

Math 2551 Learning Target G2 Practice

G2: Calculus of Curves. I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.

Concept Checks

1. **T/F:** If a spaceship is in orbit around the Moon with a constant speed of 3750 miles per hour, then it is the case that the acceleration of the spaceship is zero.
2. **T/F:** A smooth curve in the plane that never crosses itself can have two distinct tangent lines at a given point.
3. If $\mathbf{r}(t)$ parameterizes a curve C in space, then _____ gives a direction vector for the tangent line to the curve at the point $\mathbf{r}(0)$.
4. **T/F:** If the Death Star is in an elliptical orbit around the forest moon of Endor and is moving at a constant speed of 5770 km/h then its acceleration is zero.
5. **T/F:** A particle moving at constant speed must have zero acceleration.
6. A hummingbird with velocity vector $\mathbf{r}'(t)$ starts flying from $(2, 1, -3)$ at time $t = 0$ and flies around for 5 seconds. Where is the hummingbird located at time $t = 5$ if $\int_0^5 \mathbf{r}'(t) dt = 0$?
7. **T/F:** If a vector-valued function $\mathbf{r}(t)$ is differentiable at $t = 2$ then a direction vector for the line tangent to $\mathbf{r}(t)$ at $t = 2$ is $\mathbf{r}'(2)$.
8. **T/F:** If the speed of a particle is zero, its velocity must be zero.

Open Response

9. Find an equation of the tangent line to the space curve

$$\mathbf{r}(t) = \langle \ln(t), t - 1, t \ln(t) \rangle$$

at $t = 1$.

10. What is a vector equation for the tangent line to the curve parameterized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $(2, 4, 8)$?

11. In this problem, you will work with the differential equation

$$\mathbf{r}''(t) = 3\mathbf{i} + e^t\mathbf{j} + 49e^{7t}\mathbf{k}, \quad -\infty < t < \infty.$$

- (a) Find all vector-valued functions $\mathbf{r}(t)$ with $-\infty < t < \infty$ which are solutions to this equation.

Hint: Your answer to this part should include some undetermined constant vectors.

- (b) Find all vector-valued functions $\mathbf{r}(t)$ with $-\infty < t < \infty$ which are solutions to this equation and also have the property that $\mathbf{r}'(0) = 3\mathbf{j} + 7\mathbf{k}$.

- (c) Find all vector-valued functions $\mathbf{r}(t)$ with $-\infty < t < \infty$ which are solutions to this equation, have the property that $\mathbf{r}'(0) = 3\mathbf{j} + 7\mathbf{k}$ and also have the property that $\mathbf{r}(0) = \langle 1, 1, 5 \rangle$.

12. Find the tangent line to the space curve parameterized by

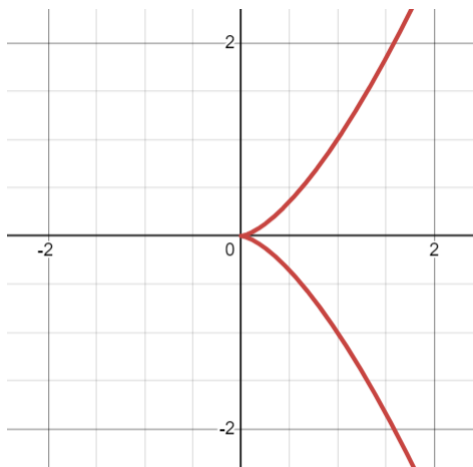
$$\mathbf{r}(t) = \langle 2\sqrt{2}\cos^3(t), 2\sqrt{2}\sin^3(t), t - \frac{\pi}{4} \rangle$$

at the point where $t = \pi/4$.

13. Suppose that a pegasus (a flying horse) is flying through the sky. Two hours after it left its stable, its velocity vector is $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ miles per hour and its position is $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ miles. The stable is not located at the origin.

- (a) Find an equation of the tangent line to the flight path of the pegasus two hours after it left its stable.
- (b) One hour after it left its stable, the position of the pegasus was $-\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ miles. Use this fact and your answer to part (a) to show that the pegasus cannot be flying in a straight line. Be sure to clearly explain your answer.
- (c) If the pegasus maintains its flight speed at the two hour mark of $\sqrt{14}$ miles per hour for the rest of its flight, is it possible for it to return to its stable? Be sure to clearly explain your answer.

14. Compute the tangent line for the curve $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ at $t = 0$ or explain why this does not exist. A picture of the curve is below.



15. Find the tangent line to the curve parameterized by

$$\mathbf{r}(t) = \langle t^3 + 2t, t + 1, 4 - t - t^2 \rangle, \quad 0 \leq t \leq 3$$

at the point $(2, 2, 2)$.

16. Solve the initial value problem

$$\mathbf{r}'(t) = \langle 2t, e^t, 3 \rangle, \quad \mathbf{r}(0) = \langle 2, 2, 0 \rangle$$

for $t \in \mathbb{R}$.

