MATH 2551 C/HP Notes

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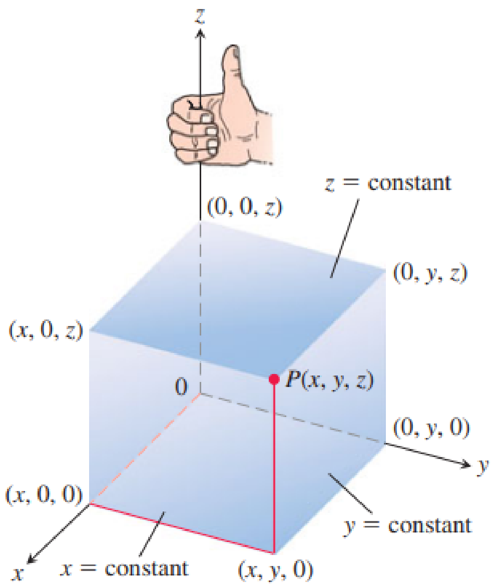
Day 1 - Course Introduction and Cross Products

Pre-Lecture

12.1: Three-Dimensional Coordinates



Day 1 - Lecture

Daily Announcements & Reminders:

Goals for Today:

Sections 12.1, 12.3, 12.4

- Set classroom norms
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Class Values/Norms:

- Mistakes are a learning opportunity
- Mathematics is collaborative
- Make sure everyone is included
- Criticize ideas, not people
- Be respectful of everyone
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Introduction to the Course

Big Idea: Extend differential & integral calculus.

What are some key ideas from these two courses?

Differential Calculus

Integral Calculus

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

Example 1. What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation

$$x^2 + y^2 = 1?$$

Example 2. What is the distance from $(3, 2, -2)$ to the xy-plane? What is the distance to the x-axis?

Section 12.3/4: Dot & Cross Products

Definition 3. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \dots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \dots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = \underline{\hspace{8cm}}$$

This product tells us about ____.

In particular, two vectors are **orthogonal** if and only if their dot product is ______.

Example 4. Are $\mathbf{u} = \langle 1, 1, 4 \rangle$ and $\mathbf{v} = \langle -3, -1, 1 \rangle$ orthogonal?

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.

1.

2.

Definition 5. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ <u>in $\mathbb{R}^3$</u> is

$$\mathbf{u} \times \mathbf{v} = \underline{\hspace{8cm}}$$

Example 6. Find $\langle 1, 2, 0\rangle \times \langle 3, -1, 0\rangle$.

Day 2 - Lines, Planes, and Quadrics

Pre-Lecture

12.5: Lines

Lines in $\mathbb{R}^2$, a new perspective:

Example 7. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.

Day 2 Lecture

Daily Announcements & Reminders:

Goals for Today:

Sections 12.5-12.6

- Apply the cross product to solve problems
- Learn the equations that describe lines, planes, and quadric surfaces in $\mathbb{R}^3$
- Solve problems involving the equations of lines and planes
- Sketch quadric surfaces in $\mathbb{R}^3$

Example 8. Find a set of parametric equations for the line through the point $(1, 10, 100)$ which is parallel to the line with vector equation

$$\mathbf{r}(t) = \langle 1, 4, -3 \rangle t + \langle 0, -1, 1 \rangle$$

Section 12.5 Planes

Planes in $\mathbb{R}^3$

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction $\mathbf{n}$, called the *normal vector*.

Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

In $\mathbb{R}^3$, a pair of lines can be related in three ways:

parallel **skew** **intersecting**

On the other hand, a pair of planes can be related in just two ways:

parallel **intersecting**

Example 9. Consider the planes $y - z = -2$ and $x - y = 0$. Show that the planes intersect and find an equation for the line passing through the point $P = (-8, 0, 2)$ which is parallel to the line of intersection of the planes.

