Worksheet # 1: Precalculus review: functions and inverse functions

1. Find the domain and range of $f(x) = \frac{x+1}{x^2+x-2}$.

2. For each of the following conditions, find the equation of the line that satisfies those conditions.

   (a) the line passes through the point $(1, 3)$ with slope 13.

   (b) the line passes through the points (π, π) and $(-8, -4)$.

   (c) the line has y-intercept 3 and has slope -2.

3. Let f be a linear function with slope m where $m \neq 0$. What is the slope of the inverse function f^{-1}? Why is your answer correct?

4. If $f(x) = 5x+7$ and $g(x) = x^2$, find $f \circ g$ and $g \circ f$. Are the functions $f \circ g$ and $g \circ f$ the same function? Why or why not?

5. Let $f(x) = 2^{\cos(3(x+1)^2+9)} - 7$.

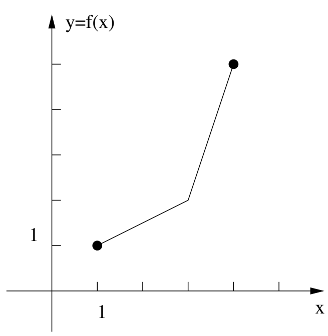
   (a) Can you find functions g and h such that $f = g \circ h$?

   (b) Can you find functions g, h, and s such that $f = g \circ h \circ s$?

   (c) Can you do this with four functions? Five functions? What is the largest number of functions you can find so that f can be written as a composition of those functions?

6. Let $f(x) = 2 + \frac{1}{x+3}$. Determine the inverse function of f, which we write as f^{-1}. Give the domain and range of f and the inverse function f^{-1}. Verify that $f \circ f^{-1}(x) = x$.

7. Consider the function whose graph appears below.



   (a) Find $f(3)$, $f^{-1}(2)$ and $f^{-1}(f(2))$.

   (b) Give the domain and range of f and of f^{-1}.

   (c) Sketch the graph of f^{-1}.

8. A ball is thrown in the air from ground level. The height of the ball in meters at time t seconds is given by the function $h(t) = -4.9t^2 + 30t$. At what time does the ball hit the ground? (Be sure to use the proper units!)

9. True or False: (justify your answer!)

   (a) Every function has an inverse.

   (b) If $f \circ g(x) = x$ for all x in the domain of g, then f is the inverse of g.

   (c) If $f \circ g(x) = x$ for all x in the domain of g and $g \circ f(x) = x$ for all x in the domain of f, then f is the inverse of g.

   (d) The function $f(x) = \sin(x)$ is one to one.

   (e) The function $f(x) = 1/(x+2)^3$ is one to one.

10. We form a box by removing squares of side length x centimeters from the four corners of a rectangle of width 100 cm and length 150 cm and then folding up the flaps between the squares that were removed. a) Write a function which gives the volume of the box as a function of x. b) Give the domain for this function.

11. Create a function that is the composition of ten functions. Can you do this in a "sneaky" way so that it is hard for someone else to figure out the ten functions you used? (Hint: try using different compositions of $g(x) = x + 1$ and $h(x) = 2x + 3$, for example $f = g \circ h \circ g \circ h \circ h$. What happens?)

Worksheet # 2: The Exponential Function and the Logarithm

1. Many students find statements like $2^0 = 1$ and $2^{1/3} = \sqrt[3]{2}$ a bit mysterious, even though most of us have used them for years, so let's start there. Write down the list of numbers $2^1 = 2$, $2^2 = 2 \times 2 = 4$, $2^3 = 2 \times 2 \times 2 = 8$, thus
$$2^1, 2^2, 2^3, 2^4, 2^5, \ldots .$$
   (a) What do you multiply by to get from a number on this list to the next number to the right? Starting from any number *except* 2^1, what do you divide by to get from that number to the previous number on the left?
   (b) If we start at 2^1 and move to the left following this patten, it suggests how we should define 2^0. What do you get for 2^0 if you follow the pattern?
   (c) If we now move from 2^0 another number to the left following the pattern, it suggests how we should define 2^{-1}, and then 2^{-2}, etc. What do you get for these values if you follow the pattern?
   (d) Do these patterns help you make sense of the rule $2^{a+b} = 2^a \times 2^b$? Discuss this with the students in your group. (Bonus question: discuss whether or not these patterns and this rule help us make sense of the equations $2^{1/2} = \sqrt{2}$ and $2^{1/3} = \sqrt[3]{2}$.)
   (e) Does it matter for your reasoning that the base of the exponential was the number 2? Why or why not?

2. Find the solutions to the following computational problems by using properties of exponentials and logarithms.
   (a) Solve $10^{2x+1} = 100$.
   (b) Solve $2^{(x^2)} = 16$.
   (c) Solve $2^x = 4^{x+2}$.
   (d) Find $\log_2(8)$.
   (e) Find $\ln(e^2)$.
   (f) Solve $e^{3x} = 3$.
   (g) Solve $4^x = e$.
   (h) Solve $\ln(x+1) + \ln(x-1) = \ln(3)$. Be sure to check your answer.

3. (a) Graph the functions $f(x) = 2^x$ and $g(x) = 2^{-x}$ and give the domains and range of each function.
   (b) Determine if each function is one-to-one. Determine if each function is increasing or decreasing.
   (c) Graph the inverse function to f. Give the domain and range of the inverse function.

4. Since e^x and $\ln(x)$ are inverse functions, we can write $\heartsuit = e^{\ln(\heartsuit)}$ if the value of $\heartsuit$ is positive. This is super useful when dealing with exponential functions with complicated bases, because then you can use log laws to simplify the exponent on e. But, you have to be careful when you apply this rule, as the following examples show.
   (a) Explain why $b = e^{\ln(b)}$ is only true when $b > 0$. (Hint: think about the domain of natural log.)
   (b) Explain why for any $b > 0$, we have $b^a = e^{a \ln(b)}$.
   (c) Explain why $(\cos(x) + 3)^{\sin(x)+2} = e^{(\sin(x)+2)\ln(\cos(x)+3)}$ is true.
   (d) Explain why $\cos(x)^{\sin(x)} = e^{\sin(x)\ln(\cos(x))}$ is not true.
   (e) Let f be the function $f(x) = 4^x$. Find the value of k that allows you to write the function f in the form $f(x) = e^{kx}$.
   (f) Let f be the function $f(x) = 5 \cdot 3^x$. Find a k that allows you to write the function f in the form Ae^{kx}.

5. Evaluate the expressions $4^{(3^2)}$ and $(4^3)^2$. Are they equal?

6. Suppose a and b are positive real numbers and $\ln(ab) = 3$ and $\ln(ab^2) = 5$. Find $\ln(a)$, $\ln(b)$, and $\ln(a^3/\sqrt{b})$.

7. Suppose that a population doubles every two hours. If we have one hundred critters at 12 noon, how many will there be after 1 hour? after 2 hours? How many were there at 11am? Give a formula for the number of critters at t hours after 12 noon.

8. Suppose that f is a function of the form $f(x) = Ae^{kx}$. If $f(2) = 20$ and $f(5) = 10$, will we have $k > 0$ or $k < 0$? Find A and k so that $f(2) = 20$ and $f(5) = 10$.

9. The number e is mysterious and arises in many different ways.

   (a) Use your calculator to compute $(1 + \frac{1}{n})^n$ for n equal to $1, 2, 3, \ldots$. Compute this for larger and larger n until it does not make a difference in the decimal you get. What is the value you reach?

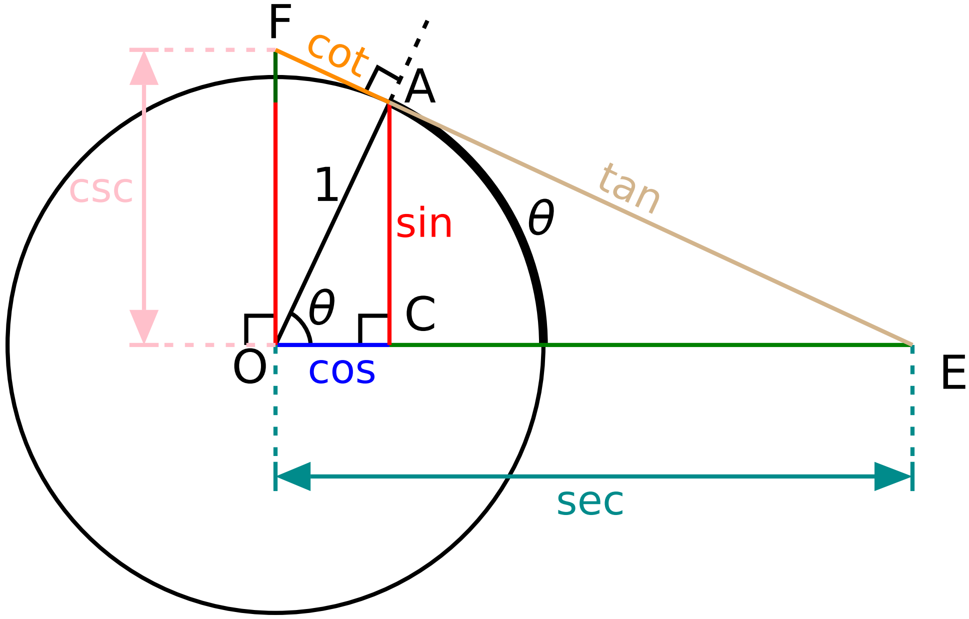
   (b) Use your calculator to compute the sums

$$\frac{1}{1} + \frac{1}{1 \cdot 2},$$
$$\frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3},$$
$$\frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4},$$

   and so on, until it no longer makes a difference in the decimal you get. What is the value you reach?

   (c) The first two problems showed you two ways to approximate the value of e. The first way involved limits, which we will talk about soon in this course. The second involved "infinite series," which is a topic covered in Calculus II. As we will see later in this course, there is another way to define e — we will see that the area of the region enclosed by the line $x = 1$, the x-axis, the graph of $y = 1/x$, and the line $x = a$ is equal to $\ln(a)$. So, the value of a for which this area is 1 is $a = e$. Amazing!

Worksheet # 3: Review of Trigonometry



1. The key to understanding trig functions is to understand the unit circle — given an angle θ between 0 and $\pi/2$ (measured in radians!), each of the six trig functions measures a length related to the unit circle.

   (a) The word *radian* is an abbreviation of the phrase "radial angle." In a circle of radius r, one radian is defined to be the angle given by an arc of the circle having length r. Draw the unit circle in the plane and for each $m = 1, 2, 3, 4, 5, 6$, place a dot at the *approximate* point on the circle that is m radians counterclockwise from the point $(1, 0)$ (you won't be able to measure this precisely, just estimate it as best you can).

   (b) Recall that the definition of π is the ratio of the circumference to the diameter in a circle. Use your picture from the previous problem to explain why the value of π is greater than 3 but less than 3.5.

   (c) Define the functions $\sin(\theta)$ and $\cos(\theta)$ to be the lengths of the arcs AC and OC, respectively, on the diagram above. Explain why this definition of $\sin(\theta)$ and $\cos(\theta)$ agrees with the usual triangle-based definitions, i.e. that $\sin(\theta)$ is equal to "opposite" over "hypotenuse" in a right triangle.

   (d) Use different pairs of similar triangles to explain why each of the functions $\tan(\theta)$, $\cot(\theta)$, $\csc(\theta)$, and $\sec(\theta)$ measure the correspondingly labeled length in the picture above.

   (e) Explain why $\sin^2(\theta)+\cos^2(\theta) = 1$ is equivalent to the Pythagorean theorem applied to the triangle OAC above, and why $1 + \tan^2(\theta) = \sec^2(\theta)$ is equivalent to the Pythagorean theorem applied to the triangle OAE above.

   (f) For the unit circle, the radial angle of θ corresponds to an arc of length θ. Inverse trig functions are sometimes written $\arcsin(x)$ and $\arccos(x)$. Discuss with the other students in your group why it makes sense that the function $\sin(\theta)$ gives the length of the vertical line AC in the diagram above, while the function $\arcsin(x)$ is equal to the length of the circular arc for which the vertical line AC has length x; conclude that if $\sin(\theta) = x$, then $\arcsin(x) = \theta$. Discuss arccos similarly.

