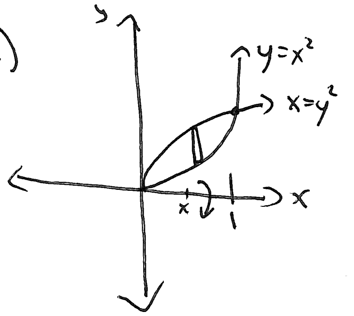
Final Review Solutions

1] a)



$x^2 = \sqrt{x} \iff x^{3/2} = 1$ or $x=0$ $\iff x = 1$ or $x = 0$

Using washers:

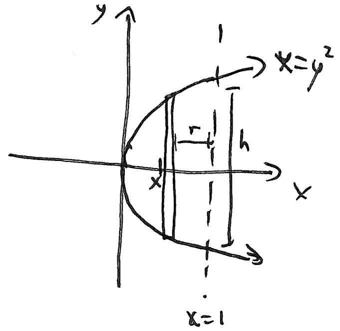
$R_{out} = \sqrt{x}$

$R_{in} = x^2$

$$V = \pi \int_0^1 (\sqrt{x})^2 - (x^2)^2 \, dx = \pi \int_0^1 x - x^4 \, dx$$

$$= \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right) \Big|_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} \right) = \boxed{\frac{3\pi}{10}}$$

b)



Using shells:

$r = 1 - x$

$h = \sqrt{x} - (-\sqrt{x})$

$= 2\sqrt{x}$

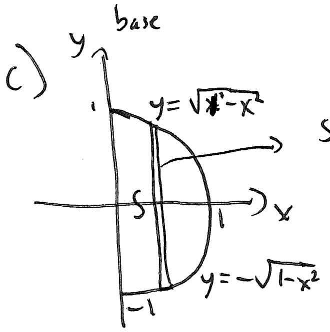
$$V = 2\pi \int_0^1 (1-x) \, 2\sqrt{x} \, dx = 4\pi \int_0^1 x^{1/2} - x^{3/2} \, dx$$

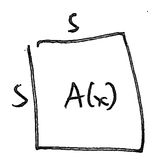
$$= 4\pi \left(\frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} \right) \Big|_0^1$$

$$= 4\pi \left(\frac{2}{3} - \frac{2}{5} \right)$$

$$= \boxed{\frac{16\pi}{15}}$$

c)





$s = 2\sqrt{1-x^2}$

So $A(x) = s^2 = 4(1-x^2)$.

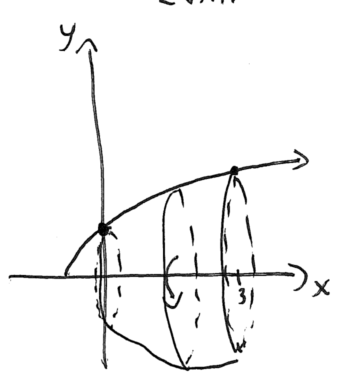
$$\text{So } V = \int_0^1 A(x) \, dx$$

$$= \int_0^1 4(1-x^2) \, dx$$

$$= 4 \left(x - \frac{1}{3} x^3 \right) \Big|_0^1$$

$$= \boxed{\frac{8}{3}}$$

2] $y' = \frac{1}{2\sqrt{x+1}}$, so $SA = \int_0^3 2\pi y \, ds$



$$= \int_0^3 2\pi \sqrt{x+1} \sqrt{1 + (y')^2} \, dx$$

$$= \int_0^3 2\pi \sqrt{x+1} \sqrt{1 + \left(\frac{1}{2\sqrt{x+1}} \right)^2} \, dx$$

