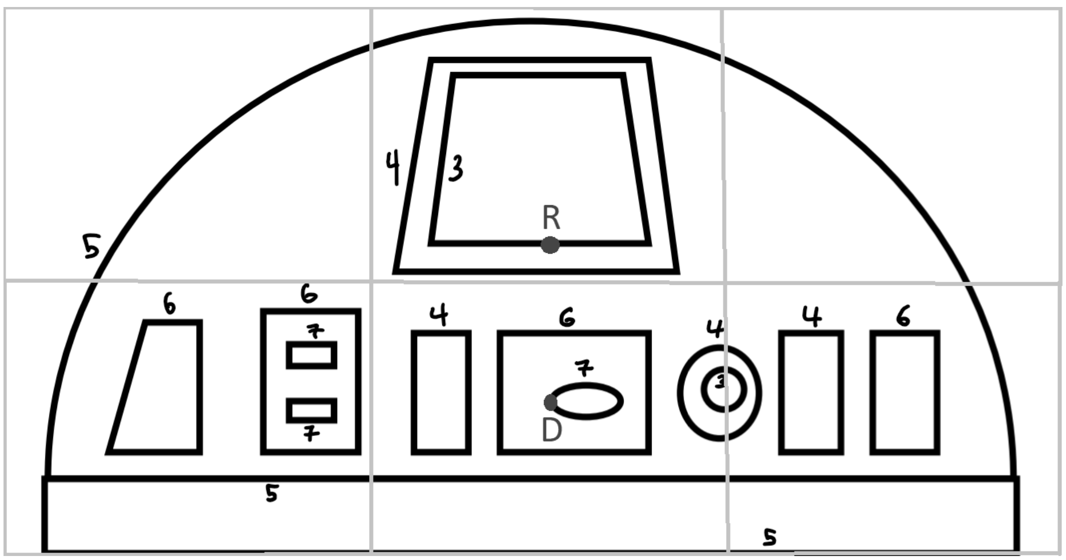
Math 2551 Learning Target A1 Practice

A1: Interpreting Derivatives. I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.

Concept Checks

1. **T/F:** Suppose we have a function $f(x, y)$ and $\mathbf{u} \in \mathbb{R}^2$ is a unit vector. Then $D_{\mathbf{u}}f(a, b)$ is a vector.

2. What is the sign of the directional derivative of f at the point R in the direction from R to D?



   (a) Positive
   (b) Negative
   (c) Zero
   (d) Undefined
   (e) Cannot be determined.

3. **T/F:** The rate of change of a differentiable function f at any point P in the direction of $\nabla f(P)$ is always positive if P is not a critical point of f.

4. **T/F:** Suppose that the price P in dollars to purchase a used car is a function of C, its original cost in dollars, and its age A in years, i.e. $P = f(C, A)$. Then the sign of $\dfrac{\partial P}{\partial C}$ is negative.

5. **T/F:** A function $f(x, y)$ can be an increasing function of x with y held fixed, and be a decreasing function of y with x held fixed.