2. When θ is not between 0 and $\pi/2$, then we extend the definition of the trig functions as you have seen in your previous courses, allowing us to answer the following questions.

   (a) Suppose that $\sin(\theta) = 5/13$ and $\cos(\theta) = -12/13$. Find the values of $\tan(\theta)$, $\cot(\theta)$, $\csc(\theta)$, $\sec(\theta)$, and $\tan(2\theta)$.

   (b) If $\pi/2 \leq \theta \leq 3\pi/2$ and $\tan\theta = 4/3$, find $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

   (c) Find all solutions of the equations (a) $\sin(x) = -\sqrt{3}/2$ and (b) $\tan(x) = 1$.

   (d) A ladder that is 6 meters long leans against a wall so that the bottom of the ladder is 2 meters from the base of the wall. Make a sketch illustrating the given information and answer the following questions: How high on the wall is the top of the ladder located? What angle does the top of the ladder form with the wall?

3. Let O be the center of a circle whose circumference is 48 centimeters. Let P and Q be two points on the circle that are endpoints of an arc that is 6 centimeters long. Find the angle between the segments OQ and OP. Express your answer in radians.

   Find the distance between P and Q.

4. Show that $\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$.

5. Find the exact values of the following expressions. Do not use a calculator.

   (a) $\tan^{-1}(1)$

   (b) $\tan(\tan^{-1}(10))$

   (c) $\sin^{-1}(\sin(7\pi/3))$

   (d) $\tan(\sin^{-1}(0.8))$

6. Find all solutions to the following equations in the interval $[0, 2\pi]$. You will need to use some trigonometric identities.

   (a) $\sqrt{3}\cos(x) + 2\tan(x)\cos^2(x) = 0$

   (b) $3\cot^2(x) = 1$

   (c) $2\cos(x) + \sin(2x) = 0$

Worksheet # 4: Average and Instantaneous Velocity

1. A ball is thrown vertically into the air from ground level with an initial velocity of 15 m/s. Its height at time t is $h(t) = 15t - 4.9t^2$.

   (a) How far does the ball travel during the time interval $[1, 3]$?

   (b) Compute the ball's average velocity over the time interval $[1, 3]$.

   (c) Graph the curve $y = h(t)$ and the line between the points $(1, h(1))$ and $(3, h(3))$. How does the slope of this line relate to your answer in part (b)?

   (d) Compute the ball's average velocity over the time intervals $[1, 1.01]$, $[1, 1.001]$,$[0.99, 1]$, and $[0.999, 1]$.

   (e) Estimate the instantaneous velocity when $t = 1$.

2. A particle moves along a line and its position $p(t)$ in meters after time t seconds is given by the following table.

t	0.0	0.2	0.5	0.65	0.9	1.1	1.15	1.3
$p(t)$	3	4.2	5.7	8.8	7.6	8.0	9.0	9.5

   (a) Describe the motion of the particle between 0 seconds and 1.3 seconds. Justify your description by representing the table as points $(t, p(t))$ plotted in the plane.

   (b) Consider the average velocity of the particle across different time intervals. Based on your calculations, can you conclude at what point in time the particle is moving with the largest positive *instantaneous* velocity? Why or why not? Based on this data, when do you believe that the particle is likely to be moving with the largest positive instantaneous velocity? Why?

3. With the other members of your group, compare and contrast the first two problems on this worksheet.

   (a) Which of these two problems do you believe is most representative of the type of problem a scientist or engineer will encounter when analyzing a real-world experiment? Why?

   (b) Which of these two problems do you believe is most representative of the type of problem a scientist or engineer will encounter when setting up a mathematical model of a physical situation? Why?

4. Let $p(t) = t^3 - 45t$ denote the distance (in meters) to the right of the origin of a particle at time t minutes after noon.

   (a) Find the average velocity of the particle on the intervals $[2, 2.1]$ and $[2, 2.01]$.

   (b) Use this information to guess a value for the instantaneous velocity of particle at 12:02pm.

5. A particle moves along a line and its position after time t seconds is $p(t) = 3t^3 + 2t$ meters to the right of the origin. Approximate the instantaneous velocity of the particle at $t = 2$.

6. A particle is moving along a straight line so that its position at time t seconds is given by $s(t) = 4t^2 - t$ meters.

   (a) Find the average velocity of the particle over the time interval $[1, 2]$.

   (b) Determine the average velocity of the particle over the time interval $[2, t]$ where $t > 2$. Simplify your answer. [Hint: Factor the numerator.]

   (c) Based on your answer in (b) can you guess a value for the instantaneous velocity of the particle at $t = 2$?

Worksheet # 5: Limits: A Numerical and Graphical Approach

1. For each task or question below, provide a specific example of a function $f(x)$ that supports your answer.

   (a) In words, briefly describe what "$\lim_{x\to a} f(x) = L$" means.

   (b) In words, briefly describe what "$\lim_{x\to a} f(x) = \infty$" means.

   (c) Suppose $\lim_{x\to 1} f(x) = 2$. Does this imply $f(1) = 2$?

   (d) Suppose $f(1) = 2$. Does this imply $\lim_{x\to 1} f(x) = 2$?

2. Compute the value of the following functions near the given $x-$value. Use this information to guess the value of the limit of the function (if it exists) as x approaches the given value.

   (a) $f(x) = 2^{x-1} + 3$, $x = 1$

   (b) $f(x) = \frac{\sin(2x)}{x}$, $x = 0$

   (c) $f(x) = \sin\left(\frac{\pi}{x}\right)$, $x = 0$

   (d) $f(x) = \frac{x^2-3x+2}{x^2+x-6}$, $x = 2$

3. Let $f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ x - 1 & \text{if } 0 < x \text{ and } x \ne 2 \\ -3 & \text{if } x = 2 \end{cases}$.

   (a) Sketch the graph of f.

   (b) Compute the following:

      i. $\lim_{x\to 0^-} f(x)$

      ii. $\lim_{x\to 0^+} f(x)$

      iii. $\lim_{x\to 0} f(x)$

      iv. $f(0)$

      v. $\lim_{x\to 2^-} f(x)$

      vi. $\lim_{x\to 2^+} f(x)$

      vii. $\lim_{x\to 2} f(x)$

      viii. $f(2)$

4. In the following, sketch the functions and use the sketch to compute the limit.

   (a) $\lim_{x\to \pi} x$

   (b) $\lim_{x\to 3} \pi$

   (c) $\lim_{x\to a} |x|$

   (d) $\lim_{x\to 3} 2^x$

5. Show that $\lim_{h\to 0} \frac{|h|}{h}$ does not exist by examining one-sided limits. Then sketch the graph of $\frac{|h|}{h}$ to verify your reasoning.

6. Compute the following limits or explain why they fail to exist.

   (a) $\lim_{x\to -3^+} \frac{x+2}{x+3}$

   (b) $\lim_{x\to -3^-} \frac{x+2}{x+3}$

   (c) $\lim_{x\to -3} \frac{x+2}{x+3}$

   (d) $\lim_{x\to 0^-} \frac{1}{x^3}$

7. In the theory of relativity, the mass of a particle with velocity v is:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

   where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \to c^-$?

Worksheet # 6: Limit Laws and Continuity

Remark on Notation: When working through a limit problem, your answers should be a chain of true equalities. Make sure to keep the $\lim_{x\to a}$ operator until the very last step.

1. Given $\lim_{x\to 2} f(x) = 5$ and $\lim_{x\to 2} g(x) = 2$, use limit laws to compute the following limits or explain why we cannot find the limit.

   (a) $\lim_{x\to 2} f(x)^2 + x \cdot g(x)^2$

   (b) $\lim_{x\to 2} \dfrac{f(x)-5}{g(x)-2}$

   (c) $\lim_{x\to 2} \dfrac{f(x)g(x)}{x}$

   (d) $\lim_{x\to 2}(f(x)g(2))$

2. For each limit, evaluate the limit or or explain why it does not exist. Use the limit rules to justify each step. It is good practice to sketch a graph to check your answers.

   (a) $\lim_{x\to 2} \dfrac{x+2}{x^2-4}$

   (b) $\lim_{x\to 2} \left(\dfrac{1}{x-2} - \dfrac{3}{x^2-x-2}\right)$

   (c) $\lim_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$

   (d) $\lim_{x\to 0} \dfrac{(2+x)^3-8}{x}$

3. Let $f(x) = 1 + x^2 \sin\left(\dfrac{1}{x}\right)$ for $x \neq 0$. Consider $\lim_{x\to 0} f(x)$.

   (a) Find two simpler functions, g and h, that satisfy the hypothesis of the Squeeze Theorem.

   (b) Determine $\lim_{x\to 0} f(x)$ using the Squeeze Theorem.

   (c) Use a calculator to produce a graph that illustrates this application of the Squeeze Theorem.

4. For each of the following tasks/problems, provide a specific example of a function $f(x)$ that supports your answer.

   (a) State the definition of continuity.

   (b) List the three things required to show f is continuous at a.

   (c) What does it mean for $f(x)$ to be continuous on the interval $[a, b]$? What does it mean to say only that "$f(x)$ is continuous"?

   (d) Identify the three possible types of discontinuity of a function at a point. Provide a sketch of each type.

5. Show that the following functions are continuous at the given point a using problem 4b.

   (a) $f(x) = \pi$, $a = 1$

   (b) $f(x) = \dfrac{x^2+3x+1}{x+3}$, $a = -1$

   (c) $f(x) = \sqrt{x^2-9}$, $a = 4$

6. Give the intervals of continuity for the following functions.

   (a) $f(x) = \dfrac{x+1}{x^2+4x+3}$

   (b) $f(x) = \dfrac{x}{x^2+1}$

   (c) $f(x) = \sqrt{2x-3} + x^2$

(d) $f(x) = \begin{cases} x^2+1 & \text{if } x \le 0 \\ x+1 & \text{if } 0 < x < 2 \\ -(x-2)^2 & \text{if } x \ge 2 \end{cases}$

7. Let c be a number and consider the function $f(x) = \begin{cases} cx^2 - 5 & \text{if } x < 1 \\ 10 & \text{if } x = 1 \\ \dfrac{1}{x} - 2c & \text{if } x > 1 \end{cases}$.

   (a) Find all numbers c such that $\lim_{x \to 1} f(x)$ exists.

   (b) Is there a number c such that $f(x)$ is continuous at $x = 1$? Justify your answer.

8. Find parameters a and b so that the following function is continuous

$$f(x) = \begin{cases} 2x^2 + 3x & \text{if } x \le -4 \\ ax + b & \text{if } -4 < x < 3 \\ -x^3 + 4x^2 - 5 & \text{if } 3 \le x \end{cases}$$

9. Suppose that

$$f(x) = \begin{cases} \frac{x-6}{|x-6|} & \text{if } x \ne 6, \\ 1 & \text{if } x = 6 \end{cases}$$

   Determine the points at which the function $f(x)$ is discontinuous and state the type of discontinuity.

Worksheet # 7: Intermediate Value Theorem and Limits at Infinity

1. State the Intermediate Value Theorem. Show $f(x) = x^3 + x - 1$ has a zero in the interval $(0, 1)$.

2. Using the Intermediate Value Theorem, find an interval of length 1 in which a solution to the equation $2x^3 + x = 5$ must exist.

3. Let $f(x) = \dfrac{e^x}{e^x - 2}$.

   (a) Show that $f(0) < 1 < f(\ln(4))$.

   (b) Can you use the Intermediate Value Theorem to conclude that there is a solution of $f(x) = 1$?

   (c) Can you find a solution to $f(x) = 1$?

4. (a) Show that the equation $xe^x = 2$ has a solution in the interval $(0, 1)$.

   (b) Determine if the solution lies in the interval $(0, 1/2)$ or $(1/2, 1)$.

   (c) Continue in this manner to find an interval of length $1/8$ which contains a solution of the equation $xe^x = 2$.

5. Consider the following piecewise function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

   Although $f(-1) = 0$ and $f(1) = 1$, $f(x) \neq 1/2$ for all x in its domain. Why doesn't this contradict the Intermediate Value Theorem?

