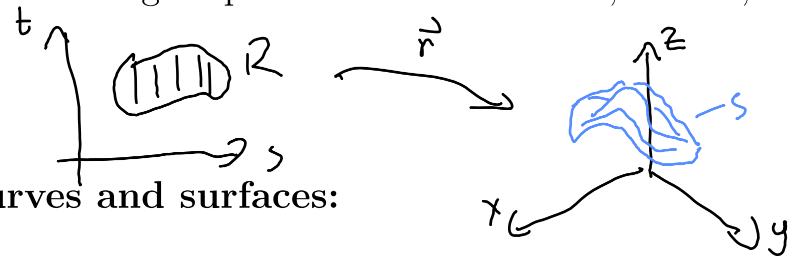
Daily Announcements & Reminders:

- HW 16.4 due tonight
- Office hours at 3 pm today instead of right after class
- Do warmup problem on Ed ⟶

Goals for Today: Sections 16.5/16.6

- Describe surfaces in $\mathbb{R}^3$ with a parameterization
- Define and compute surface integrals
- Use surface integrals to compute meaningful quantities: surface areas, masses, flux, etc.



Different ways to think about curves and surfaces:

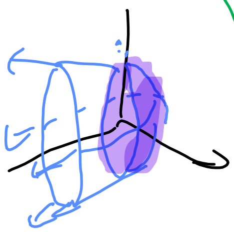
	Curves	Surfaces
Explicit:	$y = f(x)$ $y = \sqrt{4 - x^2}$	$z = f(x, y)$ $z = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$
Implicit:	$F(x, y) = 0$ $x^2 + y^2 = 4$	$F(x, y, z) = 0$ $x^2 + y^2 + z^2 = 4$
Parametric Form:	$\mathbf{r}(t) = \langle x(t), y(t) \rangle$ $\vec{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$ $0 \le t \le 2\pi$	$\vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$ $\vec{r}: \mathbb{R}^2 \to \mathbb{R}^3$ $s\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + t\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \vec{r}(s,t)$

- You have already done this; plane is: (through origin)

Example 128. Give parameteric representations for the surfaces below. S

Goal: $\vec{r}(s,t)$ $(\mathbb{R}^2 \to \mathbb{R}^3)$ with domain R s.t. $\vec{r}$ describes S

a) $x = y^2 + \frac{1}{2}z^2 - 2$



$\vec{r}(s,t) = \langle s^2 + \frac{1}{2}t^2 - 2, s, t \rangle$ $\quad s,t \in \mathbb{R}$

$\vec{r}_2(s,t) = \langle t^2 + \frac{1}{2}s^2 - 2, t, s \rangle$

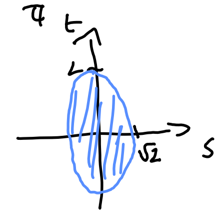
$\vec{r}_3(s,t) = \langle s^2 - 2, s\cos(t), \sqrt{2}s \cdot \sin(t) \rangle$ $\quad s \geq 0$, $0 \leq t \leq 2\pi$

b) The portion of the surface $x = y^2 + \frac{1}{2}z^2 - 2$ which lies behind the yz-plane.

$\vec{r}(s,t) = \langle s^2 + \frac{1}{2}t^2 - 2, s, t \rangle$ $\quad s^2 + \frac{1}{2}t^2 - 2 < 0 \Rightarrow \frac{s^2}{2} + \frac{t^2}{4} < 1$

$\vec{r}_3(s,t) = \langle s^2 - 2, s\cos(t), \sqrt{2}s \cdot \sin(t) \rangle$

$0 \leq s \leq \sqrt{2}$ $\quad 0 \leq t \leq 2\pi$



c) $x^2 + y^2 + z^2 = 9$ Think in spherical coords: $\rho^2 = 9 \Rightarrow \rho = 3$

$\vec{r}(\varphi, \theta) = \langle 3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi \rangle$

$0 \leq \varphi \leq \pi$

$0 \leq \theta \leq 2\pi$

d) $x^2 + y^2 = 25$ $\xrightarrow{\text{cylindrical}}$ $r = 5$

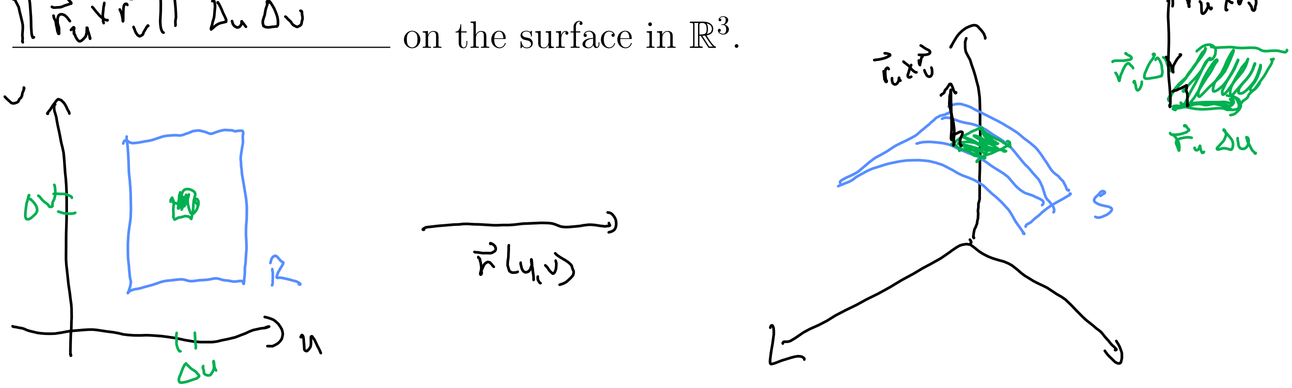
$\vec{r}(\theta, z) = \langle 5\cos\theta, 5\sin\theta, z \rangle$ $\quad 0 \leq \theta \leq 2\pi$, $z \in \mathbb{R}$