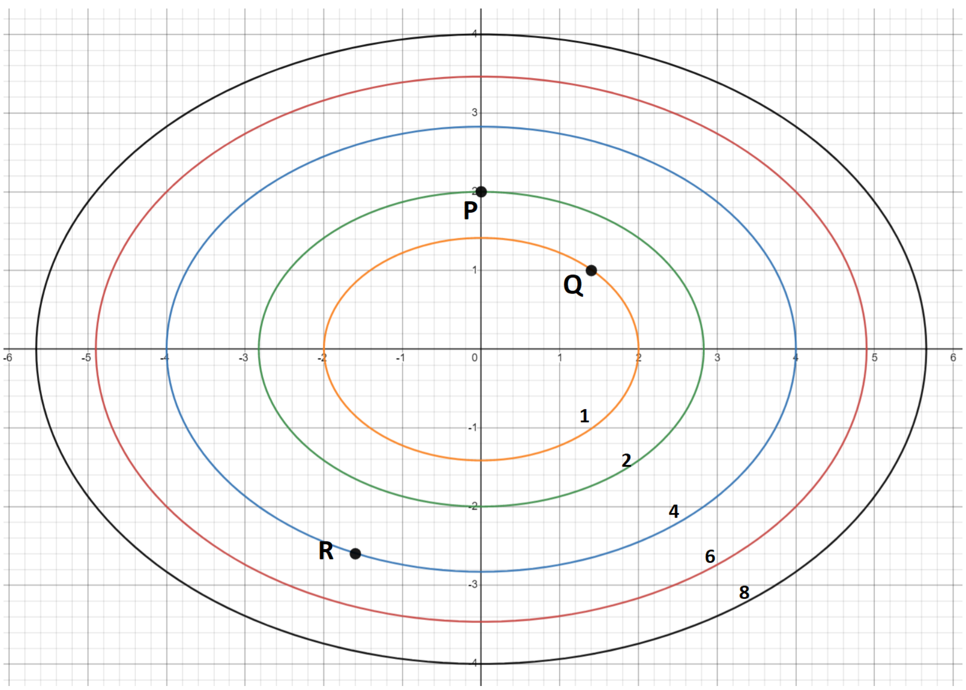
6. **T/F:** The function $f(x, y)$ has gradient ∇f at the point (a, b). The vector ∇f is perpendicular to the level curve $f(x, y) = f(a, b)$.

7. **T/F:** The function $f(x, y)$ has gradient ∇f at the point (a, b). The vector ∇f is perpendicular to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$.

8. **T/F:** The function $f(x, y)$ has gradient ∇f at the point (a, b). The vector $f_x(a, b)\mathbf{i} + f_y(a, b)\mathbf{j} + \mathbf{k}$ is perpendicular to the surface $z = f(x, y)$.

Open Response

1. Below is a contour plot for a smooth function $f(x, y)$. Use this contour plot to answer parts (a)-(c) below.



   (a) Determine the sign $(+, -, 0)$ of f_x at the point Q.

   (b) Determine the sign $(+, -, 0)$ of the directional derivative of f in the direction $\mathbf{i} + \mathbf{j}$ at the point R.

   (c) Draw a vector in the direction of greatest increase of f at the point P.

2. In this problem, you will work with the function $g : \mathbb{R}^3 \to \mathbb{R}$, $g(x, y, z) = x^4 + y^3 + z^2$ and the point $P = (-2, 1, 2)$ in the domain of g.

   (a) Suppose that you are only able to travel away from P in one of the following directions. Which direction (assuming you move with unit speed) will yield the greatest instantaneous increase in g?