6. Describe the behavior of the function $f(x)$ if $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = M$.

7. Explain the difference between "$\lim_{x\to-3} f(x) = \infty$" and "$\lim_{x\to\infty} f(x) = -3$".

8. Evaluate the following limits, or explain why the limit does not exist:

   (a) $\displaystyle\lim_{x\to\infty} \frac{3x^2 - 7x}{x - 8}$

   (b) $\displaystyle\lim_{x\to\infty} \frac{2x^2 - 6}{x^4 - 8x + 9}$

   (c) $\displaystyle\lim_{x\to-\infty} \frac{x}{x^6 - 4x^2}$

   (d) $\displaystyle\lim_{x\to-\infty} 3$

   (e) $\displaystyle\lim_{x\to\pm\infty} \frac{5x^3 - 7x^2 + 9}{x^2 - 8x^3 - 8999}$

   (f) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^{10} + 2x}}{x^5}$

9. Find the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ if $f(x) = \left(\dfrac{x^2}{x+1} - \dfrac{x^2}{x-1}\right)$.

10. Sketch a graph with all of the following properties:

    - $\lim_{t\to\infty} f(t) = 2$
    - $\lim_{t\to-\infty} f(t) = 0$
    - $\lim_{t\to 0^+} f(t) = \infty$
    - $\lim_{t\to 0^-} f(t) = -\infty$
    - $\lim_{t\to 4} f(t) = 3$
    - $f(4) = 6$

Worksheet # 8: Review for Exam I

1. Find all real numbers of the constant a and b for which the function $f(x) = ax + b$ satisfies:
   (a) $f \circ f(x) = f(x)$ for all x.
   (b) $f \circ f(x) = x$ for all x.

2. Simplify the following expressions.
   (a) $\log_5 125$
   (b) $(\log_4 16)(\log_4 2)$
   (c) $\log_x(x(\log_y y^x))$
   (d) $\log_\pi(1 - \cos x) + \log_\pi(1 + \cos x) - 2\log_\pi \sin x$

3. Suppose that $\tan(x) = \frac{3}{4}$ and $-\pi < x < 0$. Find $\cos(x)$, $\sin(x)$, and $\sin(2x)$.

4. (a) Solve the equation $3^{2x+5} = 4$ for x. Show each step in the computation.
   (b) Express the quantity $\log_2(x^3 - 2) + \dfrac{1}{3}\log_2(x) - \log_2(5x)$ as a single logarithm. For which x is the resulting identity valid?

5. Suppose that the height of an object at time t is $h(t) = 5t^2 + 40t$.
   (a) Find the average velocity of the object on the interval $[3, 3.1]$.
   (b) Find the average velocity of the object on the interval $[a, a + h]$.
   (c) Find the instantaneous velocity of the object time a.

6. Calculate the following limits using the limit laws. Carefully show your work and use only one limit law per step.
   (a) $\lim_{x\to 0}(2x - 1)$
   (b) $\lim_{x\to 0} \dfrac{\sqrt{x+4} - 2}{x}$

7. Calculate the following limits if they exist or explain why the limit does not exist.
   (a) $\lim_{x\to 1} \dfrac{x - 2}{\frac{1}{x} - \frac{1}{2}}$
   (b) $\lim_{x\to 2} \dfrac{x - 2}{\frac{1}{x} - \frac{1}{2}}$
   (c) $\lim_{x\to 2^+} \dfrac{x^2 - 1}{x - 2}$
   (d) $\lim_{x\to a}(xa - a^2)$
   (e) $\lim_{x\to 4} \dfrac{\sqrt{x} - 2}{x^2 - 16}$
   (f) $\lim_{x\to 2} \dfrac{x + 1}{x - 2}$
   (g) $\lim_{x\to 2} \dfrac{x^3 - 8}{x - 2}$
   (h) $\lim_{a\to x}(xa - a^2)$

8. State the Squeeze Theorem. Use it to find the following limits.
   (a) $\lim_{x\to 0} x \sin \dfrac{1}{x^2}$
   (b) $\lim_{x\to \frac{\pi}{2}} \cos x \cos(\tan x)$

9. If $f(x) = \dfrac{|x - 3|}{x^2 - x - 6}$, find $\lim_{x\to 3^+} f(x)$, $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3} f(x)$.

10. If $\lim_{x\to 2} f(x) = 3$ and $\lim_{x\to 2} g(x) = 5$, find the following or explain why we cannot find the limit.

(a) $\lim_{x\to 2}(2f(x)+3g(x))$

(b) $\lim_{x\to 2} \dfrac{f(x)}{g(x)+1}$

(c) $\lim_{x\to 2} f(2)g(x)$

(d) $\lim_{x\to 2} \dfrac{x-2}{2f(x)-6}$

11. (a) State the definition of the continuity of a function $f(x)$ at $x = a$.

(b) Find the constant a so that the function is continuous on the entire real line.

$$f(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a} & \text{if } x \neq a \\ 8 & \text{if } x = a \end{cases}$$

12. If $g(x) = x^2 + 5^x - 3$, use the Intermediate Value Theorem to show that there is a number a such that $g(a) = 10$.

13. Complete the following statements:

(a) A function $f(x)$ passes the horizontal line test, if the function f is ....

(b) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{provided ....}$$

(c) $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$ if and only if ....

(d) Let $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ be a piecewise function.
The function $g(x)$ is NOT continuous at $x = 2$ since ....

(e) Let $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$ be a piecewise function.
The function $f(x)$ is NOT continuous at $x = 0$ since ....

Worksheet # 9: The Derivative, Velocities, and Tangent Lines

1. Comprehension check:

   (a) What is the definition of the derivative $f'(a)$ at a point a?

   (b) What is the geometric meaning of the derivative $f'(a)$ at a point a?

   (c) True or false: If $f(1) = g(1)$, then $f'(1) = g'(1)$?

2. Part of Problem #1 on Worksheet #4 is given below. Rewrite each of these questions as a problem about the graph of $h(t)$, secant lines to the graph of $h(t)$, and/or tangent lines to the graph of $h(t)$.

   A ball is thrown vertically into the air from ground level with an initial velocity of 15 m/s. Its height at time t is $h(t) = 15t - 4.9t^2$.

   (a) How far does the ball travel during the time interval $[1, 3]$?

   (b) Compute the ball's average velocity over the time interval $[1, 3]$.

   (c) Compute the ball's average velocity over the time intervals $[1, 1.01]$, $[1, 1.001]$,$[0.99, 1]$, and $[0.999, 1]$.

   (d) Estimate the instantaneous velocity when $t = 1$.

3. (a) Find a function f and a number a so that the following limit represents a derivative $f'(a)$.

$$\lim_{h\to 0} \frac{(4+h)^3 - 64}{h}$$

   (b) Using your function f, set $h = -2$ and draw the graph of f and the secant line whose slope is given by $\frac{(4+h)^3-64}{h}$.

   (c) Create a real-world scenario that is modeled by f, and write a problem about this scenario for which the answer is given by $\lim_{h\to 0} \dfrac{(4+h)^3 - 64}{h}$.

4. Let $f(x) = |x|$. Find $f'(1)$, $f'(0)$ and $f'(-1)$ or explain why the derivative does not exist.

5. The point $P = (3, 1)$ lies on the curve $y = \sqrt{x-2}$.

   (a) If Q is the point $(x, \sqrt{x-2})$, find a formula for the slope of the secant line PQ.

   (b) Using your formula from part (a) and a calculator, find the slope of the secant line PQ for the following values of x (do not round until you get to the final answer):

   2.9, 2.99, 2.999, 3.1, 3.01, and 3.001

   TI-8x Calculator Tip: Enter the formula under "y=" and then use "Table".

   (c) Using the results of part (b), guess the value of the slope of the tangent line to the curve at $P = (3, 1)$.

   (d) Verify that your guess is correct by computing an appropriate derivative.

   (e) Using the slope from part (d), find the equation of the tangent line to the curve at $P = (3, 1)$.

6. Let

$$g(t) = \begin{cases} at^2 + bt + c & \text{if } t \le 0 \\ t^2 + 1 & \text{if } t > 0 \end{cases}.$$

   Find all values of a, b, and c so that g is differentiable at $t = 0$.

7. Let $f(x) = e^x$ and estimate the derivative $f'(0)$ by considering difference quotients $(f(h) - f(0))/h$ for small values of h.

8. Suppose that $f'(0)$ exists. Does the limit

$$\lim_{h \to 0} \frac{f(h) - f(-h)}{h}$$

exist? Can you express the limit in terms of $f'(0)$?

9. Can you find a function f so that

$$\lim_{h \to 0} \frac{f(h) - f(-h)}{h}$$

exists, but f is not differentiable at 0?

10. Find A and B so that the limit

$$\lim_{x \to 1} \frac{x^2 + 2x - (Ax + B)}{(x - 2)^2}$$

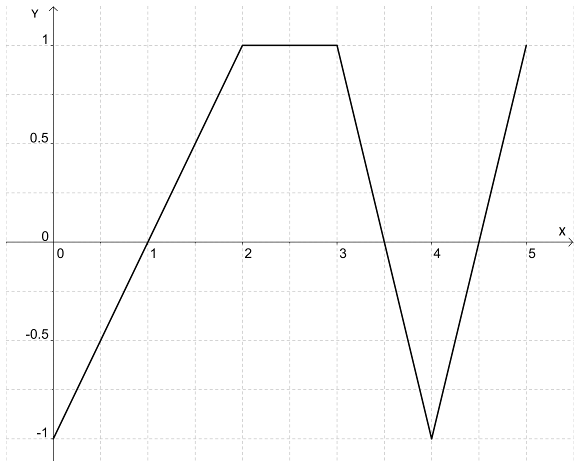
is finite. Give the value of the limit.

11. Find the specified derivative for each of the following using the limit definition of derivative.

    (a) If $f(x) = 1/x$, find $f'(2)$.
    (b) If $g(x) = \sqrt{x}$, find $g'(2)$.
    (c) If $h(x) = x^2$, find $h'(s)$.
    (d) If $f(x) = x^3$, find $f'(-2)$.
    (e) If $g(x) = 1/(2 - x)$, find $g'(t)$.

Worksheet # 10: The Derivative as a Function, Polynomials and Exponentials

1. Consider the graph below of the function $f(x)$ on the interval $[0, 5]$.



   (a) For which x values would the derivative $f'(x)$ not be defined?

   (b) Sketch the graph of the derivative function f'.

2. Water temperature affects the growth rate of brook trout. The table shows the amount of weight gained by brook trout after 24 days in various water temperatures.

Temperature (Celsius)	15.5	17.7	20.0	22.4	24.4
Weight gained (grams)	37.2	31.0	19.8	9.7	-9.8

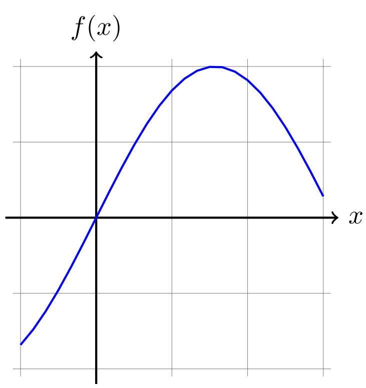
   (a) If $W(x)$ is the weight gain at temperature x, construct a table of estimated values for W'.

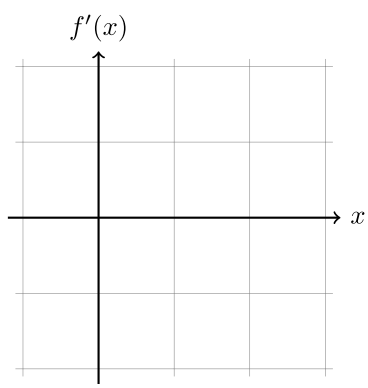
   (b) Plot the points for $W(x)$ and $W'(x)$ in the $x-y$-plane. Sketch one possible version of the graph for W, and use this to sketch a possible version of the graph for W'.

