MATH 2551 Guided Notes

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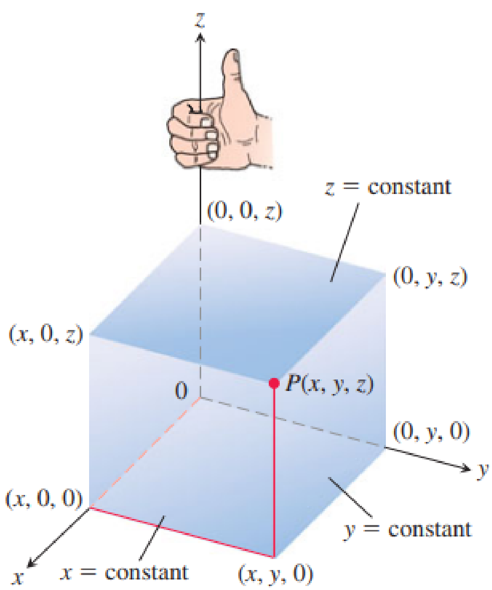
Fall 2025

Day 1 - Course Introduction and Cross Products

Pre-Lecture

Section 12.1: Three-Dimensional Coordinate Systems



Day 1 - Lecture

Daily Announcements & Reminders:

Goals for Today:

Sections 12.1, 12.3, 12.4

- Set classroom norms
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Icebreaker on PollEverywhere

Introduction to the Course

Purposes:

- Pre-Lecture: Get in math headspace, first exposure to the day's topic
- Lecture: Fill in the rest of the new ideas for the week
- Studio: Guided group practice with new ideas under supervision, develop independence
- WeBWorK: Basic skills practice for the topics
- LT Practice: Extra practice problems for assessments
- Quizzes, Checkpoints, & Exams: Demonstrate learning & get feedback

Feedback loops: Lecture -> practice -> studio -> practice -> assessment -> practice -> assessment -> ...

Important Canvas Items: Back to Canvas!

Big Idea: Extend differential & integral calculus.

What are some key ideas from these two courses?

<u>Differential Calculus</u> <u>Integral Calculus</u>

Poll

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

Example 1. What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation

$$x^2 + y^2 = 1?$$

Section 12.3/4: Dot & Cross Products

Definition 2. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \dots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \dots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = \underline{\hspace{8cm}}$$

This product tells us about ____.

In particular, two vectors are **orthogonal** if and only if their dot product is ________.

Example 3. Are $\mathbf{u} = \langle 1, 1, 4 \rangle$ and $\mathbf{v} = \langle -3, -1, 1 \rangle$ orthogonal?

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.

1.

2.

Definition 4. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$ is

$$\mathbf{u} \times \mathbf{v} = \underline{\hspace{8cm}}$$

Example 5. Find $\langle 1, 2, 1\rangle \times \langle 3, -1, 0\rangle$.

Day 2 - Lines, Planes, and Quadrics

Pre-Lecture

12.5: Lines

Lines in $\mathbb{R}^2$, a new perspective:

Example 6. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.

Day 2 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **G1: Lines and Planes.** I can describe lines using the vector equation of a line. I can describe planes using the general equation of a plane. I can find the equations of planes using a point and a normal vector. I can find the intersections of lines and planes. I can describe the relationships of lines and planes to each other. I can solve problems with lines and planes.
- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.

Goals for Today:

Sections 12.5-12.6

- Apply the cross product to solve problems
- Learn the equations that describe lines, planes, and quadric surfaces in $\mathbb{R}^3$
- Solve problems involving the equations of lines and planes
- Sketch quadric surfaces in $\mathbb{R}^3$

Example 7. Find a set of parametric equations for the line through the point $(1, 10, 100)$ which is parallel to the line with vector equation

$$\mathbf{r}(t) = \langle 1, 4, -3 \rangle t + \langle 0, -1, 1 \rangle$$

Section 12.5 Planes

Planes in $\mathbb{R}^3$

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction $\mathbf{n}$, called the *normal vector*.

Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

In $\mathbb{R}^3$, a pair of lines can be related in three ways:

parallel **skew** **intersecting**

On the other hand, a pair of planes can be related in just two ways:

parallel **intersecting**

Example 8. [Poll] The lines

$$\ell_1(t) = \langle 1, 1, 1\rangle t + \langle 0, 0, 1\rangle$$

and

$$\ell_2(t) = \langle 2, 2, 2\rangle t + \langle 0, 0, 1\rangle$$

are related in what way?