   (A) parallel to the x-axis, with x increasing

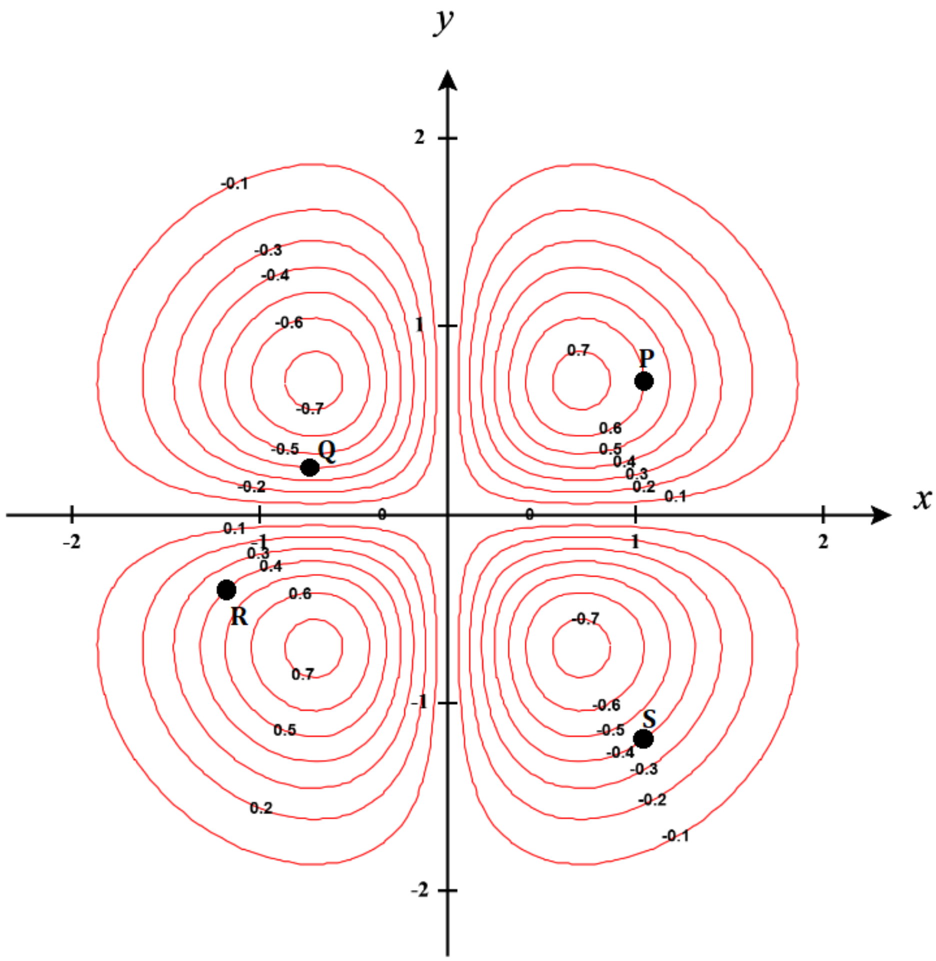
   (B) parallel to the y-axis, with y increasing

   (C) parallel to the z-axis, with z increasing

   (D) directly away from the origin

   (b) Justify your answer to part (a).

3. In this problem, you will work with the function $f : \mathbb{R}^2 \to \mathbb{R}$ whose contour plot near the origin is shown below.



(a) Determine the sign $(+, -, 0)$ of the directional derivative at the point P in the direction towards the point $(2, 0)$.

(b) Determine the sign $(+, -, 0)$ of the directional derivative at the point Q in the x-direction.

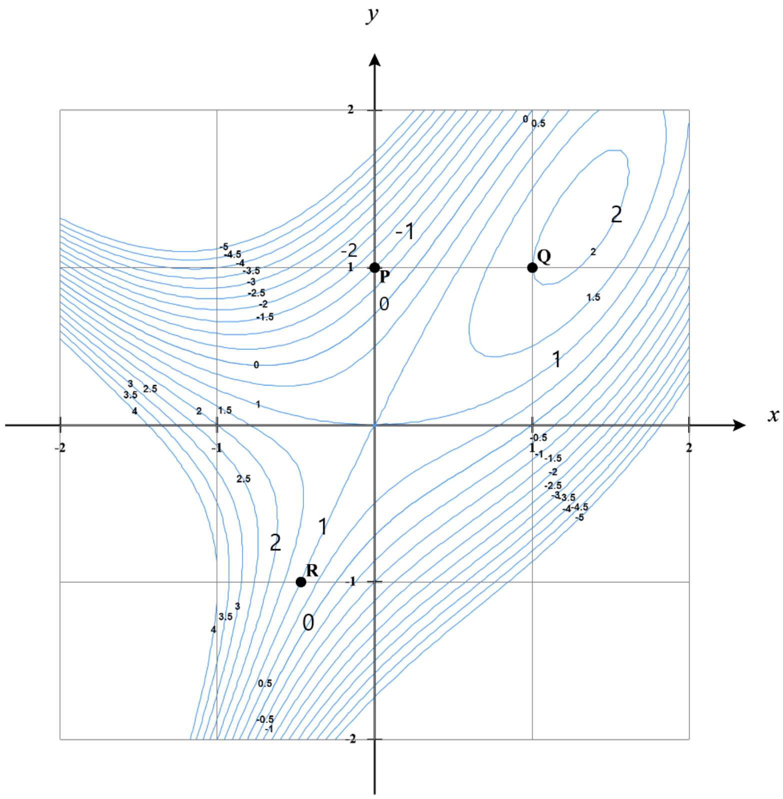
(c) Determine the sign $(+, -, 0)$ of the directional derivative at the point R in the direction of $\nabla f(R)$.