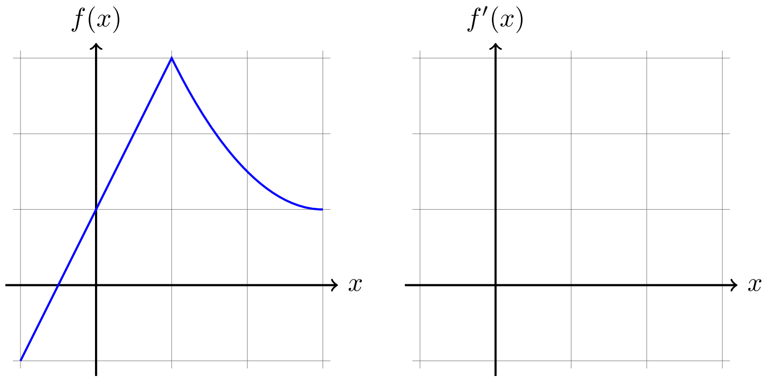
   (c) What are the units for W'?

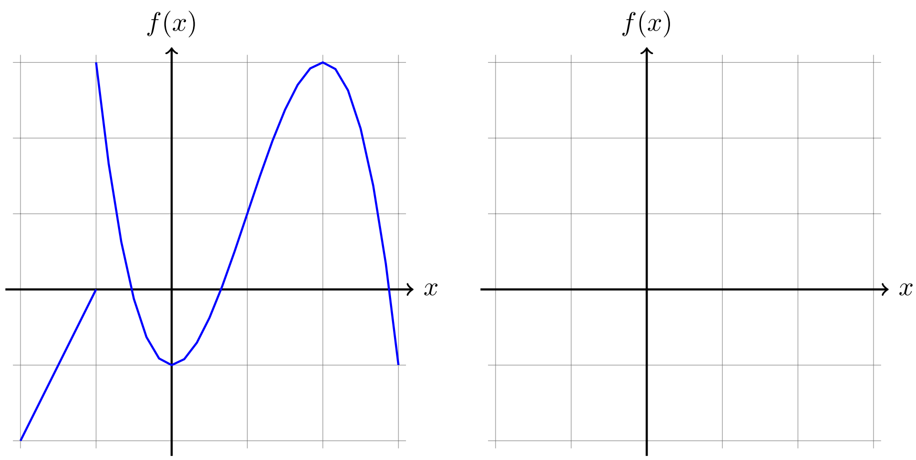
3. For each function f whose graph is given below, identify points where $f'(x)$ does not exist and sketch the graph of f'.

(a)





(b) 

(c) 

4. Compute the derivative of the following functions using both the limit definition and the rules for polynomials and exponentials.

   (a) $f(x) = 4 + 8x - 10x^3$

   (b) $g(x) = -7x^2 + x - 2$

5. Suppose N is the number of people in the United States who travel by car to another state for a vacation this year when the average price of gasoline is p dollars per gallon. Do you expect dN/dp to be positive or negative? Explain your answer. What about d^2N/dp^2?

6. Find a formula for the n-th derivative of x^n.

7. Find the first, second, and third derivatives of the following functions using the rules for polynomials and exponentials.

   (a) $f(x) = 3^{30}$

   (b) $g(t) = (t + 1)(t + 2)(t + 3)$

   (c) $h(a) = \dfrac{\sqrt{a} + a}{a^3}$

   (d) $y = e^{x+2} + 1$

   (e) $F(x) = \dfrac{2}{x^3} + 3e^x - x^7$

Worksheet # 11: Product and Quotient Rules

1. Show by way of example that, in general,

$$\frac{d}{dx}(f \cdot g) \neq \frac{df}{dx} \cdot \frac{dg}{dx}$$

and

$$\frac{d}{dx}\left(\frac{f}{g}\right) \neq \frac{\frac{df}{dx}}{\frac{dg}{dx}}.$$

2. (a) If $f'(x) = g'(x)$ for all x, then does $f = g$? Explain your answer.
   (b) If $f(x) = g(x)$ for all x, then does $f' = g'$? Explain your answer.
   (c) How is the number e defined?
   (d) Are differentiable functions also continuous? Are continuous functions also differentiable? Provide several concrete examples to support your answers.

3. Calculate the derivatives of the following functions in the two ways that are described.
   (a) $f(r) = r^3/3$
      i. using the constant multiple rule and the power rule
      ii. using the quotient rule and the power rule

   Which method should we prefer?
   (b) $f(x) = x^5$
      i. using the power rule
      ii. using the product rule by considering the function as $f(x) = x^2 \cdot x^3$
   (c) $g(x) = (x^2 + 1)(x^4 - 1)$
      i. first multiply out the factors and then use the power rule
      ii. by using the product rule

4. State the quotient and product rule and be sure to include all necessary hypotheses.

5. Compute the first derivative of each of the following:
   (a) $f(x) = (3x^2 + x)e^x$
   (b) $f(x) = \dfrac{\sqrt{x}}{x-1}$
   (c) $f(x) = \dfrac{e^x}{2x^3}$
   (d) $f(x) = (x^3 + 2x + e^x)\left(\dfrac{x-1}{\sqrt{x}}\right)$
   (e) $f(x) = \dfrac{2x}{4 + x^2}$
   (f) $f(x) = \dfrac{ax + b}{cx + d}$
   (g) $f(x) = \dfrac{(x^2 + 1)(x^3 + 2)}{x^5}$
   (h) $f(x) = (x - 3)(2x + 1)(x + 5)$

6. Find an equation of the tangent line to the given curve at the specified point. Sketch the curve and the tangent line to check your answer.
   (a) $y = x^2 + \dfrac{e^x}{x^2 + 1}$ at the point $x = 3$.
   (b) $y = 2xe^x$ at the point $x = 0$.

7. Let $f(x) = (3x-1)e^x$. For which x is the slope of the tangent line to f positive? Negative? Zero?

8. Suppose that $f(2) = 3,\ g(2) = 2,\ f'(2) = -2$, and $g'(2) = 4$. For the following functions, find $h'(2)$.

   (a) $h(x) = 5f(x) + 2g(x)$

   (b) $h(x) = f(x)g(x)$

   (c) $h(x) = \dfrac{f(x)}{g(x)}$

   (d) $h(x) = \dfrac{g(x)}{1+f(x)}$

9. Calculate the first three derivatives of $f(x) = xe^x$ and use these to guess a general formula for $f^{(n)}(x)$, the n-th derivative of f.

10. Is there a formula for the derivative of $f \cdot g \cdot h$? What about $f \cdot g \cdot h \cdot k$? What about a product of five functions? Of six functions?

Worksheet # 12: Derivatives of Trigonometric Functions and the Chain Rule

1. For each of these problems, explain why it is true or give an example showing it is false.
   (a) True or False: If $f'(\theta) = -\sin(\theta)$, then $f(\theta) = \cos(\theta)$.
   (b) True or False: If θ is one of the non-right angles in a right triangle and $\sin(\theta) = \dfrac{2}{3}$, then the hypotenuse of the triangle must have length 3.
2. Calculate the first five derivatives of $f(x) = \sin(x)$. Then determine $f^{(8)}$ and $f^{(37)}$
3. Let $f(t) = t + 2\cos(t)$.
   (a) Find all values of t where the tangent line to f at the point $(t, f(t))$ is horizontal.
   (b) What are the largest and smallest values for the slope of a tangent line to the graph of f?
4. (a) Carefully state the chain rule using complete sentences.
   (b) Suppose f and g are differentiable functions so that $f(2) = 3$, $f'(2) = -1$, $g(2) = \dfrac{1}{4}$, and $g'(2) = 2$. Find each of the following:
      i. $h'(2)$ where $h(x) = \sqrt{[f(x)]^2 + 7}$.
      ii. $l'(2)$ where $l(x) = f(x^3 \cdot g(x))$.
5. Differentiate both sides of the double-angle formula for the cosine function, $\cos(2x) = \cos^2(x) - \sin^2(x)$. Do you obtain a familiar identity?
6. Differentiate each of the following and simplify your answer.
   (a) $r(\theta) = \theta^3 \sin(\theta)$
   (b) $s(t) = \tan(t) + \csc(t)$
   (c) $h(x) = \sin(x)\csc(x)$
   (d) $g(x) = \sec(x) + \cot(x)$
   (e) $f(x) = \sqrt[3]{2x^3 + 7x + 3}$
   (f) $g(t) = \tan(\sin(t))$
   (g) $h(u) = \sec^2(u) + \tan^2(u)$
   (h) $f(x) = xe^{(3x^2+x)}$
   (i) $g(x) = \sin(\sin(\sin(x)))$
7. Given the following functions: $f(x) = \sec(x)$, and $g(x) = x^3 - 2x + 1$. Find:
   (a) $f(g(x))$
   (b) $f'(x)$
   (c) $g'(x)$
   (d) $f'(g(x))$
   (e) $(f \circ g)'(x)$
8. Find an equation of the tangent line to the curve at the given point.
   (a) $f(x) = x^2 e^{3x}$, $x = 2$
   (b) $f(x) = \sin(x) + \sin^2(x)$, $x = 0$
9. Compute the derivative of $\frac{x}{x^2+1}$ in two ways:
   (a) Using the quotient rule.
   (b) Rewrite the function $\frac{x}{x^2+1} = x(x^2+1)^{-1}$ and use the product and chain rule.

   Check that both answers give the same result.

10. Suppose that $k(x) = \sqrt{\sin^2(x) + 4}$. Find three functions f, g, and h so that $k(x) = f(g(h(x)))$.

11. Let $h(x) = f \circ g(x)$ and $k(x) = g \circ f(x)$ where some values of f and g are given by the table

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	4	4	-1	-1
2	3	4	3	-1
3	-1	-1	3	-1
4	3	2	2	-1

Find: $h'(-1), h'(3)$ and $k'(2)$.

12. Find all x values so that $f(x) = 2\sin(x) + \sin^2(x)$ has a horizontal tangent at x.

13. Suppose that at the instant when the radius of a circle of a circle is 5 cm, the radius is decreasing at a rate of 3 cm/sec. Find the rate of change of the area of the circle when the the radius is 5 cm.

14. A particle's distance from the origin (in meters) along the x-axis is modeled by $p(t) = 2\sin(t) - \cos(t)$, where t is measured in seconds.

    (a) Determine the particle's speed (speed is defined as the absolute value of velocity) at π seconds.

    (b) Is the particle moving towards or away from the origin at π seconds? Explain.

    (c) Now, find the velocity of the particle at time $t = \dfrac{3\pi}{2}$. Is the particle moving toward the origin or away from the origin?

    (d) Is the particle increasing speed at $\frac{\pi}{2}$ seconds?

Worksheet # 13: Implicit Differentiation and Inverse Functions

1. Find the derivative of y with respect to x:

   (a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \pi^{\frac{2}{3}}$.

   (b) $e^y \sin(x) = x + xy$.

   (c) $\cos(xy) = 1 + \sin(y)$.

2. Consider the ellipse given by the equation $\dfrac{(x-2)^2}{25} + \dfrac{(y-3)^2}{81} = 1$.

   (a) Find the equation of the tangent line to the ellipse at the point (u, v) where $u = 4$ and $v > 0$.

   (b) Sketch the ellipse and the line to check your answer.

3. Find the derivative of $f(x) = \pi^{\tan^{-1}(\omega x)}$, where ω is a constant.

4. Let (a, b) be a point in the circle $x^2 + y^2 = 144$. Use implicit differentiation to find the slope of the tangent line to the circle at (a, b).

5. Let $f(x)$ be an invertible function such that $g(x) = f^{-1}(x)$, $f(3) = \sqrt{5}$ and $f'(3) = -\frac{1}{2}$. Using only this information find the equation of the tangent line to $g(x)$ at $x = \sqrt{5}$.

6. Let $y = f(x)$ be the unique function satisfying $\dfrac{1}{2x} + \dfrac{1}{3y} = 4$. Find the slope of the tangent line to $f(x)$ at the point $(\frac{1}{2}, \frac{1}{9})$.

7. The equation of the tangent line to $f(x)$ at the point $(2, f(2))$ is given by the equation $y = -3x + 9$. If $G(x) = \dfrac{x}{4f(x)}$, find $G'(2)$.

8. Differentiate both sides of the equation, $V = \dfrac{4}{3}\pi r^3$, with respect to V and find $\dfrac{dr}{dV}$ when $r = 8\sqrt{\pi}$.

9. Use implicit differentiation to find the derivative of $\arctan(x)$. Thus if $x = \tan(y)$, use implicit differentiation to compute dy/dx. Can you simplify to express dy/dx in terms of x?

10. (a) Compute $\dfrac{d}{dx} \arcsin(\cos(x))$.

