**Daily Announcements & Reminders:**

**Goals for Today:** Section 14.7

- Define local & global extreme values for functions of two variables
- Learn how to find local extreme values for functions of two variables
- Learn how to classify critical points for functions of two variables
- Learn how to find global extreme values on a closed & bounded domain

**Last time:** If $f(x, y)$ is a function of two variables, we said $\nabla f(a, b)$ points in the direction of greatest change of $f$.

Back to the hill $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$! What should we expect to get if we compute $\nabla h(0, 0)$? Why? What does the tangent plane to $z = h(x, y)$ at $(0, 0, 4)$ look like?

**Definition 63.** Let $f(x, y)$ be defined on a region containing the point $(a, b)$. We say

- $f(a, b)$ is a ____________________ value of $f$ if $f(a, b)$ ____ $f(x, y)$ for all domain points $(x, y)$ in a disk centered at $(a, b)$
- $f(a, b)$ is a ____________________ value of $f$ if $f(a, b)$ ____ $f(x, y)$ for all domain points $(x, y)$ in a disk centered at $(a, b)$

In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $z = x^2 - y^2$ (a hyperbolic paraboloid!) near $(0, 0)$.

This is called a ____________________

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e.

**Definition 64.** If $f(x, y)$ is a function of two variables, a point $(a, b)$ in the domain of $f$ with $Df(a, b) =$________________ or where $Df(a, b)$ ____________________ is called a ____________________ of $f$.

**Example 65.** Find the critical points of the function $f(x, y) = x^3 + y^3 - 3xy$.

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix**. The **Hessian matrix** of $f(x, y)$ at $(a, b)$ is

$$Hf(a, b) =$$

**Theorem 66** (2nd Derivative Test)**.** *Suppose $(a, b)$ is a critical point of $f(x, y)$ and $f$ has continuous second partial derivatives. Then we have:*

- *If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$, $f(a, b)$ is a local minimum*
- *If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$, $f(a, b)$ is a local maximum*
- *If $\det(Hf(a, b)) < 0$, $f$ has a saddle point at $(a, b)$*
- *If $\det(Hf(a, b)) = 0$, the test is inconclusive.*

More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at $\mathbf{p}$ then

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, $f$ is concave up in every direction from $\mathbf{p}$ and so has a local minimum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, $f$ is concave down in every direction from $\mathbf{p}$ and so has a local maximum at $\mathbf{p}$.
- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, $f$ is concave up in some directions from $\mathbf{p}$ and concave down in others, so has neither a local minimum or maximum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, $f$ may have either a local minimum or neither at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, $f$ may have either a local maximum or neither at $\mathbf{p}$.

**Example 67.** Classify the critical points of $f(x, y) = x^3 + y^3 - 3xy$ from Example 65.

**Two Local Maxima, No Local Minimum:** The function $g(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2 + 2$ has two critical points, at $(-1, 0)$ and $(1, 2)$. Both are local maxima, and the function never has a local minimum!

A global maximum of $f(x, y)$ is like a local maximum, except we must have $f(a, b) \geq f(x, y)$ for **all** $(x, y)$ in the domain of $f$. A global minimum is defined similarly.

**Theorem 68.** *On a closed & bounded domain, any continuous function $f(x, y)$ attains a global minimum & maximum.*

**Closed:**

**Bounded:**

**Strategy for finding global min/max of $f(x, y)$ on a closed & bounded domain $R$**

1. Find all critical points of $f$ inside $R$.
2. Find all critical points of $f$ on the boundary of $R$
3. Evaluate $f$ at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

**Example 69.** Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region $R$ bounded by $y = x^2$ and $y = 4$.