{\sqrt{2}}{9e^t}.$$

9. The correct answer is D). Only B) and D) are unit vectors, and only D) is normal to the curve, since the dot product between it and the tangent vector $\langle 1, 2, 1\rangle$ at $t = 2$ is 0, while the dot product with B is non-zero.

10. (a) $\mathbf{r}(2) = \langle 4, 8/3, 2\rangle$

(b)

$$\begin{aligned}
s(t) &= \int_0^t |\mathbf{r}'(\tau)| d\tau \\
&= \int_0^t |\langle 2\tau, \tau^2, 0\rangle| d\tau \\
&= \int_0^t \sqrt{4\tau^2 + \tau^4 + 0} d\tau \\
&= \int_0^t \tau\sqrt{4 + \tau^2} d\tau
\end{aligned}$$

Let $u = 4 + \tau^2$, $du = 2\tau\, d\tau$. Then we have

$$\begin{aligned}
s(t) &= \int_4^{4+t^2} \frac{1}{2}\sqrt{u} du \\
&= \frac{1}{3} u^{3/2}|_4^{4+t^2} \\
&= \frac{(4+t^2)^{3/2} - 8}{3}
\end{aligned}$$

(c) From a) and b), this is just $s(2)$.

$$s(2) = \frac{8^{3/2} - 8}{3} = \frac{8}{3}(2\sqrt{2} - 1)$$

11. (a) The value of t corresponding to $(0, 2, \pi/2)$ is $t = \pi/2$. So we have

$$\begin{aligned} s(t) &= \int_{t_0}^{t} \|\mathbf{r}'(T)\| \, dT \\ &= \int_{\pi/2}^{t} \|\langle -2\sin(T), 2\cos(T), 1\rangle\| \, dT \\ &= \int_{\pi/2}^{t} \sqrt{4\sin^2(T) + 4\cos^2(T) + 1} \, dT \\ &= \int_{\pi/2}^{t} \sqrt{5} \, dT \\ &= \sqrt{5}\left(t - \frac{\pi}{2}\right) \end{aligned}$$

(b) These points correspond to $t_1 = \pi$ and $t_2 = 5\pi/2$ respectively. So the length of this part of the curve is

$$L = s(t_2) - s(t_1) = s(5\pi/2) - s(\pi) = \sqrt{5}(\frac{5\pi}{2} - \frac{\pi}{2}) - \sqrt{5}(\pi - \frac{\pi}{2}) = \frac{3\pi\sqrt{5}}{2}$$

12. We compute

$$\begin{aligned} \mathbf{r}'(t) &= \langle -3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t)\rangle, \text{ so} \\ \|\mathbf{r}'(t)\| &= \langle\sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)} \\ &= \sqrt{9\cos^2(t)\sin^2(t)(\cos^2(t) + \sin^2(t))} \\ &= 3\cos(t)\sin(t). \end{aligned}$$

Therefore

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \langle -\cos(t), \sin(t)\rangle.$$

Now we compute

$$\mathbf{T}'(t) = \langle \sin(t), \cos(t)\rangle \quad \text{and} \quad \|\mathbf{T}'(t)\| = 1$$

and we can combine these to get the curvature.

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{3\cos(t)\sin(t)}$$

13. (a) $\mathbf{r}(0) = \langle 0, 0, 0\rangle$
$\mathbf{r}(4) = \langle 4^2/2, 4/3(4)^{(3/2)}, 2(4)\rangle = \langle 8, \frac{32}{3}, 8\rangle$.

(b)

$$\begin{aligned} s(t) &= \int_{t_0}^{t} |\mathbf{r}'(T)| \, dT \\ &= \int_0^t |\langle T, 2\sqrt{T}, 2\rangle| \, dT \\ &= \int_0^t \sqrt{T^2+4T+4} \, dT \\ &= \int_0^t T+2 \, dT \\ &= \frac{1}{2}t^2+2t \end{aligned}$$

(c) The distance is $s(4)-s(0) = 1/2 \cdot 4^2 + 2(4) - (1/2 \cdot 0^2 + 2(0)) = 16$ units.

14. (a) $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$. Therefore we have

$$\mathbf{r}'(t) \cdot \mathbf{n}(t) = (f'(t)\mathbf{i} + g'(t)\mathbf{j}) \cdot (-g'(t)\mathbf{i} + f'(t)\mathbf{j}) = -f'(t)g'(t) + f'(t)g'(t) = 0.$$

Since this dot product is 0 for every value of t, $\mathbf{n}(t)$ and $\mathbf{r}'(t)$ are orthogonal for every value of t.