    (b) Compute $\frac{d}{dx}(\arcsin(x) + \arccos(x))$. Give a geometric explanation as to why the answer is 0.

    (c) Compute $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{1}{x}\right) + \tan^{-1}(x)\right)$ and simplify to show that the derivative is 0. Give a geometric explanation of your result.

11. Consider the line through $(0, b)$ and $(2, 0)$. Let θ be the directed angle from the x-axis to this line so that $\theta > 0$ when $b < 0$. Find the derivative of θ with respect to b.

12. Let f be defined by $f(x) = e^{-x^2}$.

    (a) For which values of x is $f'(x) = 0$

    (b) For which values of x is $f''(x) = 0$

13. The notation $\tan^{-1}(x)$ is ambiguous. It is not clear if the exponent -1 indicates the reciprocal or the inverse function. If we allow both interpretations, how many different ways can you (correctly) compute the derivative $f'(x)$ for

    $$f(x) = (\tan^{-1})^{-1}(x)?$$

    In order to avoid this ambiguity, we will generally use $\cot(x)$ for the reciprocal of $\tan(x)$ and $\arctan(x)$ for the inverse of the tangent function restricted to the domain $(-\pi/2, \pi/2)$.

Worksheet # 14: Derivatives of Logarithms and Rates of Change

1. Find the derivative of $f(x) = 3^x$. Compute the derivative of $\log_3(x)$. Explain the relationship between your answers.

2. Find the derivatives of the following functions.

   (a) $f(x) = \sqrt{x}\ln(x)$

   (b) $g(x) = \dfrac{\ln(x)}{1 + \ln(x)}$

3. A particle moves along a line so that its position at time t is $p(t) = 3t^3 - 12t$ where $p(t)$ represents the distance to the right of the origin. Recall that *speed* is given by the absolute value of velocity.

   (a) Find the velocity and speed at time $t = 1$.

   (b) Find the acceleration at time $t = 1$.

   (c) Is the velocity increasing or decreasing when $t = 1$?

   (d) Is the speed increasing or decreasing when $t = 1$?

4. An object is thrown upward so that its height at time t seconds after being thrown is $h(t) = -4.9t^2 + 20t + 25$ meters. Give the position, velocity, and acceleration at time t.

5. An object is thrown upward with an initial velocity of 5 m/s from an initial height of 40 m. Find the velocity of the object when it hits the ground. Assume that the acceleration of gravity is 9.8 m/s^2.

6. An object is thrown upward so that it returns to the ground after 4 seconds. What is the initial velocity? Assume that the acceleration of gravity is 9.8 m/s^2.

7. Suppose that an object is shot into the air vertically with an initial velocity v_0 and initial height s_0, with acceleration due to gravity denoted by g. Let $s(t)$ denote the height of the object after t time units.

   (a) Explain why $s(t) = s_0 + v_0 t - \dfrac{1}{2}gt^2$ (this is sometimes called "Galileo's formula").

   (b) If time is measured in seconds and distance in meters, what are the units for s_0, v_0, and g?

8. Suppose that height of a triangle is equal to its base b. Find the instantaneous rate of change in the area respect to the base b when the base is 7.

9. The cost in dollars of producing x bicycles is $C(x) = 4000 + 210x - x^2/1000$.

   (a) Explain why $C'(40)$ is a good approximation for the cost of the 41st bicycle.

   (b) How can you use the values of $C(40)$ and $C'(40)$ to approximate the cost of 42 bicycles?

   (c) Explain why the model for $C(x)$ is not a good model for cost. What happens if x is very large?

10. Suppose that a population of bacteria triples every hour and starts with 400 bacteria.

    (a) Find an expression for the number n of bacteria after t hours.

    (b) Use this expression to estimate the rate of growth of the bacteria population after 2.5 hours.

11. Think about the other science courses you are currently taking (or have taken in the past). Identify three to five examples of problems from those courses that involve computing rates of change where methods from calculus might be useful.

Worksheet # 15: Related Rates

1. Let a and b denote the length in meters of the two legs of a right triangle. At time $t = 0$, $a = 20$ and $b = 20$. If a is decreasing at a constant rate of 2 meters per second and b is increasing at a constant rate of 3 meters per second. Find the rate of change of the area of the triangle at time $t = 5$ seconds.

2. A person 6 feet tall walks along a straight path at a rate of 4 feet per second away from a streetlight that is 15 feet above the ground. Assume that at time $t = 0$ the person is touching the streetlight.

   (a) Draw a picture to represent the situation.

   (b) Find an equation that relates the length of the person's shadow to the person's position (relative to the streetlight).

   (c) Find the rate of change in the length of the shadow when $t = 3$.

   (d) Find how fast is the tip of the person's shadow is moving when $t = 4$.

   (e) Does the precise time make a difference in these calculations?

3. A spherical snow ball is melting. The rate of change of the surface area of the snow ball is constant and equal to -7 square centimeters per hour. Find the rate of change of the radius of the snow ball when $r = 5$ centimeters.

4. The height of a cylinder is a linear function of its radius (i.e. $h = ar + b$ for some a, b constants). The height increases twice as fast as the radius r and $\frac{dr}{dt}$ is constant. At time $t = 1$ seconds the radius is $r = 1$ feet, the height is $h = 3$ feet and the rate of change of the volume is 16π cubic feet/second.

   (a) Find an equation to relate the height and radius of the cylinder.

   (b) Express the volume as a function of the radius.

   (c) Find the rate of change of the volume when the radius is 4 feet.

5. A water tank is shaped like a cone with the vertex pointing down. The height of the tank is 5 meters and diameter of the base is 2 meters. At time $t = 0$ the tank is full and starts to be emptied. After 3 minutes the height of the water is 4 meters and it is decreasing at a rate of 0.5 meters per minute. At this instant, find the rate of change of the volume of the water in the tank. What are the units for your answer? Recall that the volume of a right-circular cone whose base has radius r and of height h is given by $V = \frac{1}{3}\pi r^2 h$.

6. A plane flies at an altitude of 5000 meters and a speed of 360 kilometers per hour. The plane is flying in a straight line and passes directly over an observer.

   (a) Sketch a diagram that summarizes the information in the problem.

   (b) Find the angle of elevation 2 minutes after the plane passes over the observer.

   (c) Find rate of change of the angle of elevation 2 minutes after the plane passes over the observer.

7. A car moves at 50 miles per hour on a straight road. A house is 2 miles away from the road. What is the rate of change in the angle between the house and the car and the house and the road when the car passes the house.

8. A car moves along a road that is shaped like the parabola $y = x^2$. At what point on the parabola are the rates of change for the x and y coordinates equal?

9. Let $f(x) = \dfrac{1}{1 + x^3}$ and $h(x) = \dfrac{1}{1 + f(x)}$

   (a) Find $f'(x)$.

   (b) Use the previous result to find $h'(x)$.

   (c) Let $x = x(t)$ be a function of time t with $x(1) = 1$ and set $F(t) = h(x(t))$. If $F'(1) = 18$, find $x'(1)$.

Worksheet # 16: Review for Exam II

1. (a) State the definition of the derivative of a function $f(x)$ at a point a.

   (b) Find a function f and a number a such that

   $$\frac{f(x) - f(a)}{x - a} = \frac{\ln(2x-1)}{x-1}$$

   (c) Evaluate the following limit by using (a) and (b),

   $$\lim_{x \to 1} \frac{\ln(2x-1)}{x-1}$$

2. State the following rules with the hypotheses and conclusion.

   (a) The product rule and quotient rule.

   (b) The chain rule.

3. A particle is moving along a line so that at time t seconds, the particle is $s(t) = \frac{1}{3}t^3 - t^2 - 8t$ meters to the right of the origin.

   (a) Find the time interval(s) when the particle's velocity is negative.

   (b) Find the time(s) when the velocity is zero.

   (c) Find the time interval(s) when the particle's acceleration is positive.

   (d) Find the time interval(s) when the particle is speeding up. Hint: What do we need to know about velocity and acceleration in order to know that the derivative of the speed is positive?

4. Compute the first derivative of each of the following functions:

   (a) $f(x) = \cos(4\pi x^3) + \sin(3x+2)$

   (b) $b(x) = x^4 \cos(3x^2)$

   (c) $y(\theta) = e^{\sec(2\theta)}$

   (d) $k(x) = \ln(7x^2 + \sin(x) + 1)$

   (e) $u(x) = (\sin^{-1}(2x))^2$

   (f) $h(x) = \dfrac{8x^2 - 7x + 3}{\cos(2x)}$

   (g) $m(x) = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}}$

   (h) $q(x) = \dfrac{e^x}{1+x^2}$

   (i) $n(x) = \cos(\tan(x))$

   (j) $w(x) = \arcsin(x) \cdot \arccos(x)$

5. Let $f(x) = \cos(2x)$. Find the fourth derivative at $x = 0$, $f^{(4)}(0)$.

6. Let f be a one to one, differentiable function such that $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$ and $f'(2) = 5$. Find the derivative of the inverse function, $(f^{-1})'(2)$.

7. The tangent line to $f(x)$ at $x = 3$ is given by $y = 2x - 4$. Find the tangent line to $g(x) = \dfrac{x}{f(x)}$ at $x = 3$. Put your answer in slope-intercept form.

8. Consider the curve $xy^3 + 12x^2 + y^2 = 24$. Assume this equation can be used to define y as a function of x (i.e. $y = y(x)$) near $(1, 2)$ with $y(1) = 2$. Find the equation of the tangent line to this curve at $(1, 2)$.

9. Each side of a square is increasing at a rate of 5 cm/s. At what rate is the area of the square increasing when the area is 14 cm^2?

10. The sides of a rectangle are varying in such a way that the area is constant. At a certain instant the length of a rectangle is 16 m, the width is 12 m and the width is increasing at 3 m/s. What is the rate of change of the length at this instant?

11. Suppose f and g are differentiable functions such that $f(2) = 3$, $f'(2) = -1$, $g(2) = \frac{1}{4}$, and $g'(2) = 2$. Find:

    (a) $h'(2)$ where $h(x) = \ln([f(x)]^2)$;

    (b) $l'(2)$ where $l(x) = f(x^3 \cdot g(x))$.

12. Abby is driving north along Ash Road. Boris driving west on Birch Road. At 11:57 am, Boris is 5 km east of Oakville and traveling west at a speed of 60 km/h and Abby is 10 km north of Oakville and traveling north at a speed of 50 km/h.

    (a) Make a sketch showing the location and direction of travel for Abby and Boris.

    (b) Find the rate of change of the distance between Abby and Boris at 11:57 AM.

    (c) At 11:57 AM, is the distance between Abby and Boris increasing, decreasing, or not changing?

Worksheet #17: Exponential Growth and Decay

1. Solve the following equations for α:

   (a) $500 = 1000e^{20\alpha}$

   (b) $40 = \alpha e^{10k}$, where $k = \frac{\ln(2)}{7}$.

   (c) $100,000 = 40,000e^{0.06\alpha}$.

   (d) $\alpha = 2,000e^{36k}$, where $k = \frac{\ln(0.5)}{18}$.

2. Choose two of the previous four equations, write a problem which may be represented by the given equation. Be sure to explicitly state what each quantity represents, using correct units. Use a different type of scenario for each equation.

3. The mass of substance X decays exponentially. Let $m(t)$ denote the mass of substance X at time t where t is measured in hours. If we know $m(1) = 100$ grams and $m(10) = 50$ grams, find an expression for the mass at time t.

4. A certain cell culture grows at a rate proportional to the number of cells present. If the culture contains 500 cells initially and 800 after 24 hours, how many cells will there be after a further 12 hours?

5. Suppose that the rate of change of the mosquito population in a pond is directly proportional to the number of mosquitoes in the pond.
$$\frac{dP}{dt} = KP$$
   where $P = P(t)$ is the number of mosquitoes at time t, t is measured in days and the constant of proportionality $K = .007$

   (a) Give the units of K.

   (b) If the population of mosquitoes at time $t = 0$ is $P(0) = 200$. How many mosquitoes will there be after 90 days?

6. Suppose that $P(t)$ gives the number of individuals in a population at time t where t is measured in years. Each year 23 out of 1000 individuals give birth and 11 out of 1000 individuals die.

