Worksheet # 11: Product and Quotient Rules

1. Show by way of example that, in general,

$$\frac{d}{dx}(f \cdot g) \neq \frac{df}{dx} \cdot \frac{dg}{dx}$$

and

$$\frac{d}{dx}\left(\frac{f}{g}\right) \neq \frac{\frac{df}{dx}}{\frac{dg}{dx}}.$$

2. (a) If $f'(x) = g'(x)$ for all x, then does $f = g$? Explain your answer.
   (b) If $f(x) = g(x)$ for all x, then does $f' = g'$? Explain your answer.
   (c) How is the number e defined?
   (d) Are differentiable functions also continuous? Are continuous functions also differentiable? Provide several concrete examples to support your answers.

3. Calculate the derivatives of the following functions in the two ways that are described.
   (a) $f(r) = r^3/3$
      i. using the constant multiple rule and the power rule
      ii. using the quotient rule and the power rule

   Which method should we prefer?
   (b) $f(x) = x^5$
      i. using the power rule
      ii. using the product rule by considering the function as $f(x) = x^2 \cdot x^3$
   (c) $g(x) = (x^2 + 1)(x^4 - 1)$
      i. first multiply out the factors and then use the power rule
      ii. by using the product rule

4. State the quotient and product rule and be sure to include all necessary hypotheses.

5. Compute the first derivative of each of the following:
   (a) $f(x) = (3x^2 + x)e^x$
   (b) $f(x) = \dfrac{\sqrt{x}}{x-1}$
   (c) $f(x) = \dfrac{e^x}{2x^3}$
   (d) $f(x) = (x^3 + 2x + e^x)\left(\dfrac{x-1}{\sqrt{x}}\right)$
   (e) $f(x) = \dfrac{2x}{4+x^2}$
   (f) $f(x) = \dfrac{ax+b}{cx+d}$
   (g) $f(x) = \dfrac{(x^2+1)(x^3+2)}{x^5}$
   (h) $f(x) = (x-3)(2x+1)(x+5)$

6. Find an equation of the tangent line to the given curve at the specified point. Sketch the curve and the tangent line to check your answer.
   (a) $y = x^2 + \dfrac{e^x}{x^2+1}$ at the point $x = 3$.
   (b) $y = 2xe^x$ at the point $x = 0$.

7. Let $f(x) = (3x-1)e^x$. For which x is the slope of the tangent line to f positive? Negative? Zero?

8. Suppose that $f(2) = 3,\ g(2) = 2,\ f'(2) = -2$, and $g'(2) = 4$. For the following functions, find $h'(2)$.

   (a) $h(x) = 5f(x) + 2g(x)$

   (b) $h(x) = f(x)g(x)$

   (c) $h(x) = \dfrac{f(x)}{g(x)}$

   (d) $h(x) = \dfrac{g(x)}{1+f(x)}$

9. Calculate the first three derivatives of $f(x) = xe^x$ and use these to guess a general formula for $f^{(n)}(x)$, the n-th derivative of f.

10. Is there a formula for the derivative of $f \cdot g \cdot h$? What about $f \cdot g \cdot h \cdot k$? What about a product of five functions? Of six functions?

Supplemental Worksheet # 11

11. Compute the first derivative of each of the following:

    (a) $f(x) = (ax+b)(cx+d)$

    (b) $g(x) = \frac{e^x(2x+7)}{x+1}$

    (c) $h(x) = (e^x + x^2 + 7)(5x + \sqrt{x} + \pi)$

    (d) $k(x) = \frac{x^2+3x+6}{e^x}$

12. (Review) Suppose that $f(x) = \sqrt{ax}$ where a is a constant. Use the limit definition of the derivative to calculate $f'(x)$.

13. (Challenge) A number c is a root of a polynomial $f(x)$ if and only if $f(x) = (x-c)g(x)$ for some other polynomial $g(x)$. We say that c is a multiple root of $f(x)$ if $f(x) = (x-c)^2h(x)$ for some polynomial $h(x)$.

    (a) Show that if c is a multiple root of $f(x)$, then c is also a root of $f'(x)$.

    (b) Show that if c is a root of both $f(x)$ and $f'(x)$, then c is a multiple root of $f(x)$.