(d) Draw a vector parallel to the gradient vector of f at the point S.

(e) This function has a critical point at the center of the concentric contours near S. Based on the plot and the gradient at S, classify this critical point.

4. In this problem, you will work with the function $g : \mathbb{R}^2 \to \mathbb{R}$, whose contour plot is given below.

$$g(x, y) = -x^3 + 4xy - 2y^2 + 1.$$



In parts (a)-(c), write the sign $(+, -, 0)$ of the directional derivative at the given point in the given direction in the answer box provided. You do not need to show work for these parts.

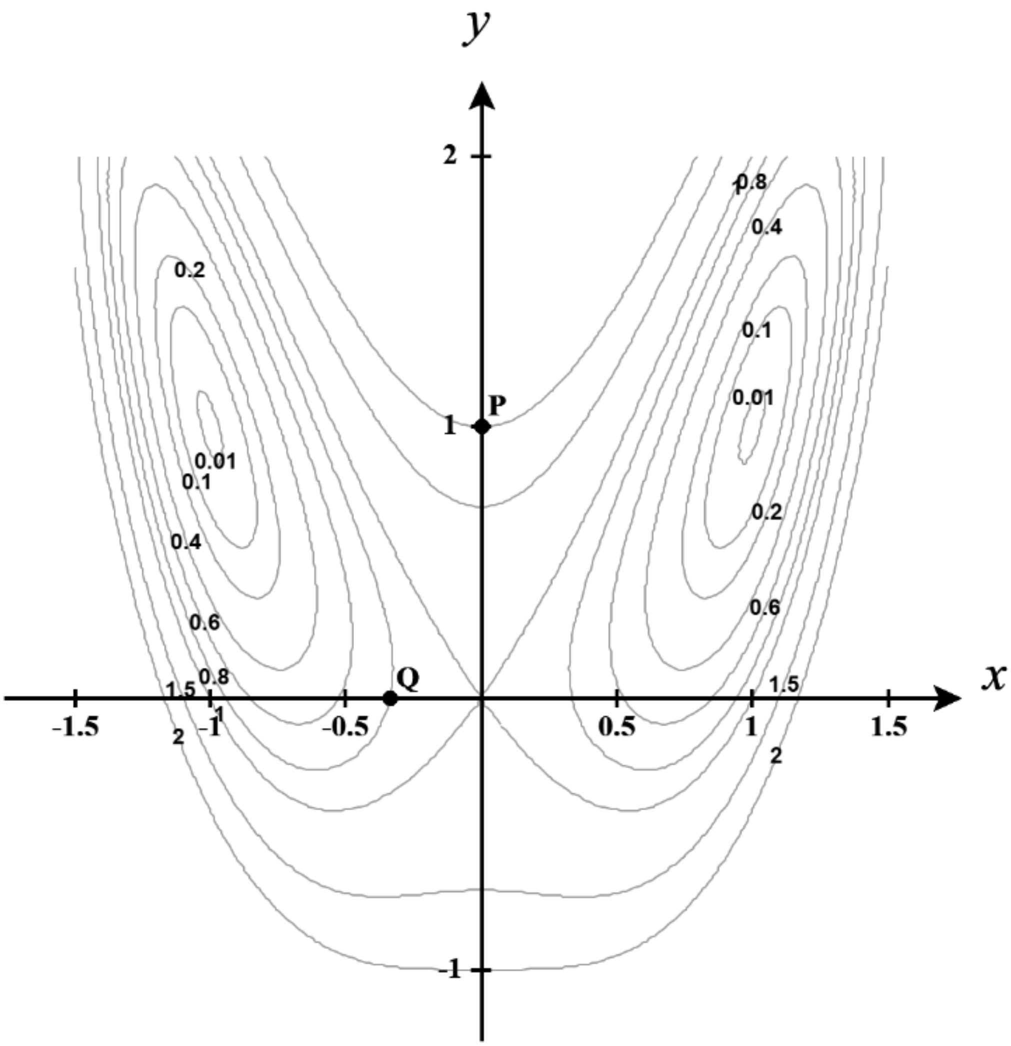
(a) At the point P in the direction $\langle 1, -1 \rangle$:

(b) At the point Q in the direction $\langle -1, 0 \rangle$:

(c) At the point R in a direction orthogonal to $\nabla g(R)$:

(d) Let $\mathbf{u} = \langle 4/5, 3/5 \rangle$. Compute $D_{\mathbf{u}} g(1, 0)$.

5. In this problem, you will analyze the derivatives of the temperature function $T(x, y)$ whose contour plot is shown below.



(a) Is the temperature increasing, decreasing, or staying the same as x increases from the point P?

(b) Determine the sign $(+, -, 0)$ of the rate of change of T at Q in the direction from Q towards $(-1, 1)$.

(c) Draw a point R where the temperature is at a local extreme value.

(d) Draw a vector which points in a direction of no change in temperature at the point Q.

Math 2551 LT A1 Practice Answers

Concept Checks

1. False.
2. a) Positive
3. True
4. False
5. True
6. True
7. False
8. False

Open Response

1. (a) $f_x(Q) > 0$ since the contours are increasing in the positive x-direction from Q
   (b) This derivative is negative since the contours are decreasing in this direction from R
   (c) The vector is in the $\mathbf{j}$ direction.
2. In this problem, you will work with the function $g : \mathbb{R}^3 \to \mathbb{R}$, $g(x, y, z) = x^4 + y^3 + z^2$ and the point $P = (-2, 1, 2)$ in the domain of g.
   (a) (D) is correct.
   (b) This problem is asking in which of the given directions is the directional derivative of g greatest. So we compute.

   $Dg(x, y, z) = \begin{bmatrix} 4x^3 & 3y^2 & 2z \end{bmatrix}$, so $Dg(P) = \begin{bmatrix} -32 & 3 & 4 \end{bmatrix}$.