   Find a differential equation of the form $P' = kP$ that the function P satisfies.

7. A lucky colony of rabbits is brought to a large island where there are no predators and unlimited food. Under these conditions, they will reproduce at such a rate that the population doubles every 9 years. After 3 years, a team of scientists determines that there are 7000 rabbits on the island.

   (a) How many rabbits were brought to the island originally?

   (b) How many rabbits will there be 12 years after their introduction to the island?

8. Suppose that f is a solution of the differential equation $f' = kf$ on an open interval (a, b) where k is a constant. Compute the derivative of $g(x) = e^{-kx}f(x)$ and show that g is constant. Explain why $f(x) = Ae^{kx}$.

Worksheet # 18: Extreme Values and the Mean Value Theorem

1. Comprehension check:

   (a) True or False: If $f'(c) = 0$ then f has a local maximum or local minimum at c.

   (b) True or False: If f is differentiable and has a local maximum or minimum at $x = c$ then $f'(c) = 0$.

   (c) A function continuous on an open interval may not have an absolute minimum or absolute maximum on that interval. Give an example of continuous function on $(0, 1)$ which has no absolute maximum.

   (d) True or False: If f is differentiable on the open interval (a, b), continuous on the closed interval $[a, b]$, and $f'(x) \neq 0$ for all x in (a, b), then we have $f(a) \neq f(b)$.

2. Sketch the following:

   (a) The graph of a function defined on $(-\infty, \infty)$ with three local maxima, two local minima, and no absolute minima.

   (b) The graph of a continuous function with a local maximum at $x = 1$ but which is not differentiable at $x = 1$.

   (c) The graph of a function on $[-1, 1)$ which has a local maximum but not an absolute maximum.

   (d) The graph of a function on $[-1, 1]$ which has a local maximum but not an absolute maximum.

   (e) The graph of a discontinuous function defined on $[-1, 1]$ which has both an absolute minimum and absolute maximum.

3. State the definition of a critical number. Use this definition to find the critical numbers for the following functions:

   (a) $f(x) = x^4 + x^3 + 1$

   (b) $g(x) = e^{3x}(x^2 - 7)$

   (c) $h(x) = |5x - 1|$

   (d) $j(x) = (4 - x^2)^{1/3}$

4. Find the absolute maximum and absolute minimum values of the following functions on the given intervals. Specify the x-values where these extrema occur.

   (a) $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$

   (b) $h(x) = x + \sqrt{1 - x^2}$, $[-1, 1]$

   (c) $f(x) = 2\cos(x) + \sin(2x)$, $[0, \frac{\pi}{2}]$

   (d) $f(x) = x^{-2} \ln x$, $[\frac{1}{2}, 4]$

5. State the Mean Value Theorem. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

   (a) $f(x) = \dfrac{x}{x + 2}$ on the interval $[1, 4]$

   (b) $f(x) = \sin(x) - \cos(x)$ on the interval $[0, 2\pi]$

6. Use the Mean Value Theorem to show that $\sin(x) \leq x$ for $x \geq 0$. What can you say for $x \leq 0$?

7. Suppose that $g(x)$ is differentiable for all x and that $-5 \leq g'(x) \leq 3$ for all x. Assume also that $g(0) = 4$. Based on this information, use the Mean Value Theorem to determine the largest and smallest possible values for $g(2)$.

8. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 miles on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why did she deserve the ticket?

9. Suppose that we know that $1 \leq f(2) \leq 4$ and that $2 \leq f'(x) \leq 5$ for all x. What are the largest and smallest possible values for $f(6)$? What about $f(-1)$?

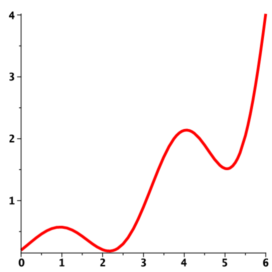
10. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1 - x)^b$ on the interval $0 \leq x \leq 1$.

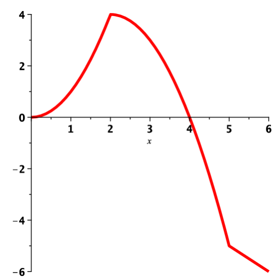
Worksheet # 19: The Shape of a Graph

1. Comprehension Check:
   (a) Explain what the First Derivative Test reveals about a continuous function $f(x)$ including when and how to use it.
   (b) Explain what the Second Derivative Test reveals about a twice differentiable function $f(x)$ and include how to use it. Does the test always work? What should you do if it fails?
   (c) Identify the similarities and differences between these two tests.

2. (a) Consider the function $f(x) = 2x^3 - 9x^2 - 24x + 5$ on $(-\infty, \infty)$.
      i. Find the critical number(s) of $f(x)$.
      ii. Find the intervals on which $f(x)$ is increasing or decreasing.
      iii. Find the local extrema of $f(x)$.
   (b) Repeat with the function $f(x) = \dfrac{x}{x^2+4}$ on $(-\infty, \infty)$.

3. Below are the graphs of two functions.





   (a) Find the intervals where each function is increasing and decreasing respectively.
   (b) Find the intervals of concavity for each function.
   (c) For each function, identify all local extrema and inflection points on the interval (0,6).

4. (a) Consider the function $f(x) = x^4 - 8x^3 + 5$.
      i. Find the intervals on which the graph of $f(x)$ is increasing or decreasing.
      ii. Find the inflection points of $f(x)$.
      iii. Find the intervals of concavity of $f(x)$.
   (b) Repeat with the function $f(x) = 2x + \sin(x)$ on $\left(-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.
   (c) Repeat with the function $f(x) = xe^x$.

5. Find the local extrema of the following functions using the second derivative test (if possible):
   (a) $f(x) = x^5 - 5x + 4$
   (b) $g(x) = 5x - 10\ln(2x)$
   (c) $h(x) = 3x^5 - 5x^3 + 10$

6. Sketch a graph of a continuous function $f(x)$ with the following properties:
   - f is increasing on $(-\infty, -3) \cup (1, 7) \cup (7, \infty)$
   - f is decreasing on $(-3, 1)$
   - f is concave up on $(0, 3) \cup (7, \infty)$
   - f is concave down on $(-\infty, 0) \cup (3, 7)$

Worksheet # 20: L'Hôpital's Rule & Optimization

1. Suppose we know:

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} p(x) = \infty \qquad \lim_{x\to a} q(x) = \infty$$

   Which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

   (a) $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$

   (b) $\displaystyle\lim_{x\to a} \frac{f(x)}{p(x)}$

   (c) $\displaystyle\lim_{x\to a} f(x)p(x)$

   (d) $\displaystyle\lim_{x\to a} \frac{p(x)}{f(x)}$

   (e) $\displaystyle\lim_{x\to a} p(x)q(x)$

   (f) $\displaystyle\lim_{x\to a} \frac{p(x)}{q(x)}$

2. Carefully state l'Hôpital's Rule.

3. Compute the following limits. Use l'Hôpital's Rule where appropriate, but first check that no easier method will solve the problem.

   (a) $\displaystyle\lim_{x\to 1} \frac{x^9 - 1}{x^5 - 1}$

   (b) $\displaystyle\lim_{x\to \infty} \frac{3x + 2\sqrt{x}}{1 - x}$

   (c) $\displaystyle\lim_{x\to 0} \frac{\sin(4x)}{\tan(5x)}$

   (d) $\displaystyle\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2}$

   (e) $\displaystyle\lim_{x\to \infty} \frac{e^x}{x^3}$

   (f) $\displaystyle\lim_{x\to -\infty} \frac{2x - 5}{|3x + 2|}$

   (g) $\displaystyle\lim_{x\to \infty} \frac{5x^2 + \sin x}{3x^2 + \cos x}$

   (h) $\displaystyle\lim_{x\to 1} \frac{x^2 + 2x - 2}{x^2 - 2x + 2}$

   (i) $\displaystyle\lim_{x\to -\infty} x^2 e^x$

   (j) $\displaystyle\lim_{x\to \infty} x^3 e^{-x^2}$

   (k) $\displaystyle\lim_{x\to \pi} \frac{\cos(x) + 1}{x^2 - \pi^2}$

   (l) $\displaystyle\lim_{x\to \infty} x \cdot \left(\arctan(x) - \frac{\pi}{2}\right)$

4. Find the value A for which we can use l'Hôpital's rule to evaluate the limit

$$\lim_{x\to 2} \frac{x^2 + Ax - 2}{x - 2}.$$

   For this value of A, give the value of the limit.

5. Find the dimensions of x and y of the rectangle of maximum area that can be formed using 3 meters of wire.

   (a) What is the constraint equation relating x and y?

   (b) Find a formula for the area in terms of x alone.

   (c) Solve the optimization problem.

6. Find two numbers whose difference is 100 and whose product is a minimum.

7. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

8. A farmer wants to fence in an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can she do this so as to minimize the cost of the fence?

Worksheet # 21: Optimization

1. Suppose that f is a function on an open interval $I = (a, b)$ and c is in I. Suppose that f is continuous at c, $f'(x) > 0$ for $x > c$ and $f'(x) < 0$ for $x < c$. Is $f(c)$ an absolute minimum value for f on I? Justify your answer.

2. A farmer has 2400 feet of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

3. A hockey team plays in an arena with a seating capacity of 15000 spectators. With the ticket price set at \$12, average attendance at a game has been 11000. A market survey indicates that for each dollar the ticket price is lowered, average attendance will increase by 1000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?

4. An oil company needs to run a pipeline to a nearby station. The station and oil company are on opposite sides of a river that is 1 km wide, and that runs exactly west-east and the station is 10 km east along the river from the the oil company. The cost of building pipe on land is \$200 per meter and the cost of building pipe in water is \$300 per meter. Set up an equation whose solution(s) are the critical points of the cost function for this problem.

   Find the least expensive way to construct the pipe.

5. A flexible tube of length 4 m is bent into an L-shape. Where should the bend be made to minimize the distance between the two ends?

6. A 10 meter length of rope is to be cut into two pieces to form a square and a circle. How should the rope be cut to maximize the enclosed area?

7. Find the point(s) on the hyperbola $y = \dfrac{16}{x}$ that is (are) closest to $(0, 0)$. Be sure to clearly state what function you choose to minimize or maximize and why.

8. Consider a can in the shape of a right circular cylinder. The top and bottom of the can is made of a material that costs 4 cents per square centimeter, and the side is made of a material that costs 3 cents per square centimeter. We want to find the dimensions of the can which has volume 72 π cubic centimeters, and whose cost is as small as possible.

   (a) Find a function $f(r)$ which gives the cost of the can in terms of radius r. Be sure to specify the domain.

   (b) Give the radius and height of the can with least cost.

   (c) Explain how you known you have found the can of least cost.

9. Find the point on the line $y = x$ closest to the point $(1, 0)$. Find the point on the line $y = x$ closest to the point $(r, 1 - r)$. What does the collection of points $(r, 1 - r)$ look like graphically?

10. A box is to have a square base, no top, and a volume of 10 cubic centimeters. What are the dimensions of the box with the smallest possible total surface area? Provide an exact answer; do not convert your answer to decimal form. Make a sketch and introduce all the notation you are using.

Worksheet # 22: Antiderivatives & Areas and Distances

1. Comprehension Check:

   (a) If F is an antiderivative of a continuous function f, is F continuous? What if f is not continuous?

   (b) Let $g(x) = \dfrac{x^3}{3} + 1$. Find $g'(x)$. Now give two antiderivatives of $g'(x)$.

   (c) Let $h(x) = x^2 + 1$, and let $H(x)$ be any antiderivative of h. What is $H'(x)$?

2. Find the most general antiderivative of the function $f(x) = x^2 - 3x + 2 - \frac{5}{x}$.

3. Find f given that

$$f'(x) = \sin(x) - \sec(x)\tan(x), \qquad f(\pi) = 1.$$

4. Find g given that

$$g''(t) = -9.8, \qquad g'(0) = 1, \qquad g(0) = 2.$$

   On the surface of the earth, the acceleration of an object due to gravity is approximately $-9.8\,\mathrm{m/s^2}$. What situation could we describe using the function g? Be sure to specify what g and its first two derivatives represent.

