Daily Announcements & Reminders:

Goals for Today: Section 14.3

- Learn how to compute partial derivatives of functions of multiple variables
- Learn how to compute higher-order partial derivatives
- Understand Clairaut's theorem
- Define the total derivative

14.3: Partial Derivatives

Goal: Describe how a function of two (or three, later) variables is changing at a point (a, b).

Example 52. Let's go back to our example of the small hill that has height

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point (x, y). If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive y-direction), at what rate will our height change? What if we walk due east (the positive x-direction)?

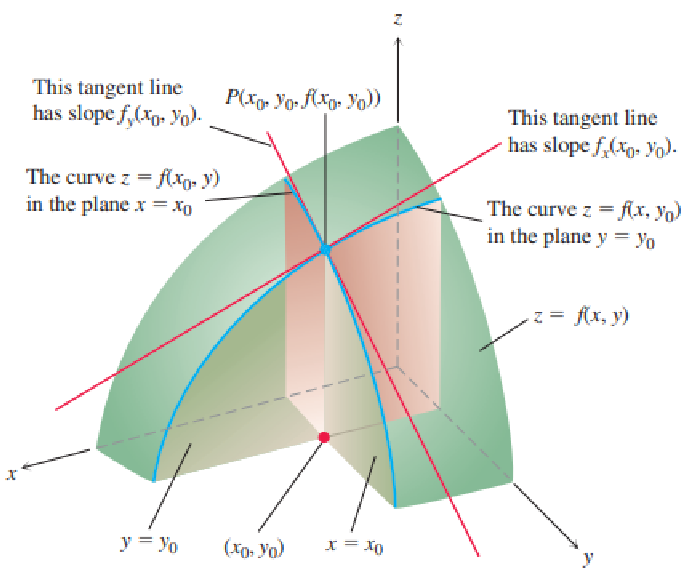
Let's investigate graphically.

Definition 53. If f is a function of two variables x and y, its ____________________ are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \_\_\_\_\_\_\_\_\_\_\_\_ \qquad f_y(x, y) = \lim_{h \to 0} \_\_\_\_\_\_\_\_\_\_\_\_$$

Notations:

Interpretations:



Example 54. Find $f_x(1,2)$ and $f_y(1,2)$ of the functions below.

a) $f(x, y) = \sqrt{5x - y}$

b) $f(x, y) = \tan(xy)$

Question: How would you define the second partial derivatives?

Example 55. Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} of the function below.

a) $f(x, y) = \sqrt{5x - y}$

What do you notice about f_{xy} and f_{yx} in the previous example?

Theorem 56 (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions $f, f_x, f_y, f_{xy}, f_{yx}$ are all continuous on D, then*

Example 57. What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

Example 58. How many rates of change should the function $f(s, t) = \begin{bmatrix} s^2 + t \\ 2s - t \\ st \end{bmatrix}$ have? Compute them.

In the previous example, we computed ________ partial derivatives. How might we **organize** this information?

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$,

we have ________ inputs, ________ output, and ________ partial derivatives, which we can use to form the **total derivative**.

This is a ____________ map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an ____________________, with one column per input and one row per output.

It has the formula $Df_{ij} =$

Example 59. Find the total derivatives of each function:

a) $f(x) = x^2 + 1$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

c) $f(x, y) = \sqrt{5x - y}$

d) $f(x, y, z) = 2xyz - z^2 y$

e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)