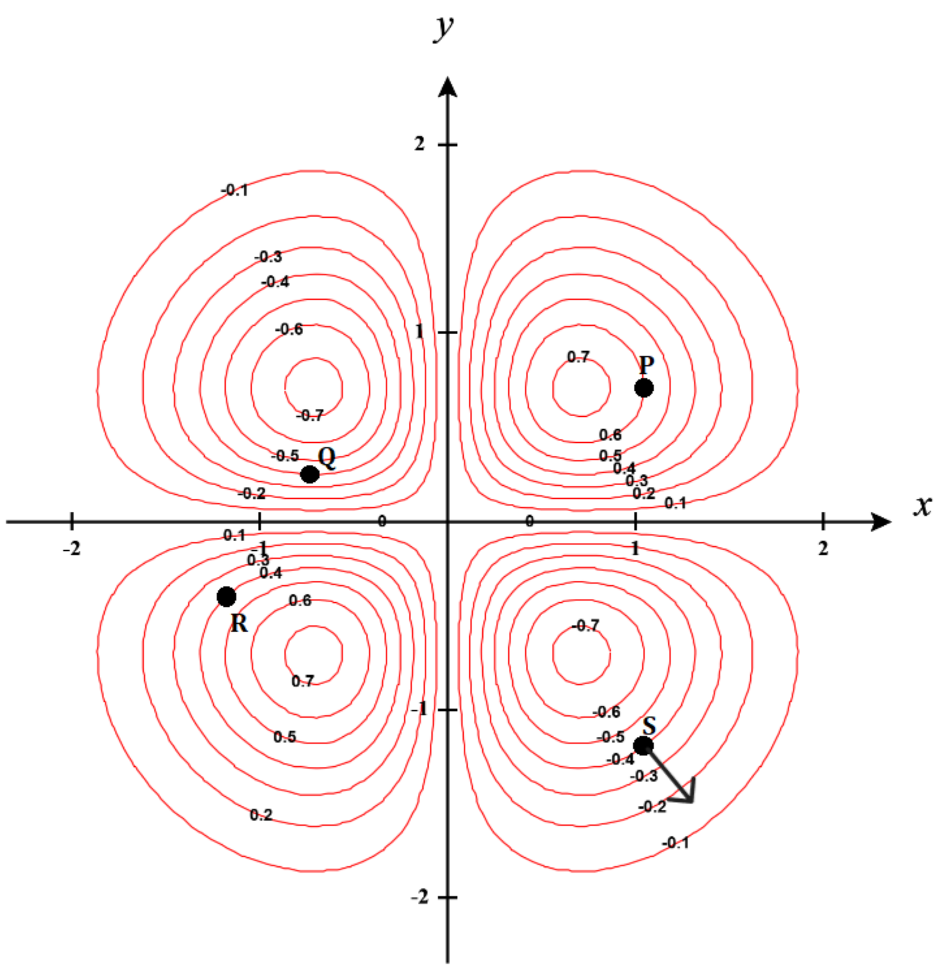
   We also need a unit vector in each direction; for (A)-(C) these are the standard unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and for (D) it is the vector $\mathbf{u} = \vec{OP}/|\vec{OP}| = \frac{1}{3}\langle -2, 1, 2\rangle$.

We then have:

$$\begin{aligned}
D_{\mathbf{i}}g(P) &= Dg(P)\mathbf{i} = -32 \\
D_{\mathbf{j}}g(P) &= Dg(P)\mathbf{j} = 3 \\
D_{\mathbf{k}}g(P) &= Dg(P)\mathbf{k} = 4 \\
D_{\mathbf{u}}g(P) &= Dg(P)\mathbf{u} = \frac{1}{3}(-2(-32) + 1(3) + 2(4)) = 25
\end{aligned}$$

Of these, 25 is the largest value, so the direction (D) yields the greatest instantaneous increase in g.

3. In this problem, you will work with the function $f : \mathbb{R}^2 \to \mathbb{R}$ whose contour plot near the origin is shown below.