Section 12.6 Quadric Surfaces

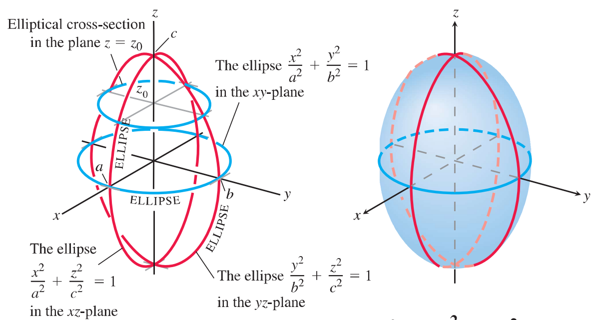
Definition 10. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

You know several examples already:

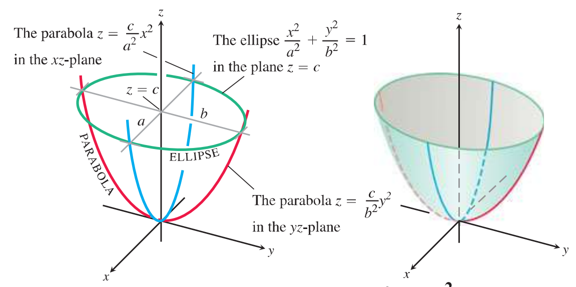
The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections.

Example 11. Use cross-sections to sketch and identify the quadric surface $x = z^2 + y^2$.

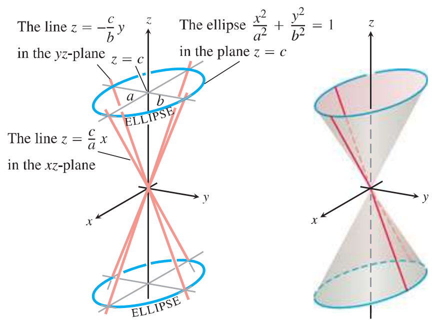
TABLE 12.1 Graphs of Quadric Surfaces



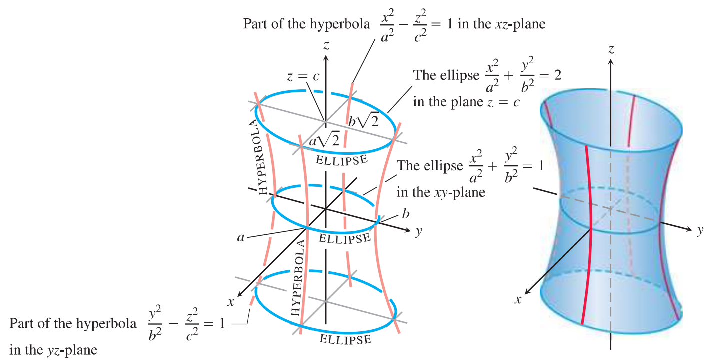
ELLIPSOID $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$



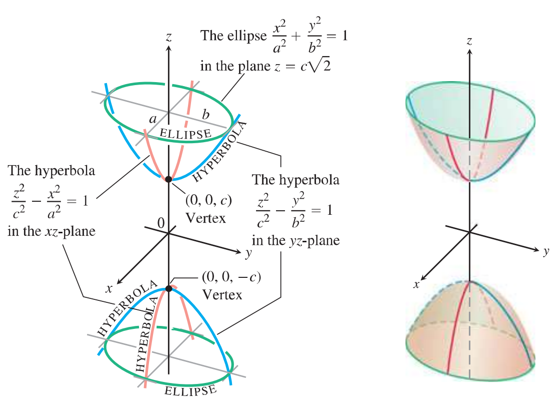
ELLIPTICAL PARABOLOID $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$