Example 9. [Poll] The lines

$$\ell_1(t) = \langle 1, 1, 1\rangle t + \langle 0, 0, 1\rangle$$

and

$$\ell_2(t) = 2t\mathbf{i} - t\mathbf{j} - t\mathbf{k}$$

are related in what way?

Example 10. Consider the planes $y - z = -2$ and $x - y = 0$. Show that the planes intersect and find an equation for the line of intersection of the planes.

Section 12.6 Quadric Surfaces

Definition 11. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

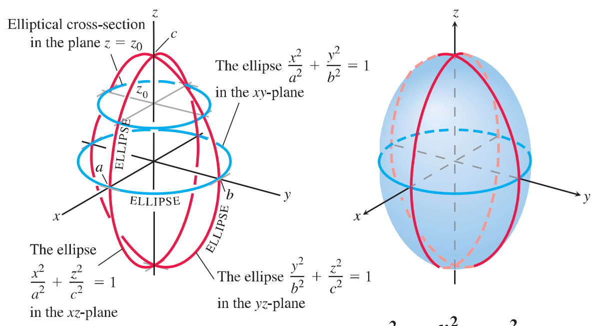
The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections. We'll also make heavy use of 3d graphing technology to get comfortable with these new objects.

Example 12. Use a 3d graphing utility to plot the quadric surface

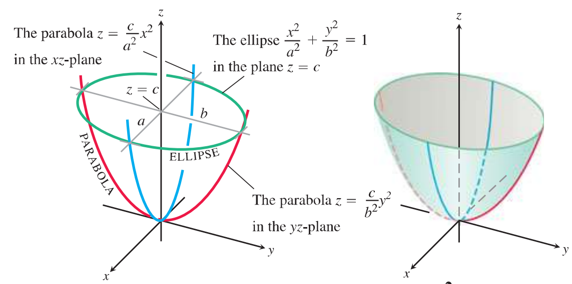
$$z = x^2 + y^2.$$

This surface is called a __________________________, because it has two coordinate directions with cross sections that are __________________ and one with cross sections that are _____________.

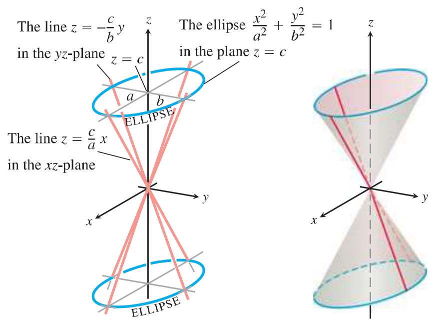
TABLE 12.1 Graphs of Quadric Surfaces



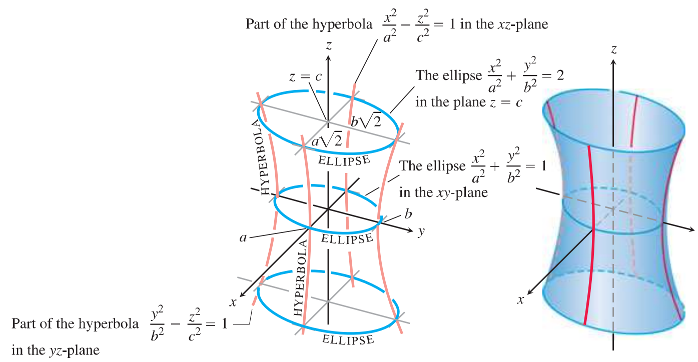
ELLIPSOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



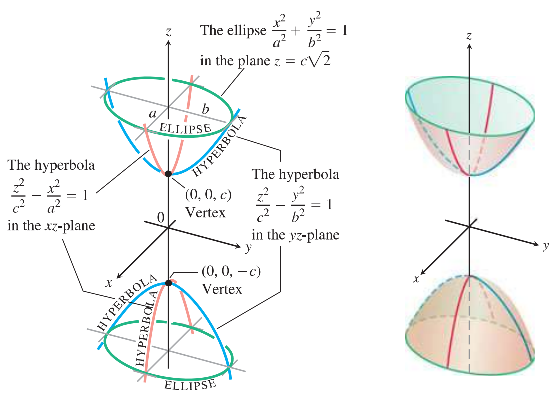
ELLIPTICAL PARABOLOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