$$= 2\pi \int_0^3 \sqrt{x+1} \cdot \frac{\sqrt{4(x+1)+1}}{\sqrt{4(x+1)}} \, dx$$

$$= 2\pi \int_0^3 \frac{\sqrt{x+1}}{2\sqrt{x+1}} \sqrt{4x+5} \, dx$$

$$= \pi \int_0^3 \sqrt{4x+5} \, dx$$

$u = 4x + 5$ $\quad x=0 : u = 5$

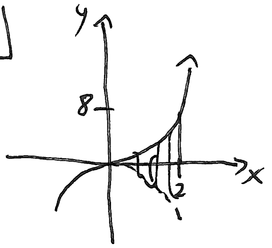
$du = 4 dx$ $\quad x=3 : u = 17$

$$= \frac{\pi}{4} \int_5^{17} \sqrt{u} \, du$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_5^{17}$$

$$= \boxed{\frac{\pi}{6} \left(17^{3/2} - 5^{3/2} \right)}$$

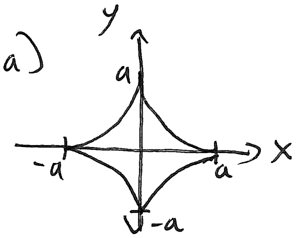
3)



$$SA = \int_a^b 2\pi y\, ds = \int_0^8 2\pi y\sqrt{1+(x')^2}\,dy = \boxed{\int_0^8 2\pi y\sqrt{1+\left(\frac{1}{3y^{2/3}}\right)^2}\,dy}$$

$$= \int_0^2 2\pi x^3\sqrt{1+(y')^2}\,dx = \boxed{\int_0^2 2\pi x^3\sqrt{1+(3x^2)^2}\,dx}$$

4) a)



$$\frac{dx}{dt} = -3\cos^2\theta\sin\theta\cdot a$$

$$\frac{dy}{dt} = 3\sin^2\theta\cos\theta\cdot a$$

Use symmetry:

$s = 4\cdot$ (length of curve in Q1)

$$= 4\int_0^{\pi/2} ds$$

$$= 4\int_0^{\pi/2}\sqrt{(x')^2+(y')^2}\,d\theta$$

$$= 4\int_0^{\pi/2}\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\,d\theta$$

$$= 4\int_0^{\pi/2}\sqrt{a^2 9\cos^2\theta\sin^2\theta\,(\underbrace{\cos^2\theta+\sin^2\theta}_{=1})}\,d\theta$$

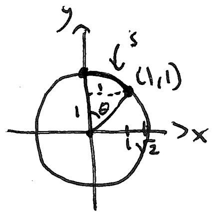
$$= 4\int_0^{\pi/2} 3a\cos\theta\sin\theta\,d\theta \qquad \text{Use } 2\cos\theta\sin\theta = \sin 2\theta$$

$$= \cancel{2}\cdot 6a\int_0^{\pi/2}\sin(2\theta)\,d\theta$$

$$= -3a\cos(2\theta)\Big|_0^{\pi/2}$$

$$= -3a(-1-1) = \boxed{6a}$$

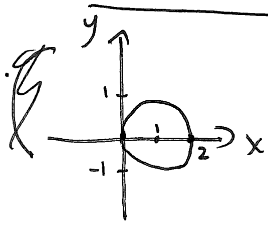
b)



The angle $\theta = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$.

So the arc length s is the length of $\frac{\left(\frac{\pi}{4}\right)}{2\pi} = \frac{1}{8}$ of the circumference of a circle of radius $\sqrt{2}$, which is $\frac{1}{8}(2\pi\sqrt{2}) = \boxed{\frac{\pi}{2\sqrt{2}}}$

c)



This arc travels around the pictured circle twice, so $s = 2\cdot 2\pi(1) = 4\pi$.

Alternatively:

$$s = \int_0^{2\pi}\sqrt{r^2+(r')^2}\,d\theta$$

$$= \int_0^{2\pi}\sqrt{4\cos^2\theta + 4\sin^2\theta}\,d\theta$$

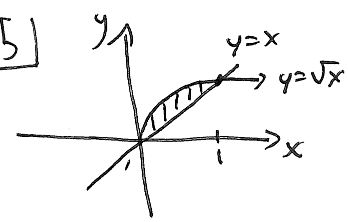
$$= \int_0^{2\pi} 2\,d\theta$$

$$= 2\theta\Big|_0^{2\pi} = \boxed{4\pi}$$

4d) $\frac{dx}{dt} = 6t \quad \frac{dy}{dt} = 6t^2$

$$s = \int_0^5 \sqrt{(x')^2+(y')^2}\,dt$$
$$= \int_0^5 \sqrt{36t^2+36t^4}\,dt$$
$$= \int_0^5 6t\sqrt{1+t^2}\,dt \qquad u = 1+t^2,\ du = 2t\,dt$$
$$= \int 3\sqrt{u}\,du$$
$$= 3\tfrac{2}{3}u^{3/2}\Big|_{t=0}^{t=5}$$
$$= 2(1+t^2)^{3/2}\Big|_0^5$$
$$= \boxed{2(26^{3/2}-1)}$$

5)



$$A = \int_0^1 \sqrt{x} - x\,dx$$
$$= \tfrac{2}{3}x^{3/2} - \tfrac{x^2}{2}\Big|_0^1$$
$$= \tfrac{2}{3} - \tfrac{1}{2}$$
$$= \tfrac{1}{6}$$

$$M_y = \int_0^1 x(\sqrt{x} - x)\,dx$$
$$= \int_0^1 x^{3/2} - x^2\,dx$$
$$= \tfrac{2}{5}x^{5/2} - \tfrac{1}{3}x^3\Big|_0^1$$
$$= \tfrac{2}{5} - \tfrac{1}{3}$$
$$= \tfrac{1}{15}$$

$$M_x = \int_0^1 \tfrac{1}{2}\left((\sqrt{x})^2 - x^2\right)dx$$
$$= \tfrac{1}{2}\int_0^1 x - x^2\,dx$$
$$= \tfrac{1}{2}\left(\tfrac{x^2}{2} - \tfrac{x^3}{3}\right)\Big|_0^1$$
$$= \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{3}\right)$$
$$= \tfrac{1}{2}\left(\tfrac{1}{6}\right) = \tfrac{1}{12}$$

$$\bar{x} = \frac{M_y}{A} = \tfrac{1}{15}\cdot 6 = \boxed{\tfrac{2}{5}} \qquad \bar{y} = \frac{M_x}{A} = \tfrac{1}{12}\cdot 6 = \boxed{\tfrac{1}{2}}$$

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6) From 4a), $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$.

So $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta}$

If $(x,y) = (1, 3\sqrt{3})$ and $a=8$, we have $1 = 8\cos^3\theta$, $3\sqrt{3} = 8\sin^3\theta$, so $\frac{1}{2} = \cos\theta$, $\frac{\sqrt{3}}{2} = \sin\theta$, so $\theta = \frac{\pi}{3}$.

So $\frac{dy}{dx} = -\tan\frac{\pi}{3} = \boxed{-\sqrt{3}.}$ The tangent line is then $\boxed{y - 3\sqrt{3} = -\sqrt{3}(x-1)}$

$\boxed{y = -\sqrt{3}\cdot x + 4\sqrt{3}}$

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7) $r = 2\cos\theta$

So $x = r\cos\theta = 2\cos^2\theta$

$y = r\sin\theta = 2\sin\theta\cos\theta = \sin 2\theta$

So $\frac{dy}{d\theta} = 2\cos 2\theta$, $\frac{dx}{d\theta} = -4\cos\theta\sin\theta$

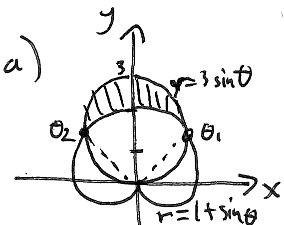
$= -2\sin 2\theta$

So $\frac{dy}{dx} = \frac{2\cos 2\theta}{-2\sin 2\theta} = -\cot(2\theta)$.