What can we do with this? $\vec{r}(u,v)$ is a parameterization of S

If our parameterization is **smooth** ($\mathbf{r}_u, \mathbf{r}_v$ not parallel in the domain), then:

- $\mathbf{r}_u \times \mathbf{r}_v$ is normal vector to the surface S
- A rectangle of size $\Delta u \times \Delta v$ in the uv-domain is mapped to a ~~rectangle~~ parallelogram of size $\|\vec{r}_u \times \vec{r}_v\| \, \Delta u \, \Delta v$ on the surface in $\mathbb{R}^3$.

$d\sigma = \|\vec{r}_u \times \vec{r}_v\| \, dA$



- Thus, Area$(S) = \displaystyle\iint_S d\sigma = \iint_R \|\vec{r}_u \times \vec{r}_v\| \, dA$

surface integral   double integral

Example 129 (Poll). Find the area of the portion of the cylinder $x^2 + y^2 = 25$ between $z = 0$ and $z = 1$. 10π

1) Parameterize;

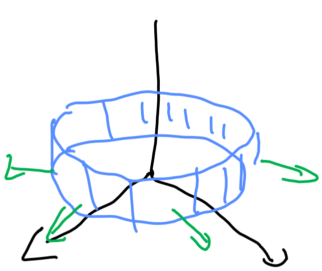
$$\vec{r}(u,v) = \langle 5\cos(u), 5\sin(u), v \rangle \qquad 0 \le u \le 2\pi, \quad 0 \le v \le 1$$

2) Compute $\|\vec{r}_u \times \vec{r}_v\|$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5\sin(u) & 5\cos(u) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 5\cos(u), 5\sin(u), 0 \rangle \quad \vec{n}$$

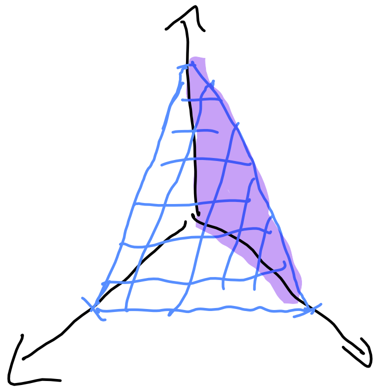
$\|\vec{r}_u \times \vec{r}_v\| = 5$

3) Compute: $SA = \displaystyle\int_0^{2\pi} \int_0^1 5 \, dv \, du = 10\pi$



Example 130. Suppose the density of a thin plate S in the shape of the portion of the plane $x + y + z = 1$ in the first octant is $\delta(x, y, z) = 6xy$. Find the mass of the plate.

kg/m^2



$$\text{mass} = \iint_S \delta(x,y,z)\, d\sigma$$

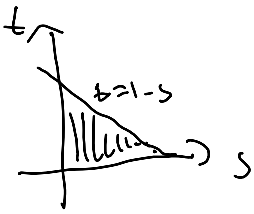
1) Parameterize S: $x = 1-y-z$

$\vec{r}(s,t) = \langle 1-s-t, s, t \rangle$

$s \geq 0$

$t \geq 0$

$1-s-t \geq 0$



$0 \leq t \leq 1-s$

$0 \leq s \leq 1$

2) Find $\|\vec{r}_s \times \vec{r}_t\|$:

$$\vec{r}_s \times \vec{r}_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

$\|\vec{r}_s \times \vec{r}_t\| = \sqrt{3}$

3) Substitute:

$$\iint_S \delta(x,y,z)\, d\sigma = \iint_R \delta(\vec{r}(s,t))\, \|\vec{r}_s \times \vec{r}_t\|\, dA$$

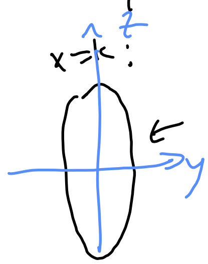
$$= \int_0^1 \int_0^{1-s} 6(1-s-t)s \cdot \sqrt{3}\, dt\, ds$$

$$= \sqrt{3}/4 \text{ kg}$$

$s^2 - 2 = k$

$$X = y^2 + \frac{1}{2}z^2 - 2$$

$x = k$:



$k+2 = y^2 + \frac{1}{2}z^2$

$$1 = \frac{y^2}{k+2} + \frac{z^2}{2(k+2)}$$

$$\langle k, \underbrace{\sqrt{k+2}}_{=s}\cos(t), \sqrt{2}\sqrt{k+2}\sin(t)\rangle$$