Math 2551 Checkpoint 2

CHECKPOINT KEY

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 25 minutes to complete the problems.
- You may not use aids of any kind.
- Show your work. Answers without work shown will receive a **Not Yet**

Learning Targets

- **D2: Tangent Planes and Linear Approximations.** I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.
- **D3: Optimization.** I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.
- **A1: Interpreting Derivatives.** I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.

Useful Formulas

- Linearization: Near $\mathbf{a}$, $f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$
- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is
$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle.$$
- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$
- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

1. [**D2: Tangent Planes and Linear Approximations**]

(a) **Fill-In:** If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at the point (a, b), then $Df(a, b)$ is the ____________________ that best approximates Δf near the point (a, b).

linear map

(b) Approximate the value of $\sqrt{9.44}$ using the linearization of

$$f(x, y) = \sqrt{x^2 + y}$$

near the point $(1, 8)$.
Hint: You can write 9.44 *as* $1.2^2 + 8$.

Solution. We have

$$\begin{aligned} f(1,8) &= \sqrt{1^2+8} = \sqrt{9} = 3, \\ Df(x,y) &= \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y}} & \dfrac{1}{2\sqrt{x^2+y}} \end{bmatrix} \\ Df(1,8) &= \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \end{bmatrix} \end{aligned}$$

Thus, the linearization is given by

$$L(x,y) = f(1,8) + Df(1,8) \cdot \begin{bmatrix} x-1 \\ y-8 \end{bmatrix} = 3 + \frac{1}{3}(x-1) + \frac{1}{6}(y-8).$$

To approximate $\sqrt{9.44}$, we can set $x = 1.2$ and $y = 8$:

$$L(1.2, 8) = 3 + \frac{1}{3}(1.2-1) + \frac{1}{6}(8-8) = 3 + \frac{1}{3}(0.2) + 0 = 3.0667.$$

Therefore, the approximation is

$$\sqrt{9.44} \approx 3.0667.$$

2. [**D3: Optimization**] Find and classify all critical points of the function

$$f(x,y) = 2x^2y - 8xy + 6y + y^2.$$

Solution. To find the critical points, we compute the total derivative and set it equal to zero:

$$Df(x,y) = \begin{bmatrix} 4xy - 8y & 2x^2 - 8x + 2y + 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

From the first entry, we can factor to get:

$$4y(x-2) = 0.$$

This gives us two cases to consider: 1. $y = 0$ 2. $x = 2$
Case 1: If $y = 0$, then substituting into the second equation gives:

$$2x^2 - 8x + 6 = 2(x-3)(x-1) = 0.$$

So we have two critical points: $(1,0)$ and $(3,0)$.
Case 2: If $x = 2$, then substituting into the second equation gives:

$$2y - 2 = 2(y-1) = 0$$

So we have another critical point: $(2,1)$.
Thus, the critical points are $(1,0)$, $(3,0)$, and $(2,1)$.
Next, we classify these critical points using the second derivative test. We compute the Hessian matrix:

$$Hf(x,y) = \begin{bmatrix} 4y & 4x-8 \\ 4x-8 & 2 \end{bmatrix}$$

Now we compute the determinant at each critical point and apply the Second Derivative Test criterion.
At $(1,0)$:

$$\det(Hf(1,0)) = 2(0) - (-4)^2 = -16 < 0$$

so f has a saddle point at $(1,0)$.
At $(3,0)$:

$$\det(Hf(3,0)) = 2(0) - (4)^2 = -16 < 0$$

so f has a saddle point at $(3,0)$.
At $(2,1)$:

$$\det(Hf(2,1)) = 2(4) - 0^2 = 8 > 0, \quad f_{xx}(2,1) = 2 > 0$$

so f has a local minimum at $(2,1)$.

3. **[A1: Interpreting Derivatives]** Let $h(x, y)$ give the height in meters on a landscape in north Georgia at the point with coordinates x, y in meters measured from the origin at the center of the town of Blue Ridge.

   (a) What is the meaning of the statement $h_y(2000, 2000) = -0.25$? Be as specific as possible, and think about units.

   Solution. The statement $h_y(2000, 2000) = -0.25$ means that at the point located 2000 meters east and 2000 meters north of the center of Blue Ridge, the slope of the landscape in the north-south direction at that point is -0.25 meters of height per meter of distance.

   (b) Suppose that at the point $(3000, -1000)$, the gradient of h points southeast. If you can only walk **due north, due south, or due west**, in which direction should you walk to most rapidly increase your altitude?

   due south

   (c) The point $(35000, 1000, 1458)$ is at the top of Brasstown Bald mountain. What will the value of $\nabla h(35000, 1000)$ be? Explain.

   Solution. At the top of Brasstown Bald mountain, the gradient $\nabla h(35000, 1000)$ will be the zero vector $\mathbf{0}$. This is because at a local maximum (the peak of the mountain), the rate of change of height in all directions is zero.