(b) We showed in (a) that $\mathbf{n}(t)$ is normal to the curve $\mathbf{r}(t)$. So $\mathbf{n}(t) = \langle -2\cos(t), -2\sin(t)$ is a normal vector to this curve. To get a unit normal vector, we divide by $|\mathbf{n}(t)| = 2$ to get the vector

$$\hat{\mathbf{n}}(t) = \langle -\cos(t), -\sin(t)\rangle.$$

(c) The principal unit normal vector to this curve (a circle of radius 2) points inward toward the origin, which this vector does do, so $\hat{\mathbf{n}}(t)$ is the principal unit normal vector.

15. (a) We have $\mathbf{r}(0) = \langle 0, 1, 1\rangle$ and $\mathbf{r}(2) = 2\sqrt{2}\mathbf{i} + e^2\mathbf{j} + \dfrac{1}{e^2}\mathbf{k}$.

(b) By part (a), this is $\int_0^2 |\mathbf{r}'(t)|\ dt$.

$$\begin{aligned}
L(t) &= \int_0^2 |\mathbf{r}'(t)|\ dt \\
&= \int_0^2 |\langle \sqrt{2}, e^t, e^{-t}\rangle|\ dt \\
&= \int_0^2 \sqrt{2 + e^{2t} + e^{-2t}}\ dt \\
&= \int_0^2 \sqrt{(e^t + e^{-t})^2}\ dt \\
&= \int_0^2 e^t + e^{-t}\ dt \\
&= e^t - e^{-t}|_0^2 \\
&= e^2 - e^{-2} - (1 - 1) \\
&= e^2 - \frac{1}{e^2}.
\end{aligned}$$

16. (a) We have $\mathbf{r}'(t) = \langle -\sin(t) + \sin(t) + t\cos(t), \cos(t) - \cos(t) + t\sin(t)\rangle = t\langle \cos(t), \sin(t)\rangle$. Thus $|\mathbf{r}'(t)| = t\sqrt{\cos^2(t) + \sin^2(t)} = t$ and we have

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \langle \cos(t), \sin(t)\rangle.$$

(b) We have $\mathbf{T}'(t) = \langle -\sin(t), \cos(t)\rangle$ and $|\mathbf{T}'(t)| = \sqrt{\sin^2(t) + \cos^2(t)} = 1$, so

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\sin(t), \cos(t)\rangle.$$

(c) Using our calculation in (b), we have

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{t}.$$

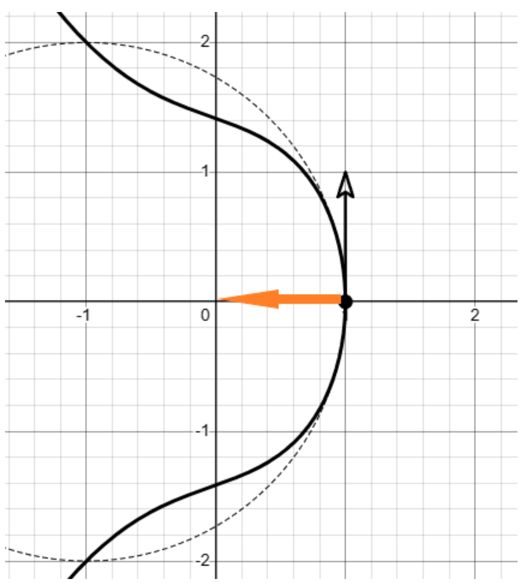
17. First, we find the parameter values t_0 and t_1 at which the parameterization passes through these points. At $(1, 1, 2)$ we have $\sqrt{2}\cos(t_0) = 1 = \sqrt{2}\sin(t_0)$, so $t_0 = \pi/4$. At $(-1, 1, 2)$ we have $-\sqrt{2}\cos(t_1) = 1 = \sqrt{2}\sin(t_1)$, so $t_1 = 3\pi/4$.