5. The velocity of a train at several times is shown in the table below. Assume that the velocity changes linearly between each time given.

t=time in minutes	0	3	6	9
v(t)=velocity in Km/h	20	80	100	140

   (a) Plot the velocity of the train versus time.

   (b) Compute the left and right-endpoint approximations to the area under the graph of v.

   (c) Explain why these approximate areas are also an approximation to the distance that the train travels.

6. Let $f(x) = \frac{1}{x}$. Divide the interval $[1, 3]$ into five subintervals of equal length and compute R_5 and L_5, the left and right endpoint approximations to the area under the graph of f in the interval $[1, 3]$. Is R_5 larger or smaller than the true area? Is L_5 larger or smaller than the true area?

7. Let $f(x) = \sqrt{1 - x^2}$. Divide the interval $[0, 1]$ into four equal subintervals and compute L_4 and R_4, the left and right-endpoint approximations to the area under the graph of f. Is R_4 larger or smaller than the true area? Is L_4 larger or smaller than the true area? What can you conclude about the value π?

8. Write each of following in summation notation:

   (a) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

   (b) $2 + 4 + 6 + 8 + 10 + 12 + 14$

   (c) $2 + 4 + 8 + 16 + 32 + 64 + 128$.

9. Compute $\displaystyle\sum_{i=1}^{4}\left(\sum_{j=1}^{3}(i + j)\right)$.

10. Let $f(x) = x^2$.

    (a) If we divide the interval $[0, 2]$ into n equal intervals of equal length, how long is each interval?

    (b) Write a sum which gives the right-endpoint approximation R_n to the the area under the graph of f on $[0, 2]$.

    (c) Use one of the formulae for the sums of powers of k to find a closed form expression for R_n.

    (d) Take the limit of R_n as n tends to infinity to find an exact value for the area.

Worksheet # 23: Definite Integrals

The following summation formulas will be useful below.

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \qquad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

1. Find the number n such that $\sum_{i=1}^{n} i = 78$.

2. Give the value of the following sums.

   (a) $\displaystyle\sum_{k=1}^{20}(2k^2+3)$

   (b) $\displaystyle\sum_{k=11}^{20}(3k+2)$

3. Recognize the sum as a Riemann sum and express the limit as an integral.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^3}{n^4}$$

4. Let $f(x) = x$ and consider the partition $P = \{x_0, x_1, \ldots, x_n\}$ which divides the interval $[1, 3]$ into n subintervals of equal length.

   (a) Find a formula for x_k in terms of k and n.

   (b) We form a rectangle whose width is $\Delta x = (x_k - x_{k-1})$ and whose height is $f(x_k)$. Give the area of the rectangle.

   (c) Choose the sample points to be the right endpoint of each subinterval. Form the Riemann sum, and use the formula for sums of powers to simplify the Riemann sum.

   (d) Take the limit as n tends to infinity to find the area of the region under $f(x)$ for $1 \leq x \leq 3$.

   (e) Find the area above using geometry to check your answer.

5. Suppose $\displaystyle\int_0^1 f(x)\,dx = 2$, $\displaystyle\int_1^2 f(x)\,dx = 3$, $\displaystyle\int_0^1 g(x)\,dx = -1$, and $\displaystyle\int_0^2 g(x)\,dx = 4$.

   Compute the following using the properties of definite integrals:

   (a) $\displaystyle\int_1^2 g(x)\,dx$

   (b) $\displaystyle\int_0^2 [2f(x) - 3g(x)]\,dx$

   (c) $\displaystyle\int_1^1 g(x)\,dx$

   (d) $\displaystyle\int_1^2 f(x)\,dx + \int_2^0 g(x)\,dx$

   (e) $\displaystyle\int_0^2 f(x)\,dx + \int_2^1 g(x)\,dx$

6. Suppose that f is a continuous function and $6 \leq f(x) \leq 7$ for x in the interval $[3, 9]$.

   (a) Find the largest and smallest possible values for $\int_3^9 f(x)\,dx$.

   (b) Find the largest and smallest possible values for $\int_8^4 f(x)\,dx$.

7. Suppose that we know $\int_0^x f(t)\,dt = \sin(x)$, find the following integrals.

(a) $\displaystyle\int_0^\pi f(t)\,dt$

(b) $\displaystyle\int_{\pi/2}^\pi f(t)\,dt$

(c) $\displaystyle\int_{-\pi}^\pi f(t)\,dt$

8. Find $\displaystyle\int_0^5 f(x)\,dx$ where $f(x) = \begin{cases} 3 & \text{if } x < 3 \\ x & \text{if } x \geq 3 \end{cases}.$

9. Simplify

$$\int_a^b f(t)\,dt + \int_b^c f(u)\,du + \int_c^a f(v)\,dv.$$

Worksheet # 24: Review for Exam III

1. Strontium-90 has a half-life of 28 days.
   (a) A sample has a mass of 50 mg initially. Find a formula for the mass remaining after t days.
   (b) Find the mass remaining after 40 days.
   (c) How long does it take the sample to decay to a mass of 2 mg?
2. Describe in words and diagrams how to use the first and second derivative tests to identify and classify extrema of a function $f(x)$.
3. Find the absolute minimum of the function $f(t) = t + \sqrt{1-t^2}$ on the interval $[-1, 1]$. Be sure to specify the value of t where the minimum is attained.
4. (a) Consider the function $f(x) = 2x^3 + 3x^2 - 72x - 47$ on $(-\infty, \infty)$.
      i. Find the critical number(s) of f.
      ii. Find the intervals on which f is increasing or decreasing.
      iii. Find the local maximum and minimum values of f.
      iv. Find the intervals of concavity and the inflection points.
   (b) Repeat with the function $f(x) = x^4 - 2x^2 + 3$.
   (c) Repeat with the function $f(x) = e^{2x} + e^{-x}$.
5. For what values of c does the polynomial $p(x) = x^4 + cx^3 + x^2$ have two inflection points? One inflection point? No inflection points?
6. (a) State the Mean Value Theorem. Use complete sentences.
   (b) Does there exist a function f such that $f(0) = -1, f(2) = 4$, and $f'(x) \leq 2$ for all x?
7. State L'Hopital's Rule for limits in indeterminate form of type $0/0$. Use complete sentences, and include all necessary assumptions. Then evaluate the following limits:
   (a) $\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$
   (b) $\lim_{x\to 0^+} x^3 \ln(x)$
   (c) $\lim_{x\to -\infty} \frac{x+2}{\sqrt{9x^2+1}}$
   (d) $\lim_{x\to 2} \frac{e^{2x}}{x+2}$
8. A poster is to have an area of 180 cm^2 with 1 cm margins at the bottom and sides and 2 cm margins at the top. What dimensions will give the largest printed area? Be sure to explain how you know you have found the largest area.
   (a) Draw a picture and write the constraint equation.
   (b) Write the function you are asked to maximize or minimize and determine its domain.
   (c) Find the maximum or minimum of the function that you found in part (b).
9. Find a positive number such that the sum of the number and twice its reciprocal is small as possible.
10. Find the most general antiderivative for each of the following:
    (a) $f(x) = 5x^{10} + 7x^2 + x + 1$
    (b) $g(x) = x + \cos(2x+1)$
    (c) $h(x) = \frac{1}{x+1}$, where $x + 1 > 0$

11. Find a function with $f''(x) = \sin(2x)$, $f(\pi) = 1$, and $f'(0) = 2$.

12. Consider the region bounded by the graph of $f(x) = \frac{1}{x}$, the x-axis, and the lines $x = 1$ and $x = 2$. Find L_3, the left endpoint approximation of this area with 3 subdivisions.

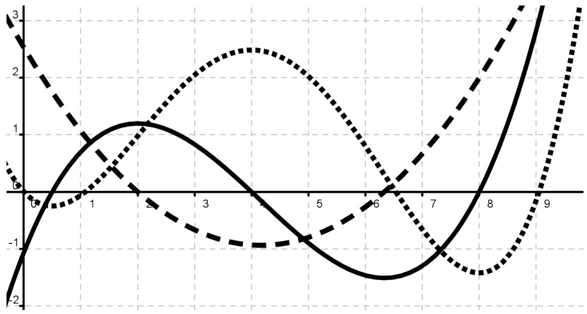
13. Suppose we know that $\sum_{k=1}^{n} a_k = n^2 + 2n$. Using this information, find the following:

    (a) $\sum_{k=1}^{20}(4a_k + 1)$.

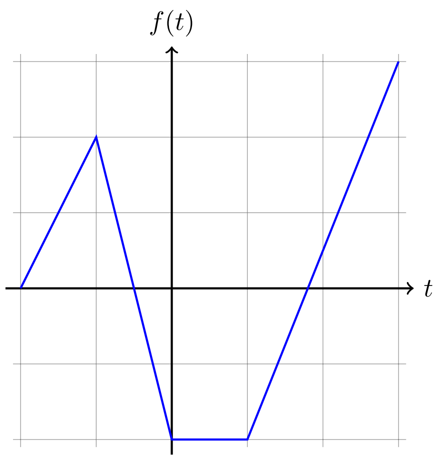
    (b) $\sum_{k=5}^{10} a_k$.

Worksheet # 25: The Fundamental Theorem of Calculus, Part 1

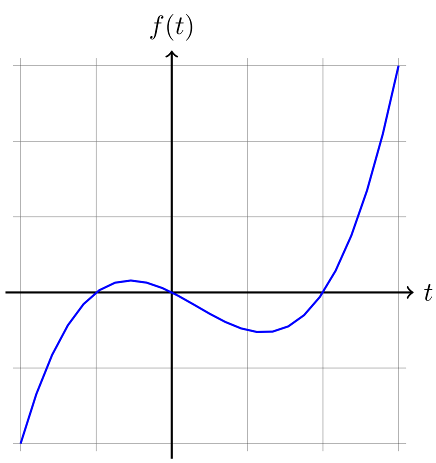
1. Below is pictured the graph of the function $f(x)$, its derivative $f'(x)$, and the integral $\int_0^x f(t)\,dt$. Identify $f(x), f'(x)$ and $\int_0^x f(t)\,dt$ and explain your reasoning.



2. Let $g(x) = \int_{-2}^x f(t)\,dt$ where f is the function whose graph is shown below.
   (a) Evaluate $g(-1)$, $g(0)$, $g(1)$, $g(2)$, and $g(3)$.
   (b) On what interval is g increasing? Why?
   (c) Where does g have a maximum value? Why?



3. Let $g(x) = \int_{-2}^x f(t)\,dt$ where f is the function whose graph is shown below. Where is $g(x)$ increasing and decreasing? Explain your answer.



4. Let $F(x) = \int_2^x e^{t^2}\, dt$. Find an equation of the tangent line to the curve $y = F(x)$ at the point with x-coordinate 2.

5. Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the following functions:

   (a) $g(x) = \displaystyle\int_1^x \left(2 + t^4\right)^5 \, dt$

   (b) $F(x) = \displaystyle\int_x^4 \cos\left(t^5\right)\, dt$

   (c) $h(x) = \displaystyle\int_0^{x^2} \sqrt[3]{1 + r^3}\, dr$

   (d) $y(x) = \displaystyle\int_{\frac{1}{x^2}}^0 \sin^3(t)\, dt$

   (e) $G(x) = \displaystyle\int_{\sqrt{x}}^{x^2} \sqrt{t}\sin(t)\, dt$

6. Find a function $f(t)$ and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\, dt = 2\sqrt{x}$$

   for all $x > 0$.

Worksheet # 26: The Fundamental Theorem of Calculus and Net Change

1. (a) State both parts of the Fundamental Theorem of Calculus using complete sentences.

   (b) Consider the function $f(x)$ on $[1, \infty)$ defined by $f(x) = \int_1^x \sqrt{t^5 - 1}\, dt$. Find the derivative of f. Explain why the function f is increasing.

   (c) Find the derivative of the function $g(x) = \int_1^{x^3} \sqrt{t^5 - 1}\, dt$ on $(1, \infty)$.