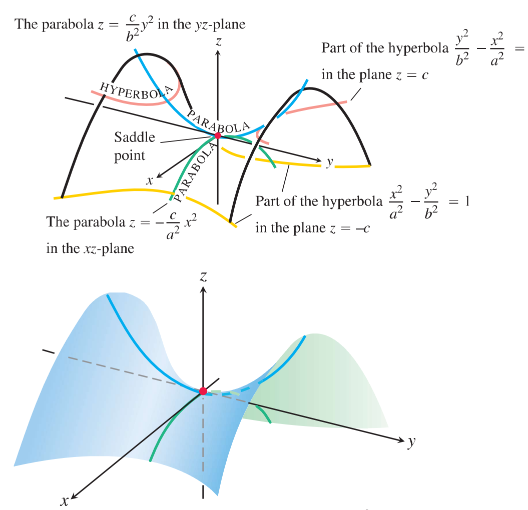
ELLIPTICAL CONE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

Example 13. [Poll] Which of the following are quadric surfaces?

1. A line
2. A sphere
3. A circle
4. An ellipse
5. The set of points (x, y, z) which solve $x^2 + y^2 - 3 = 0$.
6. The set of points (x, y, z) which solve $x^2 - y^2 - z^2 = 4$.

Example 14. [Poll] Classify the quadric surface $x + y^2 - z^2 - 3 = 0$.

Day 3 - Vector-Valued Functions & Calculus

Pre-Lecture

Section 13.1: Vector-Valued Functions

Last week, we used functions like

$$\ell(t) = \langle 2t+1, 3-t, t-1 \rangle, \quad -\infty \leq t \leq \infty$$

to produce lines in $\mathbb{R}^2$ and $\mathbb{R}^3$.

This is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

What happens when we change the component functions to be non-linear?

Given a fixed curve C in space, producing a vector-valued function $\mathbf{r}$ whose graph is C is called ________________ the curve C, and $\mathbf{r}$ is called a ____________________ of C.

Day 3 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.
- **G2: Calculus of Curves.** I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.
- **G5: Parameterization.** I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.

Goals for Today:

Sections 12.6, 13.1, 13.2

- Learn the equations that define quadric surfaces in $\mathbb{R}^3$
- Use technology to plot quadric surfaces
- Introduce vector-valued functions
- Plot vector-valued functions and construct them from a graph
- Compute limits, derivatives, and tangent lines for vector-valued functions
- Compute integrals of vector-valued functions and solve initial value problems

Example 15. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?

Check your intuition

Section 13.1: Calculus of Vector-Valued Functions

Unifying theme: Do what you already know, componentwise.

This works with limits:

Example 16. Compute $\lim_{t \to e} \langle t^2, 2, \ln(t) \rangle$.

And with derivatives:

Example 17. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1 \rangle$, find $\mathbf{r}'(t)$.

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives ______________________________
- $\|\mathbf{r}'(t)\|$ gives ______________________________
- $\mathbf{r}''(t)$ gives ______________________________

Let's see this graphically

Example 18. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

Day 4 - Geometry of Curves

Pre-Lecture

Section 13.3: Arc Length

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

Our next goal is to be able to measure distance traveled or arc length.

Motivating problem: Suppose the position of a fly at time t is

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle,$$

where $0 \leq t \leq 2\pi$.

How far does the fly travel from $t = 0$ to $t = \pi$?

Definition 19. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ from __________ to __________ that is traced out exactly once is

$$L = \underline{\hspace{6cm}}$$

Day 4 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **G2: Calculus of Curves.** I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.
- **G3: Geometry of Curves.** I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.

Goals for Today:

Sections 13.3, 13.4

- Solve initial value problems
- Compute arc lengths of curves using parameterizations
- Define and compute arc-length parameterizations

Example 20 (Poll)**. Warmup.** Compute the tangent line to

$$\mathbf{r}(t) = \langle 3t^3, \sin(t), t^2 + 1 \rangle, t \in \mathbb{R}$$

at the point $(0, 0, 1)$.