Now we compute

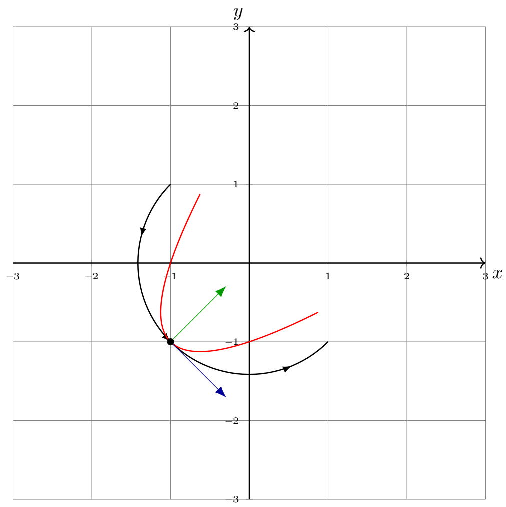
$$\begin{aligned} L &= \int_{\pi/4}^{3\pi/4} |\mathbf{r}'(t)|\ dt \\ &= \int_{\pi/4}^{3\pi/4} |\langle -\sqrt{2}\sin(t), \sqrt{2}\cos(t), 0\rangle|\ dt \\ &= \int_{\pi/4}^{3\pi/4} \sqrt{2}\ dt \\ &= \frac{\pi\sqrt{2}}{2} \end{aligned}$$

18. (a) We have

$$\begin{aligned} \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \\ &= \frac{\langle e^t - e^{-t}, 2\rangle}{|\langle e^t - e^{-t}, 2\rangle|} \\ &= \frac{\langle e^t - e^{-t}, 2\rangle}{\sqrt{(e^t - e^{-t})^2 + 4}} \\ &= \frac{\langle e^t - e^{-t}, 2\rangle}{\sqrt{e^{2t} + 2 + e^{-2t}}} \\ &= \frac{\langle e^t - e^{-t}, 2\rangle}{e^t + e^{-t}} \end{aligned}$$

(b) 

(c) The curvature of a circle of radius 2 is 1/2, so since the curve and the circle match at the marked point, the curvature of the curve at the point must also be 1/2.

19. 

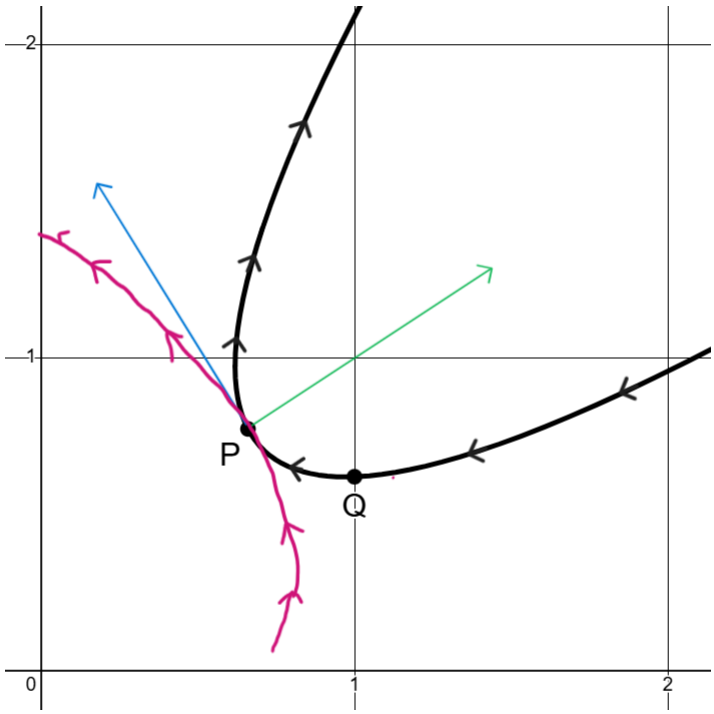
Drawn in blue is the unit tangent vector, in green the principal unit normal vector, and in red a possible solution curve for (c) (orientation should be from $(0,-1)$ towards $(-1,0)$).

This curve has the same tangent line as C at P because it must have a unit tangent vector parallel to C's since they have the same principal unit normal vector and our drawn curve is also smooth.

20. We have

$$\begin{aligned}
\text{length} &= \int_0^4 \|\mathbf{r}'(t)\|\, dt \\
&= \int_0^4 \|\langle 2t, t^2, 0\rangle\|\, dt \\
&= \int_0^4 \sqrt{4t^2+t^4}\, dt \\
&= \int_0^4 t\sqrt{4+t^2}\, dt \\
&= \int_4^{20} \frac{1}{2}\sqrt{u}\, du \\
&= \frac{1}{3}u^{3/2}\Big|_4^{20} \\
&= \frac{20^{3/2}-4^{3/2}}{3}
\end{aligned}$$

21. The unit tangent is sketched in blue, the principal unit normal in green, and a curve meeting the specified requirements in pink. The arc from P to Q is longer than the distance from P to Q.


2
1
0
1
2
P
Q