(a) —

(b) $\boxed{0}$

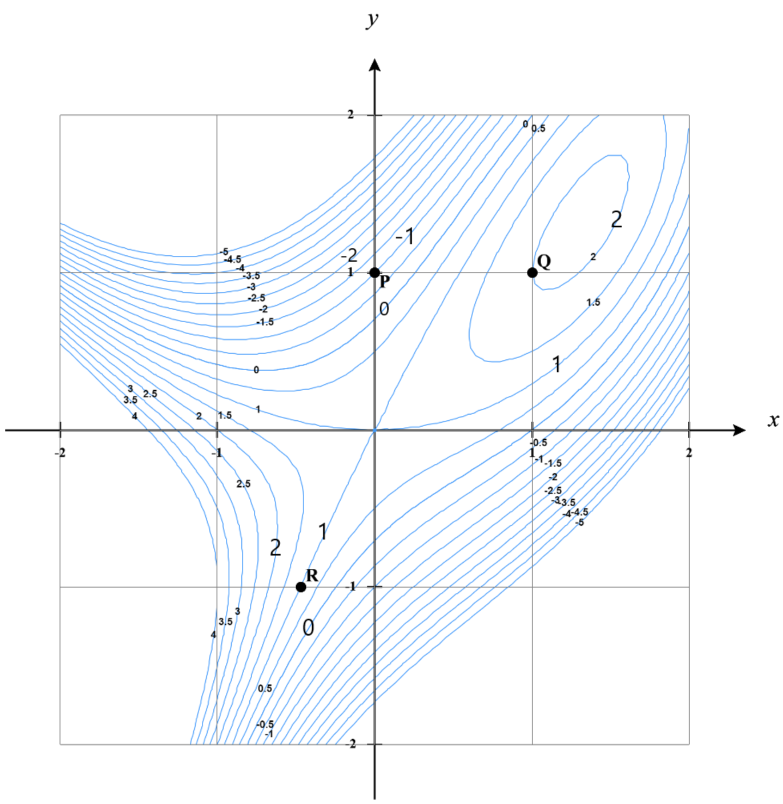
(c) $\boxed{+}$

(d) See above

(e) This critical point is a minimum since the gradient nearby points away from the critical point.

4. In this problem, you will work with the function $g : \mathbb{R}^2 \to \mathbb{R}$, whose contour plot is given below.

$$g(x, y) = -x^3 + 4xy - 2y^2 + 1.$$



In parts (a)-(c), write the sign $(+, -, 0)$ of the directional derivative at the given point in the given direction in the answer box provided. You do not need to show work for these parts.

(a) $\boxed{+}$

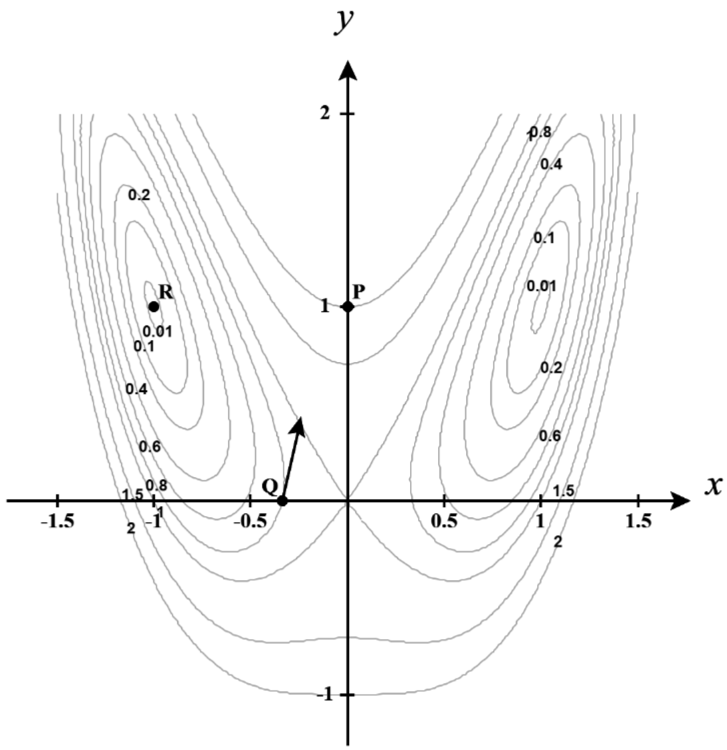
(b) $\boxed{-}$

(c) $\boxed{0}$

(d) We have $\nabla g(x,y) = \langle -3x^2+4y, 4x-4y\rangle$, so $\nabla g(1,0) = \langle -3, 4\rangle$. Then

$$D_{\mathbf{u}}g(1,0) = \nabla g(1,0)\cdot \mathbf{u} = \langle -3,4\rangle \cdot \langle 4/5, 3/5\rangle = -12/5 + 12/5 = 0.$$

5. In this problem, you will analyze the derivatives of the temperature function $T(x,y)$ whose contour plot is shown below.



(a) Is the temperature increasing, decreasing, or staying the same as x increases from the point P? $\boxed{\text{staying the same}}$

(b) Determine the sign $(+, -, 0)$ of the rate of change of T at Q in the direction from Q towards $(-1, 1)$. $\boxed{-}$

(c) Draw a point R where the temperature is at a local extreme value.
Two possible correct answers. One is shown above.

(d) Draw a vector which points in a direction of no change in temperature at the point Q.
Drawn in the picture above. The vector is tangent to the contour at Q. It can be in either direction, as long as it is tangent to the contour.