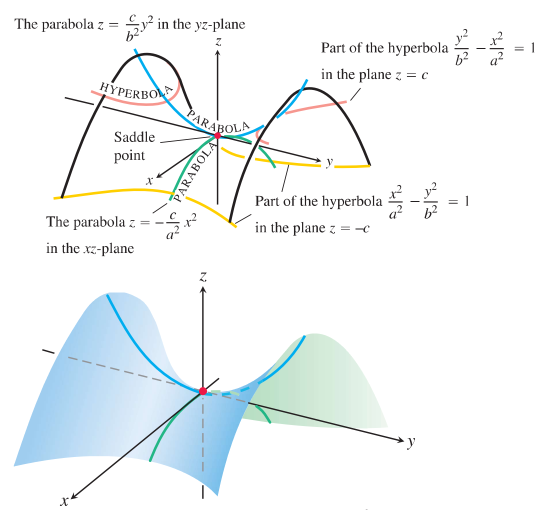
ELLIPTICAL CONE $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\quad \dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\quad \dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = \dfrac{z}{c},\ c > 0$

Day 3 - Vector-Valued Functions & Calculus

Pre-Lecture

Section 13.1: Vector-Valued Functions

Last week, we used functions like

$$\ell(t) = \langle 2t+1, 3-t, t-1 \rangle, \quad -\infty \le t \le \infty$$

to produce lines in $\mathbb{R}^2$ and $\mathbb{R}^3$.

This is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

What happens when we change the component functions to be non-linear?

Given a fixed curve C in space, producing a vector-valued function $\mathbf{r}$ whose graph is C is called ________________ the curve C, and $\mathbf{r}$ is called a __________________ of C.

Day 3 Lecture

Daily Announcements & Reminders:

Goals for Today:

Sections 13.1, 13.2

- Introduce vector-valued functions
- Plot vector-valued functions and construct them from a graph
- Compute limits, derivatives, and tangent lines for vector-valued functions
- Compute integrals of vector-valued functions and solve initial value problems

Example 12. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?

Check your intuition

Section 13.1: Calculus of Vector-Valued Functions

Unifying theme: Do what you already know, componentwise.

This works with limits:

Example 13. Compute $\lim_{t\to e}\langle t^2, 2, \ln(t)\rangle$.

And with continuity:

Example 14. Determine where the function $\mathbf{r}(t) = t\mathbf{i} - \dfrac{1}{t^2-4}\mathbf{j} + \sin(t)\mathbf{k}$ is continuous.

And with <u>derivatives</u>:

Example 15. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1 \rangle$, find $\mathbf{r}'(t)$.

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives ______________________________
- $\|\mathbf{r}'(t)\|$ gives ______________________________
- $\mathbf{r}''(t)$ gives ______________________________

Let's see this graphically

Example 16. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1 \rangle$ at time $t = 2$.

And with <u>integrals</u>:

Example 17. Find $\displaystyle\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle \; dt$.

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 18. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t}\rangle \; m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0, 0, 0\rangle$, use calculus to reconstruct his flight path.

Day 4 - Geometry of Curves

Pre-Lecture

Section 13.3: Arc Length

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Our next goal is to be able to measure distance traveled or arc length.

Motivating problem: Suppose the position of a fly at time t is

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle,$$

where $0 \le t \le 2\pi$.

How far does the fly travel from $t = 0$ to $t = \pi$?

Definition 19. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ from __________ to __________ that is traced out exactly once is

$$L = \underline{\hspace{6cm}}$$

Day 4 Lecture

Daily Announcements & Reminders:

Goals for Today:

Sections 13.3, 13.4

- Compute arc lengths of curves using parameterizations
- Define and compute arc-length parameterizations
- Define, interpret, and compute the curvature of a curve
- Compute the unit tangent and principal unit normal vectors of a curve

Example 20. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(2, 4, 8)$.

Example 21. Find the distance traveled by a particle moving along the path

$$\mathbf{r}(t) = \langle \ln(t), \sqrt{2}t, \frac{1}{2}t^2 \rangle, \qquad t > 0$$

from $t = 1$ to $t = 2$.

Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

$$s(t) = \underline{\hspace{6cm}}$$

We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.

Example 22. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \leq t \leq 2\pi$.

13.4 - Curvature, Tangents, Normals

The next idea we are going to explore is the curvature of a curve in space along with two vectors that orient the curve.