2. Use Part 2 of the Fundamental Theorem of Calculus to evaluate the following integrals or explain why the theorem does not apply:

   (a) $\displaystyle\int_{-2}^{5} 6x\, dx$

   (b) $\displaystyle\int_{-2}^{7} \frac{1}{x^5}\, dx$

   (c) $\displaystyle\int_{-1}^{1} e^{u+1}\, du$

   (d) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{\sin(2x)}{\sin(x)}\, dx$

3. Find each of the following indefinite integrals.

   (a) $\displaystyle\int 7x - 2\, dx$

   (b) $\displaystyle\int \frac{1}{x^{78}}\, dx$

   (c) $\displaystyle\int e^{u+2}\, du$

4. A population of rabbits at time t increases at a rate of $40 - 12t + 3t^2$ rabbits per year where t is measured in years. Find the population after 8 years if there are 10 rabbits at $t = 0$.

5. Suppose the velocity of a particle traveling along the x-axis is given by $v(t) = 3t^2 + 8t + 15$ m/s at time t seconds. The particle is initially located 5 meters left of the origin. How far does the particle travel from $t = 2$ seconds to $t = 3$ seconds? After 3 seconds, where is the particle with respect to the origin?

6. Suppose an object traveling in a straight line has a velocity function given by $v(t) = t^2 - 8t + 15$ km/hr. Find the displacement and distance traveled by the object from $t = 2$ to $t = 4$ hours.

7. (a) An oil storage tank ruptures and oil leaks from the tank at a rate of $r(t) = 100e^t$ liters per minute. How much oil leaks out during the first hour?

   (b) Is this model realistic? In other words, is it realistic to use this function $r(t)$ to model the leak rate in this situation? Why or why not?

8. Recognize each of the sums as a Riemann sum, express the limit as an integral and use the Fundamental Theorem to evaluate the limit.

   (a) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sqrt{3 + \frac{i}{n}}}{n}$

   (b) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} 2\frac{(2 + \frac{2i}{n})^2}{n}$

Worksheet # 27: Substitution and More Integration

1. Evaluate the following indefinite integrals, and indicate any substitutions that you use:

   (a) $\displaystyle\int \frac{4}{(1+2x)^3}\,dx$

   (b) $\displaystyle\int x^2\sqrt{x^3+1}\,dx$

   (c) $\displaystyle\int \cos^4(\theta)\sin(\theta)\,d\theta$

   (d) $\displaystyle\int \sec^3(x)\tan(x)\,dx$

   (e) $\displaystyle\int e^x \sin\left(e^x\right)\,dx$

   (f) $\displaystyle\int \frac{2x+3}{x^2+3x}\,dx$

2. Evaluate the following definite integrals, and indicate any substitutions that you use:

   (a) $\displaystyle\int_0^7 \sqrt{4+3x}\,dx$

   (b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\cos\left(\sin(x)\right)\,dx$

   (c) $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x^2}}\,dx$

   (d) $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

   (e) $\displaystyle\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$

3. (a) An oil storage tank ruptures and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the first hour?

   (b) Is this model realistic? In other words, is it realistic to use this function $r(t)$ to model the leak rate in this situation? Why or why not?

4. If f is continuous on $(-\infty, \infty)$, prove that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx\,.$$

   For the case where $f(x) \geq 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

5. Assume f is a continuous function.

   (a) If $\displaystyle\int_0^9 f(x)\,dx = 4$, find $\displaystyle\int_0^3 x \cdot f(x^2)\,dx$.

   (b) If $\displaystyle\int_0^u f(x)\,dx = 1 + e^{u^2}$ for all real numbers u, find $\displaystyle\int_0^2 f(2x)\,dx$.

6. Do you remember our technique from worksheet #2 of writing $b^x = e^{x\ln(b)}$? Use this to find the indefinite integral $\displaystyle\int b^x\,dx$.

7. Which integral should be evaluated using substitution? Evaluate both integrals:

   (a) $\displaystyle\int \frac{9\,dx}{1+x^2}$

   (b) $\displaystyle\int \frac{x\,dx}{1+9x^2}$

8. Find a so that if $x = au$, then $\sqrt{16+x^2} = 4\sqrt{1+u^2}$.

9. Evaluate the following indefinite integrals, and indicate any substitutions that you use:

(a) $\displaystyle\int \frac{dx}{x^2+3}$

(b) $\displaystyle\int \frac{\cos(\ln(t))\,dt}{t}$

(c) $\displaystyle\int \frac{x\,dx}{\sqrt{7-x^2}}$

(d) $\displaystyle\int \frac{dt}{4t^2+9}$

(e) $\displaystyle\int \frac{\ln(\arccos(x))\,dx}{\arccos(x)\sqrt{1-x^2}}$

(f) $\displaystyle\int \frac{dt}{|t|\sqrt{12t^2-3}}$

(g) $\displaystyle\int \frac{dx}{(4x-1)\ln(8x-2)}$

(h) $\displaystyle\int e^{9-2x}dx$

Worksheet # 28: Area Between Curves

1. Suppose that Alpha runs faster than Beta throughout a 1500-meter race. What is the physical meaning of the area between their velocity curves for the first minute of the race?

2. Find the area of the region between the graphs of $y = x^2$ and $y = x^3$.

3. If the birth rate of a population is $b(t) = 2200e^{0.024t}$ people per year and the death rate is $d(t) = 1460e^{0.018t}$ people per year, find the area between these curves for $0 \leq t \leq 10$. What does this area represent in the model of population growth/decline?

4. Find the area of the regions enclosed by the graphs of $y = \sqrt{x}$ and $y = \frac{1}{4}x + \frac{3}{4}$ in two ways. 1) By writing an integral in x. 2) Solve each equation to express x in terms of y and write an integral with respect to y.

5. Find the area of the region enclosed by the graphs of $y = x + 1$ and $y = x^3 + x^2 - x + 1$.

6. Use calculus to find the area of the triangle with the vertices $(2, 0)$, $(0, 2)$ and $(-1, 1)$.

7. Sketch the region enclosed by the curves $y = x^4$ and $y = 2 - |x|$ and find its area.

8. Sketch the region enclosed by the curves $y = \frac{1}{4}x^2$, $y = 2x^2$, and $x + y = 3$, with $x \geq 0$.

9. Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

10. Find the number a such that the line $x = a$ bisects the area under the curve $y = 1/x^2$ with $1 \leq x \leq 4$.

Worksheet # 29: Linear and Higher-Order Approximation and Applications

1. What is the relation between the linearization of a function $f(x)$ at $x = a$ and the tangent line to the graph of the function $f(x)$ at $x = a$ on the graph?

2. (a) Use the linearization of $\sqrt{x}$ at $a = 16$ to estimate $\sqrt{18}$.
   (b) Find a decimal approximation to $\sqrt{18}$ using a calculator.
   (c) Compute both the absolute error and the percentage error if we use the linearization to approximate $\sqrt{18}$.

3. For each of the following, use a linear approximation to the change in the function and a convenient nearby point to estimate the value:
   (a) $(3.01)^3$
   (b) $\sqrt{17}$
   (c) $8.06^{2/3}$

4. Use Taylor polynomials with $a = 0$ to approximate $\frac{1}{\sqrt[10]{e}}$ to five decimal places.

5. (a) Use Taylor polynomials with $a = 0$ to approximate $\int_0^1 \sin(x^4)\,dx$ to four decimal places.
   (b) Can you find an indefinite integral for this integrand? Why or why not?

6. Suppose we want to paint a sphere of radius 200 cm with a coat of paint 0.1 mm thick. Use a linear approximation to approximate the amount of paint we need to do the job.

7. Let $f(x) = \sqrt{16 + x}$. First, find the linearization to $f(x)$ at $x = 0$, then use the linearization to estimate $\sqrt{15.75}$. Present your solution as a rational number.

8. Find the linearization $L(x)$ to the function $f(x) = \sqrt{1 - 2x}$ at $x = -4$.

9. Find the linearization $L(x)$ to the function $f(x) = \sqrt[3]{x + 4}$ at $x = 4$, then use the linearization to estimate $\sqrt[3]{8.25}$.

10. Your physics professor tells you that you can replace $\sin(\theta)$ with θ when θ is close to zero. Explain why this is reasonable.

11. Suppose we measure the radius of a sphere as 10 cm with an accuracy of $\pm$ 0.2 cm. Use linear approximations to estimate the maximum error in:
    (a) the computed surface area.
    (b) the computed volume.

12. Suppose that $y = y(x)$ is a differentiable function which is defined near $x = 2$, satisfies $y(2) = -1$ and
$$x^2 + 3xy^2 + y^3 = 9.$$
    Use the linear approximation to the change in y to approximate the value of $y(1.91)$.

13. Use Taylor polynomials with $a = 0$ to approximate $\cos(1)$ to four decimal places.

14. (a) Use Taylor polynomials with $a = 0$ to approximate $\int_0^{0.5} x^2 e^{-x^2}\,dx$ to two decimal places.
    (b) Can you find an indefinite integral for this integrand? Why or why not?

15. If $f(x) = (1 + x^3)^{30}$, what is $f^{(58)}(0)$?

Worksheet # 30: Review for Final

1. Compute the following limits.

   (a) $\lim_{t\to 9} \dfrac{9-t}{3-\sqrt{t}}$

   (b) $\lim_{\theta\to 0} \dfrac{\sin(3\theta)}{10\theta}$

   (c) $\lim_{x\to\infty} \dfrac{e^x}{x^2}$

   (d) $\lim_{x\to\infty} \dfrac{(\ln x)^2}{x}$

2. (a) State the limit definition of the continuity of a function f at $x = a$.

   (b) State the limit definition of the derivative of a function f at $x = a$.

   (c) Given $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 4-3x & \text{if } x \geq 1 \end{cases}$. Is the function continuous at $x = 1$? Is the function differentiable at $x = 1$? Use the definition of the derivative. Graph the function to check your answer.

3. Provide the most general antiderivative of the following functions:

   (a) $f(x) = x^4 + x^2 + x + 1000$

   (b) $g(x) = (3x-2)^{20}$

   (c) $h(x) = \dfrac{\sin(\ln(x))}{x}$

4. Use implicit differentiation to find $\dfrac{dy}{dx}$, and compute the slope of the tangent line at (1,2) for the following curves:

   (a) $x^2 + xy + y^2 + 9x = 16$

   (b) $x^2 + 2xy - y^2 + x = 2$

5. An rock is thrown up the in the air and returns to the ground 4 seconds later. What is the initial velocity? What is the maximum height of the rock? Assume that the rock's motion is determined by the acceleration of gravity, 9.8 meters/second^2.

6. A conical tank with radius 5 meters and height 10 meters is being filled with water at a rate of 3 cubic meters per minute. How fast is the water level increasing when the water's depth is 3 meters?

7. A rectangular storage container with an open top is to have a volume of 10 cubic meters. The length of the container is twice its width. Material for the base costs $10 per square meter while material for the sides costs $6 per square meter. Find the cost of materials for the least expensive possible container.

8. (a) State the Mean Value Theorem.

   (b) If $3 \leq f'(x) \leq 5$ for all x, find the maximum possible value for $f(8) - f(2)$.

9. Use linearization to approximate $\cos(\frac{11\pi}{60})$

   (a) Write down $L(x)$ at an appropriate point $x = a$ for a suitable function $f(x)$.

   (b) Use part(a) to find an approximation for $\cos(\frac{11\pi}{60})$

   (c) Find the absolute error in your approximation.

10. Find the value(s) c such that $f(x)$ is continuous everywhere.

$$f(x) = \begin{cases} (cx)^3 & \text{if } x < 2 \\ \ln(x^c) & \text{if } x \geq 2 \end{cases}$$

11. (a) Find y' if $x^3 + y^3 = 6xy$.
    (b) Find the equation of the tangent line at $(3, 3)$.

12. Show that the function $f(x) = 3x^5 - 20x^3 + 60x$ has no absolute maximum or minimum.

13. Compute the following definite integrals:

    (a) $\int_{-1}^{1} e^{u+1}\, du$

    (b) $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$

    (c) $\int_{1}^{9} \frac{x-1}{\sqrt{x}}\, dx$

    (d) $\int_{0}^{10} |x - 5|\, dx$

    Hint: For some of the integrals, you will need to interpret the integral as an area and use facts from geometry to compute the integral.

14. Write as a single integral in the form $\int_a^b f(x)\, dx$:

$$\int_{-2}^{2} f(x)\, dx + \int_{2}^{5} f(x)\, dx - \int_{-2}^{-1} f(x)\, dx$$