Math 2551 Learning Target A2 Practice

A2: Integral Applications. I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.

Concept Checks

1. **T/F:** The mass of a region in the plane might be given by $\int_0^x \int_0^y 3 \, dy \, dx$.

2. **T/F:** If D is any solid region in space bounded by smooth surfaces, then we can evaluate

$$\iiint_D dV$$

in any order of integration to compute the volume of D.

3. The integral below could describe the mass of:

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{3} (x^2 + y^2 + z^2) \, dz \, dy \, dx$$

   (a) a solid ball that gets lighter away from the origin
   (b) a solid ball that is equally heavy at all points
   (c) a solid ball that gets heavier away from the origin
   (d) a solid cone that gets lighter away from the origin
   (e) a solid cone that gets heavier away from the origin

4. Match the volume integrals below to the regions described by filling in the blanks below. You will not use all of the integrals.

(A) A cylinder of radius 2 and height $\pi/4$.

(B) A cube with side lengths $2, \pi/4$ and 2π.

(C) The smaller region between a sphere of radius 2 and a cone making an angle of $\pi/4$ with the positive z-axis.

(I) $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 dy\, dx\, dz$$

(II) $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 r\, dr\, dz\, d\theta.$$

(III) $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^2\sin(\varphi)\, d\rho\, d\theta\, d\varphi.$$

(IV) $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 r\, dz\, dr\, d\theta.$$

(V) $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^2\sin(\varphi)\, d\rho\, d\varphi\, d\theta.$$

(A) corresponds to __________
(B) corresponds to __________
(C) corresponds to __________

Open Response

1. Write an integral for the average distance to the origin among points in a disk D of radius 3 centered at the origin in $\mathbb{R}^2$, $D = \{(x, y) \mid x^2 + y^2 \leq 9\}$.

2. Suppose that the mass of a thin metal plate D in the shape of a disk of radius 1 m centered at the origin is 12 kg and the center of mass of this disk is the point $(0, 1/2)$. Is it possible that the density $\rho(x, y)$ of the disk depends only on x and not on y, i.e. $\rho = f(x)$? Give an explanation for your answer. It may help to consider $\bar{y}$.

3. The integral below could describe the mass of what kind of solid? How does the mass of this solid change as you move away from the origin?

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (x^2 + y^2 + z^2)\, dz\, dy\, dx$$

4. Consider the volume D in the first octant of $\mathbb{R}^3$ which is bounded above by the sphere $x^2 + y^2 + z^2 = 12$ and below by the paraboloid $z = x^2 + y^2$. If D has a constant density function $\delta = 16$, write an integral that computes M_{xy}, the first moment of D about the xy-plane. Fully simplify your answer.

   Hint: Be careful with your choice of coordinate system.

5. Let D be the sphere of radius 3 centered at the origin in $\mathbb{R}^3$. The volume of this sphere is 36π. Suppose that the density of a liquid filling this sphere is $\delta(x, y, z) = 1.2x$ kilograms per cubic meter.

   (a) Write an integral expression for the mass of this sphere filled with liquid. **Do not evaluate your integral expression.**

   (b) Write an integral expression for the average distance of a point in this sphere from the origin. **Do not evaluate your integral expression.**

6. Set up but do not compute an integral to compute the mass of the solid D that has mass density function $\delta(x, y, z) = 4 + \cos(x)\sin(y) + z/10$ and occupies the region

$$y \leq 8 - x^2, \qquad x^2 + 2z^2 \leq y, \qquad x \geq 0.$$

   A complete solution will include a sketch of the shadow of D in the plane of the two variables that you do not integrate first.

7. (a) Write an expression involving one or more iterated integrals that computes the average distance to the origin in the region R with $0 \leq x \leq y^2 + 1, -2 \leq y \leq 2$.

   (b) Write an expression involving one or more iterated integrals that computes the x-coordinate of the center of mass of an object with density $\delta(x, y) = e^{xy^2}$ on the same region.

8. (a) Set up an integral that computes the average distance from the point $(2, 0)$ in the region inside the ellipse

$$\frac{(x-1)^2}{4} + \frac{y^2}{3} = 1.$$

   You may use the fact that the area of the ellipse $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$ is πab.

   (b) Without evaluating it, explain why the value of the integral you wrote above is less than 3.

Math 2551 LT A2 Practice Answers

Concept Checks

1. False
2. True
3. (e)
4. (A) corresponds to (II)
   (B) corresponds to (I)
   (C) corresponds to (V)

Open Response

1. $$d_{avg} = \frac{1}{9\pi} \iint_D \sqrt{x^2+y^2}\, dA$$

2. It is not possible. If the density did not depend on y at all, then because the region D is also symmetric across the x-axis the center of mass would ahve to lie on the x-axis with y-coordinate 0. But here it has y-coordinate $1/2$, so this is impossible.

3. This could compute the mass of a ball of radius 1 centered at the origin. The mass is increasing as distance to the origin increases.

4. We have $M_{xy} = \iiint_D z\delta\, dV$, so we compute $\iiint_D 16z\, dV$. This is easiest in cylindrical coordinates, in which D is described by $r^2 \leq z \leq \sqrt{12-r^2}$, $0 \leq \theta \leq \pi/2$. To find bounds on r, we solve $r^2 = \sqrt{12-r^2}$ to get $r^4 = 12 - r^2$, which then gives $(r^2+4)(r^2-3) = 0$. Thus we have $0 \leq r \leq \sqrt{3}$.