17. In this problem, you will work with the differential equation

$$\mathbf{r}''(t) = \sin(t)\mathbf{i} + -2\mathbf{j} + 16e^{4t}\mathbf{k}, \quad -\infty < t < \infty.$$

- (a) Find all vector-valued functions $\mathbf{r}(t)$ with $-\infty < t < \infty$ which are solutions to this equation.

Hint: Your answer to this part should include some undetermined constant vectors.

- (b) Find all vector-valued functions $\mathbf{r}(t)$ with $-\infty < t < \infty$ which are solutions to this equation and also have the property that $\mathbf{r}'(0) = 3\mathbf{i} + 4\mathbf{k}$.

- (c) Find all vector-valued functions $\mathbf{r}(t)$ with $-\infty < t < \infty$ which are solutions to this equation, have the property that $\mathbf{r}'(0) = 3\mathbf{i} + 4\mathbf{k}$ and also have the property that $\mathbf{r}(0) = \langle 0, 1, -3 \rangle$.

18. Parameterize the line tangent to the curve

$$\mathbf{r}(t) = \langle \cos^2(t), \sin(t) \cos(t), \cos(t) \rangle, \quad t \in [0, \pi]$$

at the point $(0, 0, 0)$.

Math 2551 LT G2 Practice Answers

Concept Checks

1. False
2. False
3. $\mathbf{r}'(0)$ or $\mathbf{T}(0)$
4. False
5. False
6. $(2, 1, -3)$
7. True
8. True

Open Response

9. The tangent line is $L(s) = \mathbf{r}(t_0) + s\mathbf{r}'(t_0)$. Here $t_0 = 1$, $\mathbf{r}(t) = \langle 0, 0, 0 \rangle$, $\mathbf{r}'(t) = \langle 1/t, 1, \ln(t) + 1 \rangle$, and $\mathbf{r}'(1) = \langle 1, 1, 1 \rangle$, so the tangent line is

$$L(s) = \langle 1, 1, 1 \rangle s.$$

10. $\ell(s) = \langle 1, 4, 12 \rangle s + \langle 2, 4, 8 \rangle$

11. In this problem, you will work with the differential equation

$$\mathbf{r}''(t) = 3\mathbf{i} + e^t\mathbf{j} + 49e^{7t}\mathbf{k}, \quad -\infty < t < \infty.$$

12. (a) From the differential equation we get (via integration with respect to t):

$$\mathbf{r}'(t) = 3t\mathbf{i} + e^t\mathbf{j} + 7e^{7t}\mathbf{k} + \mathbf{C}_1 \tag{1}$$

$$\mathbf{r}(t) = \frac{3}{2}t^2\mathbf{i} + e^t\mathbf{j} + e^{7t}\mathbf{k} + \mathbf{C}_1t + \mathbf{C}_2, \tag{2}$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are constant vectors.

(b) Applying this condition to equation (1) gives

$$3\mathbf{j} + 7\mathbf{k} = \mathbf{r}'(0) = \mathbf{j} + 7\mathbf{k} + \mathbf{C}_1.$$

So we have $\mathbf{C}_1 = 2\mathbf{j}$ and thus

$$\mathbf{r}(t) = \frac{3}{2}t^2\mathbf{i} + (e^t + 2t)\mathbf{j} + e^{7t}\mathbf{k} + \mathbf{C}_2. \quad (3)$$

(c) Applying the new condition to equation (3) gives

$$\langle 1, 1, 5 \rangle = \mathbf{r}(0) = \langle 0, 1, 1 \rangle + \mathbf{C}_2.$$

So we have $\mathbf{C}_2 = \langle 1, 0, 4 \rangle$ and thus

$$\mathbf{r}(t) = \left(\frac{3}{2}t^2 + 1\right)\mathbf{i} + (e^t + 2t)\mathbf{j} + (e^{7t} + 4)\mathbf{k}.$$

13. We have $\mathbf{r}(\pi/4) = \langle 1, 1, 0 \rangle$ and

$$\mathbf{r}'(t) = \langle -6\sqrt{2}\cos^2(t)\sin(t), 6\sqrt{2}\sin^2(t)\cos(t), 1 \rangle.$$

Then $\mathbf{r}'(t) = \langle -3, 3, 1 \rangle$ and so the tangent line to the curve at this point is

$$\langle -3, 3, 1 \rangle t + \langle 1, 1, 0 \rangle.$$

14. (a) There are at least two options here, depending on how we define t . If t is hours after leaving the stable, we have

$$\ell_1(t) = \langle 3, -2, 1 \rangle(t - 2) + \langle 2, -3, 4 \rangle.$$

If t is hours after this time, we have

$$\ell_2(t) = \langle 3, -2, 1 \rangle t + \langle 2, -3, 4 \rangle.$$

(b) One hour after leaving the stable corresponds to $t = 1$ or $t = -1$ in the solutions above, respectively. If the pegasus was flying in a straight line, then its path would be exactly the same as its tangent line! So we compute its position on the tangent line and compare with the given position. Using the first equation above, this gives:

$$\mathbf{r}(1) = \langle -1, -1, 5 \rangle,$$

but

$$\ell_1(1) = \langle 3, -2, 1 \rangle(1 - 2) + \langle 2, -3, 4 \rangle = \langle -1, -1, 3 \rangle.$$

Since these are not equal, the pegasus must not be flying in a straight line.

