Daily Announcements & Reminders:

Goals for Today: Section 16.4

- Define the divergence and curl of a vector field
- Interpret divergence and curl geometrically
- Apply Green's Theorem to compute line integrals over the boundary of a simply-connected region

Useful notation: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

So if $f(x, y, z)$ is a function of three variables, $\nabla f = \left\langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f) \right\rangle$

If $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is a vector field:

- $\nabla \cdot \mathbf{F} =$

- $\nabla \times \mathbf{F} =$

How do we measure the change of a vector field?

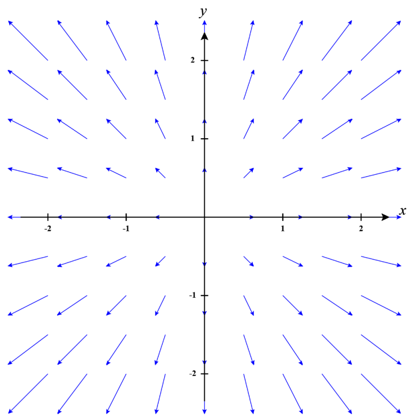
1. **Curl** (in $\mathbb{R}^3$)

   - Tells us ______________________
   - Measures ____
   - Is a _____________
   - Direction gives ______________________
   - Magnitude gives ______________________
   - $\operatorname{curl} \mathbf{F} =$
   - If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$: we use $\nabla \times \mathbf{F} = \nabla \times \langle P, Q, 0 \rangle$

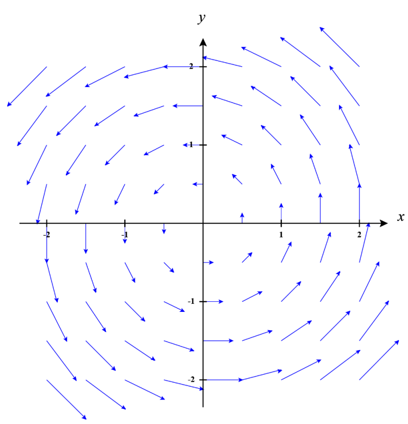
2. **Divergence** (in any $\mathbb{R}^n$)

   - Tells us ______________________
   - Measures ____
   - Is a _____________
   - $\operatorname{div} \mathbf{F} =$

Example 122. Let $\mathbf{F}(x, y) = \langle x, y\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



Example 123 (Poll)**.** Let $\mathbf{F}(x, y) = \langle -y, x\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



Question: How is this useful?

Answer: We can relate ______________________________ inside a region to the behavior of the vector field on the boundary of the region.

Theorem 124 (Green's Theorem). *Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane with outward oriented unit normal* $\mathbf{n}$. *If* $\mathbf{F} = \langle P, Q \rangle$ *has continuous partial derivatives around R, then*

a) Circulation form:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C P\, dx + Q\, dy = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_R Q_x - P_y\, dA$$

b) Flux form:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C P\, dy - Q\, dx = \iint_R (\nabla \cdot \mathbf{F})\, dA = \iint_R P_x + Q_y\, dA$$

Example 125. Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$ for the vector field $\mathbf{F} = \langle -y^2, xy \rangle$ where C is the boundary of the square bounded by $x = 0, x = 1, y = 0,$ and $y = 1$ oriented counterclockwise.

Example 126. Compute the flux out of the region R which is the portion of the annulus between the circles of radius 1 and 3 in the first octant for the vector field $\mathbf{F} = \langle \frac{1}{3}x^3, \frac{1}{3}y^3 \rangle$.

Example 127. Let R be the region bounded by the curve $\mathbf{r}(t) = \langle \sin(2t), \sin(t) \rangle$ for $0 \leq t \leq \pi$. Find the area of R, using Green's Theorem applied to the vector field $\mathbf{F} = \frac{1}{2}\langle x, y \rangle$.

Note: This is the idea behind the operation of the measuring instrument known as a planimeter.