   Now we can write our integral:

   $$M_{xy} = \iiint_D 16z\, dV = \int_0^{\pi/2} \int_0^{\sqrt{3}} \int_{r^2}^{\sqrt{12-r^2}} 16zr\, dz\, dr\, d\theta$$

5. Let D be the sphere of radius 3 centered at the origin in $\mathbb{R}^3$. The volume of this sphere is 36π. Suppose that the density of a liquid filling this sphere is $\delta(x,y,z) = 1.2x$ kilograms per cubic meter.

(a) The mass is

$$\begin{aligned}\iiint_D \delta(x,y,z)\, dV &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} 1.2x\; dz\; dy\; dx \\ &= \int_0^{2\pi}\int_0^3\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} 1.2r^2\cos(\theta)\; dz\; dr\; d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^3 1.2\rho^3\sin^2(\varphi)\cos(\theta)\; d\rho\; d\varphi\; d\theta\end{aligned}$$

(b) The average distance of a point in this sphere from the origin is:

$$\begin{aligned}d_{avg} &= \frac{1}{V_{sphere}}\iiint_D d_{(0,0,0)}(x,y,z)\; dV \\ &= \frac{1}{36\pi}\iiint_D \sqrt{x^2+y^2+z^2}\; dV \\ &= \frac{1}{36\pi}\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\; dz\; dy\; dx \\ &= \frac{1}{36\pi}\int_0^{2\pi}\int_0^3\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r\sqrt{r^2+z^2}\; dz\; dr\; d\theta \\ &= \frac{1}{36\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^3 \rho^3\sin(\varphi)\; d\rho\; d\varphi\; d\theta\end{aligned}$$

6. The given constraints suggest integrating with respect to y first since we have

$$x^2+2z^2 \le y \le 8-x^2.$$

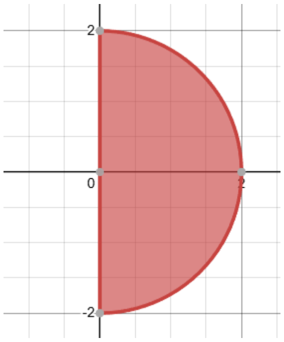
The shadow of this solid in the xz-plane is thus the region with

$$x^2+2z^2 \le 8-x^2 \text{ and } x \ge 0$$

i.e. the region

$$x^2+z^2 \le 4 \text{ and } x \ge 0.$$

A sketch of this region is given below; it is the right half of the disk of radius 2 centered at the origin in the xz-plane.



Therefore an integral to compute this mass is

$$\iiint_D \delta(x,y,z)\ dV = \int_{-2}^{2}\int_0^{\sqrt{4-z^2}}\int_{x^2+2z^2}^{8-x^2} 4+\cos(x)\sin(y)+z/10\ dy\ dx\ dz$$

$$\text{or}\ \int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+2z^2}^{8-x^2} 4+\cos(x)\sin(y)+z/10\ dy\ dz\ dx$$

$$\text{or}\ \int_0^2\int_{x^2}^{8-x^2}\int_{-\sqrt{(y-x^2)/2}}^{\sqrt{(y-x^2)/2}} 4+\cos(x)\sin(y)+z/10\ dz\ dy\ dx$$

7. (a)

$$d_{avg} = \frac{\displaystyle\int_{-2}^{2}\int_0^{y^2+1}\sqrt{x^2+y^2}\ dx\ dy}{\displaystyle\int_{-2}^{2}\int_0^{y^2+1} 1\ dx\ dy}$$

(b)

$$d_{avg} = \frac{\displaystyle\int_{-2}^{2}\int_0^{y^2+1} xe^{xy^2}\ dx\ dy}{\displaystyle\int_{-2}^{2}\int_0^{y^2+1} 1\ dx\ dy}$$

8. (a) The distance from a point (x,y) to $(2,0)$ is $f(x,y) = \sqrt{(x-2)^2+y^2}$. Then we have

$$f_{avg} = \frac{\iint_R f(x,y)\ dA}{\text{Area}(R)}.$$

Using the given fact we have $\text{Area}(R) = 2\sqrt{3}\pi$. So we get

$$f_{avg} = \frac{1}{2\sqrt{3}\pi}\int_{-1}^{3}\int_{-\sqrt{3-(3/4)(x-1)^2}}^{\sqrt{3-(3/4)(x-1)^2}}\sqrt{(x-2)^2+y^2}\ dy\ dx$$

or

$$f_{avg} = \frac{1}{2\sqrt{3}\pi} \int_{-\sqrt{3}}^{\sqrt{3}} \int_{1-\sqrt{4-(4/3)y^2}}^{1+\sqrt{4-(4/3)y^2}} \sqrt{(x-2)^2 + y^2}\ dx\ dy$$

(b) The furthest point in this ellipse from $(2, 0)$ is the point $(-1, 0)$, which is a distance of 3 from $(2, 0)$. The average distance from $(2, 0)$ to points in the ellipse must be less than this maximum distance.