Working componentwise also works with integrals:

Example 21. Find $\int_0^1 \langle t, e^{2t}, \sec^2(t) \rangle \, dt$.

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 22. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t} \rangle \ m/s.$$

l

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, use calculus to reconstruct his flight path.

l

Example 23. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(2, 4, 8)$.

Example 24. Find the distance traveled by a particle moving along the path

$$\mathbf{r}(t) = \langle \ln(t), \sqrt{2}t, \frac{1}{2}t^2 \rangle, \qquad t > 0$$

from $t = 1$ to $t = 2$.

Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

$$s(t) = \underline{\hspace{6cm}}$$

We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.

Example 25. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \leq t \leq 2\pi$.

Day 5 - Geometry of Curves Part II

Pre-Lecture

Section 13.4 - Curvature

The next idea we are going to explore is the curvature of a curve in space along with two vectors that orient the curve.

First, we need the **unit tangent vector**, denoted $\mathbf{T}(s)$: ________________

This lets us define the **curvature**, $\kappa(s) =$ ________________

Question: In which direction is $\mathbf{T}$ changing?

This is the direction of the **principal unit normal**, $\mathbf{N}(s) =$ ________________

Day 5 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **G3: Geometry of Curves.** I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.

Goals for Today:

Section 13.4

- Define, interpret, and compute the curvature of a curve
- Compute the unit tangent and principal unit normal vectors of a curve

Example 26. Last time, we found an arc length parameterization of the circle of radius 4 centered at $(0, 0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \leq s \leq 8\pi.$$

Use this to find $\mathbf{T}(s)$, $\kappa(s)$, and $\mathbf{N}(s)$.

We said that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) =$ ____________________
- $\mathbf{N}(t) =$ ____________________
- $\kappa(t) =$ ____________________ or ______________________________

Example 27. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1 \rangle$.

Day 6 - Functions of Multiple Variables

Pre-Lecture

Section 14.1: Functions of Multiple Variables

Definition 28. A ______________________________________ is a rule that assigns to each ____________ of real numbers (x, y) in a set D a ____________________ denoted by $f(x, y)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^2$$

Example 29. Three examples are

$$f(x, y) = x^2 + y^2, \qquad g(x, y) = \ln(x + y), \qquad h(x, y) = h(x, y) = \sqrt{4 - x^2 - y^2}.$$

Example 30. Find the largest possible domains of f, g, and h.

Day 6 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.

Goals for Today:

Section 14.1

- Give examples of functions of multiple variables
- Find the domain of functions of two variables
- Introduce and sketch traces and contours of functions of two variables
- Use technology to graph functions of two variables
- Find level surfaces of functions of three variables

In the pre-lecture video, we discussed the domains of the functions $f(x, y) = x^2 + y^2, g(x, y) = \ln(x + y)$, and $h(x, y) = \sqrt{4 - x^2 - y^2}$.

Definition 31. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Here are the graphs of the three functions above.

Example 32. Suppose a small hill has height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x, y). How could we draw a picture that represents the hill in 2D?

In 3D, it looks like this.

Definition 33. The ____________ (also called ____________________) of a function f of two variables are the curves with equations ________________, where k is a constant (in the range of f). A plot of ____________ for various values of z is a ____________________(or ____________________________).

Some common examples of these are:

-
-
-

Example 34. Use technology to create a contour diagram of $f(x, y) = x^2 - y^2$.

What do we notice about the contours?

Example 35. Student work: Use technology to create a contour diagram of $g(x, y) = \sqrt{16 - 4x^2 - y^2}$.

CalcPlot3D

What do you notice about your contours?

Definition 36. The ________________ of a surface are the curves of ________________________ of the surface with planes parallel to the ____________________________.

Example 37. Find the traces of the surface $z = x^2 - y^2$. Can you see these in the graph produced by CalcPlot3D?

Example 38. Use the graph of the portion of $z = f(x, y) = 4 - 2x - y^2$ in the first quadrant to identify and understand all of the traces and contours.

Definition 39. A ____________________________ is a rule that assigns to each __________ of real numbers (x, y, z) in a set D a ________________ denoted by $f(x, y, z)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 40. Describe the domain of the function $f(x, y, z) = \dfrac{1}{4 - x^2 - y^2 - z^2}$.

Example 41. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.