(c) Yes, the pegasus can still return to its stable without changing its speed, because it can change the direction of its flight while maintaining the same speed.

15. The tangent line at the origin does not exist, since the curve is not smooth at that point. There are several ways to see this: for example, because $\mathbf{r}'(t) = \langle 0, 0 \rangle$, the tangent vector at the origin does not exist. We can also observe that the direction of the tangent vector abruptly reverses as $t \rightarrow 0$: when $t < 0$, the tangent vector is approaching $-\mathbf{i}$, but when $t > 0$ it is approaching $\mathbf{i}$.
16. The tangent line at $t = t_0$ is $\ell(t) = \mathbf{r}(t_0) + \mathbf{r}'(t_0)t$ for $t \in \mathbb{R}$. We need to find the value t_0 corresponding to $(2, 2, 2)$, and by setting equations equal we see that this requires $t = 1$. So we compute

$$\mathbf{r}'(t) = \langle 3t^2 + 2, 1, -1 - 2t \rangle, \quad \mathbf{r}'(1) = \langle 5, 1, -3 \rangle.$$

Therefore the tangent line is

$$\ell(t) = \langle 5, 1, -3 \rangle t + \langle 2, 2, 2 \rangle.$$

17. We need

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{r}'(t) dt \\ &= \int \langle 2t, e^t, 3 \rangle dt \\ &= \langle t^2, e^t, 3t \rangle + \mathbf{C}. \end{aligned}$$

Now we can apply the initial condition: since $\langle 2, 2, 0 \rangle = \mathbf{r}(0) = \langle 0, 1, 0 \rangle + \mathbf{C}$, we have $\mathbf{C} = \langle 2, 1, 0 \rangle$ and so our solution is

$$\mathbf{r}(t) = \langle t^2 + 2, e^t + 1, 3t \rangle, t \in \mathbb{R}.$$

18. In this problem, you will work with the differential equation

$$\mathbf{r}''(t) = \sin(t)\mathbf{i} - 2t\mathbf{j} + 16e^{4t}\mathbf{k}, \quad -\infty < t < \infty.$$

- (a) We take two antiderivatives to find

$$\mathbf{r}'(t) = -\cos(t)\mathbf{i} - 2t\mathbf{j} + 4e^{4t}\mathbf{k} + \mathbf{C}$$

and

$$\mathbf{r}(t) = -\sin(t)\mathbf{i} - t^2\mathbf{j} + e^{4t}\mathbf{k} + t\mathbf{C} + \mathbf{D}.$$

- (b) Applying this condition to $\mathbf{r}'(t)$ gives

$$3\mathbf{i} + 4\mathbf{k} = \mathbf{r}'(0) = -\mathbf{i} + 4\mathbf{k} + \mathbf{C},$$

so we have $\mathbf{C} = 4\mathbf{i}$ and therefore

$$\mathbf{r}(t) = (4t - \sin(t))\mathbf{i} - t^2\mathbf{j} + e^{4t}\mathbf{k} + \mathbf{D}.$$

(c) Applying the new condition to our updated equation for $\mathbf{r}(t)$ gives

$$\mathbf{j} - 3\mathbf{k} = \mathbf{r}(0) = \mathbf{k} + \mathbf{D},$$

so we have $\mathbf{D} = \mathbf{j} - 4\mathbf{k}$ and therefore

$$\mathbf{r}(t) = (4t - \sin(t))\mathbf{i} + (1 - t^2)\mathbf{j} + (e^{4t} - 4)\mathbf{k}.$$

19. The curve passes through the point $(0, 0, 0)$ when $\cos^2(t) = 0$, $\sin(t)\cos(t) = 0$, and $\cos(t) = 0$. In the interval given for t , this occurs at $t = \pi/2$. The direction of the tangent line at this point is $\mathbf{r}'(\pi/2)$. Since

$$\mathbf{r}'(t) = \langle -2\cos(t)\sin(t), \cos^2(t) - \sin^2(t), -\sin(t) \rangle,$$

we have $\mathbf{r}'(\pi/2) = \langle 0, -1, -1 \rangle$. Therefore a parameterization for the tangent line is

$$\ell(t) = \langle 0, -1, -1 \rangle t + \langle 0, 0, 0 \rangle = \langle 0, -1, -1 \rangle t.$$