Daily Announcements & Reminders:

Goals for Today: Sections 14.2, 14.3

- Evaluate limits of functions of two variables
- Show that a limit does not exist using the two-path test
- Determine the set of points where a function is continuous
- Start to understand how we can measure how a function of two variables is changing

Definition 41. A ________________________________ is a rule that assigns to each __________ of real numbers (x, y, z) in a set D a ________________ denoted by $f(x, y, z)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 42. Describe the domain of the function $f(x, y, z) = \dfrac{1}{4 - x^2 - y^2 - z^2}$.

Example 43. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.

Section 14.2 Limits & Continuity

Definition 44. What is a limit of a function of two variables?

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

We won't use this definition much: the big idea is that $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ if and only if $f(x, y)$ ______________________________ regardless of how we approach (x_0, y_0).

Definition 45. A function $f(x, y)$ is **continuous** at (x_0, y_0) if

1. ______________________________
2. ______________________________
3. ______________________________

Key Fact: Adding, subtracting, multiplying, dividing, or composing two continuous functions results in another continuous function.

Example 46. Evaluate $\lim_{(x,y)\to(2,0)} \dfrac{\sqrt{2x-y}-2}{2x-y-4}$, if it exists.

Sometimes, life is harder in $\mathbb{R}^2$ and limits can fail to exist in ways that are very different from what we've seen before.

Big Idea: Limits can behave differently along different paths of approach

Example 47. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$, if it exists. Here is its graph.

This idea is called the **two-path test:**

If we can find ____________________ to (x_0, y_0) along which ________________ takes on two different values, then ____________________.

Example 48. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$

does not exist.

Example 49. **[Challenge:]** Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^4 + y^2}$$

does exist using the Squeeze Theorem.

Theorem 50 (Squeeze Theorem)**.** *If* $f(x, y) = g(x, y)h(x, y)$, *where* $\lim_{(x,y)\to(a,b)} g(x, y) = 0$ *and* $|h(x, y)| \leq C$ *for some constant* C *near* (a, b), *then* $\lim_{(x,y)\to(a,b)} f(x, y) = 0$.

14.3: Partial Derivatives

Goal: Describe how a function of two (or three, later) variables is changing at a point (a, b).

Example 51. Let's go back to our example of the small hill that has height

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point (x, y). If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive y-direction), at what rate will our height change? What if we walk due east (the positive x-direction)?

Let's investigate graphically.