

ALGEBRA PRELIM REVIEW

LINEAR ALGEBRA

- (1) Let V be an n -dimensional K -vector space, and let $U \subset V$ be a subspace of positive dimension $r < n$. Let $\varphi : V \rightarrow V$ be a K -linear map with $\varphi(U) \subset U$. Argue that:

- (a) There is an ordered basis B of V such that the coordinate matrix $A_\varphi^{B,B}$ has the form

$$A_\varphi^{B,B} = \begin{bmatrix} M & N \\ 0 & P \end{bmatrix}$$

where $M \in K^{r \times r}$ and $P \in K(n-r) \times (n-r)$.

- (b) If $\text{im}(\varphi) \subset U$, then one can choose B such that $P = 0$.

- (2) Consider the map $\varphi : \mathbb{Q}^{2 \times 2} \rightarrow \mathbb{Q}^{2 \times 2}$ given by $A \mapsto AM - MA$, where

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathbb{Q}^{2 \times 2}.$$

- (a) Show that φ is a \mathbb{Q} -linear map.
(b) Find a basis for $\ker(\varphi)$ and determine its dimension.

- (3) Consider a matrix $A \in K^{n \times n}$, where K is a field. Show:

- (a) If $\text{rk}(A) < n$, then there is a matrix $0 \neq B \in K^{n \times n}$ such that $A \cdot B = 0$
(b) If $A^k = 0$ for some $k \in \mathbb{N}$, then A is not invertible.

Proof.

□

- (4) Let V be a 4-dimensional vector space, and let $U_1, U_2 \subset V$ be two 3-dimensional subspaces of V .

- (a) Determine the possible values of $m = \dim(U_1 \cap U_2)$.
(b) For each value of m in (a), give an example of subspaces $U_1, U_2 \subset V = \mathbb{Q}^4$ whose intersection has dimension m .

- (5) Let $U \subset K^n$ be a subspace, and consider the set

$$V := \{\varphi \in \text{Hom}(K^n, K^m) \mid U \subset \ker(\varphi)\}$$

Prove:

- (a) V is a K -vector space.
(b) $\dim(V) = m(n - \dim(U))$.

- (6) (6/16) In the real vector space of continuous real-valued functions on \mathbb{R} , consider the functions $p_i, i = 0, 1, 2$ and exp defined by $p_i(x) = x^i$ and $exp(x) = e^x$ for $x \in \mathbb{R}$. Set $V := \langle p_0, p_1, p_2, exp \rangle_{\mathbb{R}}$ and consider the endomorphism $\sigma : V \rightarrow V$ defined by $(\sigma f)(x) := f(x - 1)$ for $x \in \mathbb{R}$.
- Give the matrix representation of σ with respect to the basis $\{p_0, p_1, p_2, exp\}$.
 - Determine all the eigenvalues and find the bases of all eigenspaces of σ .
 - Is σ diagonalizable?
 - Determine the minimal polynomial of σ .
- (7) Let $A \in \mathcal{M}_{m \times n}$ and $B \in \mathcal{M}_{n \times p}$. Show the following:
- $\text{rk}(AB) \leq \min\{\text{rk}(A), \text{rk}(B)\}$.
 - $\text{rk}(AB) \geq \text{rk}(A) + \text{rk}(B) - n$.
- (8) Let A, B, C be square matrices over a field K such that $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$. Argue that their minimal polynomials satisfy $\mu_A = \text{lcm}(\mu_B, \mu_C)$, where lcm denotes the least common multiple.
- (9) (6/14) Consider a linear transformation T on a vector space V of dimension 4 over \mathbb{R} . On a basis e_1, e_2, e_3, e_4 of V , the transformation is defined by $T(e_1) = e_2$, $T(e_2) = e_1$, $T(e_3) = 2e_3 + e_4$, and $T(e_4) = e_3 - 2e_4$.
- Construct the matrix A of the transformation with respect to the given basis.
 - Determine the characteristic polynomial, eigenvalues, and eigenspaces of A .
 - Determine the kernel and image of the transformation defined by the matrix $A^2 - I$ on \mathbb{R}^4 .
 - Is A diagonalizable? Would you answer differently if \mathbb{R} was replaced by \mathbb{Q} ?
- (10) (1/09) Let $n \in \mathbb{N}$ and F be a field. Suppose that $T : F \rightarrow F^n$ is a linear transformation. Show that T is injective if and only if T is not the zero map.
- (11) (6/11) Let $A \in M_{n \times n}(\mathbb{C})$ be a Hermitian matrix. Prove or disprove (with a counterexample) the following statements:
- $\det(A) \in \mathbb{R}$.
 - $|\det(A)| = 1$.
 - If A has exactly one eigenvalue, then A is a real matrix.
 - If $v = (v_1, \dots, v_n)^T$ is an eigenvector of A , then $\bar{v} = (\bar{v}_1, \dots, \bar{v}_n)^T$ is also an eigenvector of A (where \bar{v}_i denotes the complex conjugate of v_i).