First, we need the **unit tangent vector**, denoted **T**:

- In terms of an arc-length parameter s: ____________________
- In terms of any parameter t: ____________________

This lets us define the **curvature**, $\kappa(s) =$ ____________________

Example 23. Earlier, we found an arc length parameterization of the circle of radius 4 centered at $(0, 0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \le s \le 8\pi.$$

Use this to find $\mathbf{T}(s)$ and $\kappa(s)$.

Question: In which direction is $\mathbf{T}$ changing?

This is the direction of the **principal unit normal**, $\mathbf{N}(s) =$ __________________

We said that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) =$ ________________
- $\mathbf{N}(t) =$ ________________
- $\kappa(t) =$ ________________ or ________________________

Example 24. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1 \rangle$.

Day 5 - Functions of Multiple Variables

Pre-Lecture

Section 14.1: Functions of Multiple Variables

Definition 25. A ________________________________ is a rule that assigns to each __________ of real numbers (x, y) in a set D a ________________ denoted by $f(x, y)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^2$$

Example 26. Three examples are

$$f(x, y) = x^2 + y^2, \qquad g(x, y) = \ln(x + y), \qquad h(x, y) = h(x, y) = \sqrt{4 - x^2 - y^2}.$$

Example 27. Find the largest possible domains of f, g, and h.

Day 5 Lecture

Daily Announcements & Reminders:

Goals for Today: Sections 13.4-14.1

- Compute the unit tangent and principal unit normal vectors of a curve
- Give examples of functions of multiple variables
- Find the domain of functions of two variables
- Introduce and sketch traces and contours of functions of two variables
- Graph functions of two variables

In the pre-lecture video, we discussed the domains of the functions $f(x,y) = x^2 + y^2, g(x,y) = \ln(x+y)$, and $h(x,y) = \sqrt{4 - x^2 - y^2}$.

Definition 28. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Here are the graphs of the three functions above.

Example 29. Suppose a small hill has height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x, y). How could we draw a picture that represents the hill in 2D?

In 3D, it looks like this.

Definition 30. The ____________ (also called ____________________) of a function f of two variables are the curves with equations ________________, where k is a constant (in the range of f). A plot of ____________ for various values of z is a ____________________(or ____________________________).

Some common examples of these are:

-
-
-

Example 31. Create a contour diagram of $f(x, y) = x^2 - y^2$

Day 6 - Functions of Multiple Variables & Limits

Pre-Lecture

Section 14.1: Traces & Graphs

Definition 32. The ____________ of a surface are the curves of ____________________ of the surface with planes parallel to the ______________________.

Example 33. Find the traces of the surface $z = x^2 - y^2$.

Day 6 Lecture

Daily Announcements & Reminders:

Goals for Today:

Sections 14.1-14.2

- Introduce and sketch traces and contours of functions of two variables
- Graph functions of two variables
- Find level surfaces of functions of three variables
- Evaluate limits of functions of two variables

Example 34. Create a contour diagram of $g(x, y) = \sqrt{16 - 4x^2 - y^2}$.

Example 35. Use the traces and contours of $z = f(x, y) = 4 - 2x - y^2$ to sketch the portion of its graph in the first octant.

Let's check our work: https://tinyurl.com/math2551-2var-graph

Definition 36. A ______________________________ is a rule that assigns to each __________ of real numbers (x, y, z) in a set D a ________________ denoted by $f(x, y, z)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 37. Describe the domain of the function $f(x, y, z) = \dfrac{1}{4 - x^2 - y^2 - z^2}$.

Example 38. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.

Section 14.2 Limits & Continuity

Definition 39. What is a limit of a function of two variables?

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

We won't use this definition much: the big idea is that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if and only if $f(x,y)$ ______________________________ regardless of how we approach (x_0, y_0).

Definition 40. A function $f(x,y)$ is **continuous** at (x_0, y_0) if

1. ______________________________
2. ______________________________
3. ______________________________

Key Fact: Adding, subtracting, multiplying, dividing, or composing two continuous functions results in another continuous function.

Example 41. Evaluate $\displaystyle\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$, if it exists.