Math 2551 Learning Target D2 Practice

D2: Tangent Planes and Linear Approximations. I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.

Concept Checks

1. **T/F:** Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b). Then f is also continuous at (a, b).

2. Find the linearization of the function $f(x, y) = \sqrt{x^2 + y}$ at the point $(2, 5)$.

   (a) $L(x, y) = 3 + \frac{2}{3}(x - 2) + \frac{1}{6}(y - 5)$
   (b) $L(x, y) = \frac{2}{3}(x - 2) + \frac{1}{6}(y - 5)$
   (c) $L(x, y) = \frac{1}{6}(x - 2) + \frac{1}{6}(y - 5)$
   (d) $L(x, y) = 3 + \frac{x}{\sqrt{x^2 + y}}(x - 2) + \frac{1}{2\sqrt{x^2 + y}}(y - 5)$
   (e) None of the above.

3. **T/F:** A function $f(y, z)$ is differentiable at a point (b, c) if the surface $x = f(y, z)$ has a unique tangent plane at the point $(f(b, c), b, c)$.

4. **T/F:** Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable at (a, b, c) with $f(a, b, c) = d$. Then the equation of the linearization $L(x, y, z)$ of f at the point (a, b, c) is also the equation of the plane tangent to the level surface $f(x, y, z) = d$ at the point (a, b, c)

Open Response

1. Let $f(x, y, z) = x^2 + z^2 e^{y-x}$.

   (a) Find an equation of the tangent plane to the level surface $f = 13$ at the point $P = (2, 3, \frac{3}{\sqrt{e}})$.
   (b) Find the linearization $L(x, y, z)$ of f at P.

(c) Use the linearization you found to approximate the value of $f(2, 3, \frac{4}{\sqrt{e}})$

2. Let $f(x, y) = \ln(x^2 + y^2)$ and $P = (1, 0)$. Find the linear approximation of f at P.

3. The point $P = (-2, 1, 2)$ is on a level surface of $g(x, y, z) = x^4+y^3+z^2$. Find an equation for the tangent plane to this surface at P.

4. Consider the level surface $f = 1$ of the function $f(x, y, z) = \sqrt{x^2 - y^2 + z}$.

   (a) Compute the tangent plane to the point $(1, 1, 1)$ on this surface.

   (b) Use the linearization of f at $(1, 1, 1)$ to estimate the value of $f(1, 1, 1.1)$.

5. Let $f(x, y, z) = yz + x^2e^{z-y}$.

   (a) Find an equation of the tangent plane to the level surface $f = 6$ at the point $P = (\frac{2}{\sqrt{e}}, 1, 2)$.

   (b) Find the linearization $L(x, y, z)$ of f at P.

   (c) Use the linearization you found to approximate the value of $f(\frac{2}{\sqrt{e}}, 1.1, 2.1)$.

Math 2551 LT D2 Practice Answers

Concept Checks

1. True.
2. (a) is correct
3. True
4. False

Open Response

1. (a) The equation of a tangent plane to a level surface of a function of three variables at a point $P = (x_0, y_0, z_0)$ is $\nabla f(P) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$.
   $\nabla f = \langle 2x - z^2 e^{y-x}, z^2 e^{y-x}, 2ze^{y-x} \rangle$, so $\nabla f(P) = \langle -5, 9, 6\sqrt{e} \rangle$.
   Thus an equation of the tangent plane is
   $$-5(x-2) + 9(y-3) + 6\sqrt{e}(z - \frac{3}{\sqrt{e}}) = 0.$$

   (b) The linearization $L(x, y, z)$ at P is $L(x, y, z) = f(P) + \nabla f(P) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$.
   $f(P) = 13$ and we computed the rest of this in part (a).
   So the linearization is
   $$L(x, y, z) = 13 - 5(x-2) + 9(y-3) + 6\sqrt{e}(z - \frac{3}{\sqrt{e}}).$$

   (c)
   $$f(2, 3, \frac{4}{\sqrt{e}}) \approx L(2, 3, \frac{4}{\sqrt{e}}) = 13 - 5(2-2) + 9(3-3) + 6\sqrt{e}(\frac{4}{\sqrt{e}} - \frac{3}{\sqrt{e}}) = 19$$

2. First, $\nabla f = \langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \rangle$. Note $f(P) = \ln(1 + 0) = 0$, so we have the linear approximation
   $$L(x, y) = f(P) + \nabla f(P) \cdot (\mathbf{x} - \vec{OP}) = 0 + \langle 2, 0 \rangle \cdot \langle x - 1, y - 0 \rangle = 2x - 2.$$

3. The normal vector to this plane is $\nabla g(P) = \langle -32, 3, 4\rangle$. Thus the plane is:

$$-32(x+2) + 3(y-1) + 4(z-2) = 0$$

or

$$-32x + 3y + 4z = 75$$

4. (a) We have

$$\nabla f = \langle \frac{x}{\sqrt{x^2-y^2+z}}, -\frac{y}{\sqrt{x^2-y^2+z}}, \frac{1}{2\sqrt{x^2-y^2+z}} \rangle$$

so $\nabla f(1,1,1) = \langle 1, -1, 1/2\rangle$. Then the tangent plane to the surface $f = 1$ at the point $(1,1,1)$ is

$$(x-1) - (y-1) + \frac{1}{2}(z-1) = 0.$$

(b) Using our work from part (a) we have

$$L(x,y,z) = f(1,1,1) + \nabla f(1,1,1) \cdot \langle x-1, y-1, z-1\rangle = 1 + (x-1) - (y-1) + \frac{1}{2}(z-1).$$

Now

$$f(1,1,1.1) \approxeq L(1,1,1.1) = 1 + (1-1) - (1-1) + \frac{1}{2}(1.1-1) = 1.05.$$

5. Let $f(x,y,z) = yz + x^2 e^{z-y}$.

(a) The equation of a tangent plane to a level surface of a function of three variables at a point $P = (x_0, y_0, z_0)$ is $\nabla f(P) \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$.
$\nabla f = \langle 2xe^{z-y}, z - x^2e^{z-y}, y + x^2e^{z-y}\rangle$, so $\nabla f(P) = \langle 4\sqrt{e}, -2, 5\rangle$.
Thus an equation of the tangent plane is

$$4\sqrt{e}(x - \frac{2}{\sqrt{e}}) - 2(y-1) + 5(z-2) = 0$$

or

$$(4\sqrt{e})x - 2y + 5z = 16$$

(b) The linearization $L(x,y,z)$ at P is $L(x,y,z) = f(P) + \nabla f(P) \cdot \langle x - x_0, y - y_0, z - z_0\rangle$.
$f(P) = 6$ and we computed the rest of this in part (a).
So the linearization is

$$L(x,y,z) = 6 + 4\sqrt{e}(x - \frac{2}{\sqrt{e}}) - 2(y-1) + 5(z-2).$$

(c)

$$f(\frac{2}{\sqrt{e}}, 1.1, 2.1)) \approx L(\frac{2}{\sqrt{e}}, 1.1, 2.1)) = 6 + 4\sqrt{e}(0) - 2(.1) + 5(.1) = 6.3$$