So at $\theta = \frac{\pi}{3}$, $\frac{dy}{dx} = -\cot\left(\frac{2\pi}{3}\right)$

$= \boxed{\frac{1}{\sqrt{3}}}$

8) a)



Find θ_1 & θ_2:

Set $3\sin\theta = 1 + \sin\theta$

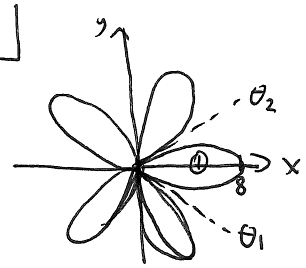
$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

$$\text{So } A = \int_{\pi/6}^{5\pi/6} \tfrac{1}{2}\left((3\sin\theta)^2 - (1+\sin\theta)^2\right)d\theta$$
$$= \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} 9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\,d\theta$$
$$= \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} 8\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) - 1 - 2\sin\theta\,d\theta$$
$$= \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} 3 - 4\cos 2\theta - 2\sin\theta\,d\theta$$
$$= \tfrac{1}{2}\left(3\theta - 2\sin 2\theta + 2\cos\theta\right)\Big|_{\pi/6}^{5\pi/6} = \pi$$

8b



$0 = 8\cos(5\theta)$

$\cos(5\theta) = 0$

$5\theta = \frac{\pi}{2} + 2\pi k$ or $-\frac{\pi}{2} + 2\pi k$

$\theta = \frac{\pi}{10} + \frac{2\pi}{5}k$ or $-\frac{\pi}{10} + \frac{2\pi}{5}k$

Take petal 1:

$$A = \int_{-\pi/10}^{\pi/10} \frac{1}{2}\left(8\cos(5\theta)\right)^2 d\theta$$

$$= 32\int_{-\pi/10}^{\pi/10} \cos^2(5\theta)\, d\theta$$

$$= 16\int_{-\pi/10}^{\pi/10} 1 + \cos(10\theta)\, d\theta$$

$$= 16\left(\theta + \tfrac{1}{10}\sin(10\theta)\right)\Big|_{-\pi/10}^{\pi/10}$$

$$= 16\left(\tfrac{\pi}{10} + \tfrac{1}{10}\sin(\pi) - -\tfrac{\pi}{10} - \tfrac{1}{10}\sin(-\pi)\right)$$

$$= \boxed{\frac{16\pi}{5}}$$

9

a) $r = \sqrt{3+1} = 2$

$\theta = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

b) $r = \sqrt{4^2+4^2} = 4\sqrt{2}$

$\tan\theta = \frac{4}{-4} = -1$ in Q2 so

$\theta = \frac{3\pi}{4}$

c) $x = r\cos\theta = 2\cdot 0 = 0$

$y = r\sin\theta = 2\cdot(-1) = -2$

d) $x = r\cos\theta = -3\cdot\frac{1}{2} = -\frac{3}{2}$

$y = r\sin\theta = -3\cdot\left(-\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$

10

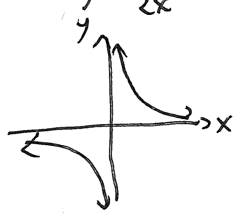
$r^2 \sin 2\theta = 1$

$2r^2\sin\theta\cos\theta = 1$

$2(r\cos\theta)(r\sin\theta) = 1$

$2xy = 1$

$y = \frac{1}{2x}$



11

$y = 7e^{4x} + 2e^{-3x}$ $\boxed{\text{Yes}}$:

$y' = 28e^{4x} - 6e^{-3x}$

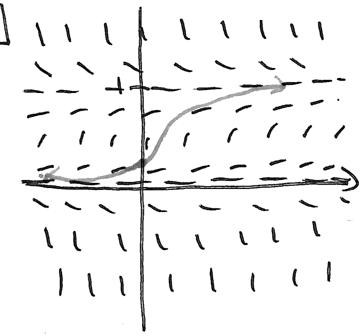
$y'' = 112e^{4x} + 18e^{-3x}$

$$y'' - y' - 12y = 112e^{4x} + 18e^{-3x} - 28e^{4x} + 6e^{-3x} - 12(7e^{4x} + 2e^{-3x})$$

$$= 112e^{4x} - 28e^{4x} - 84e^{4x} + 18e^{-3x} + 6e^{-3x} - 24e^{-3x}$$

$$= 0 \checkmark$$

12



$y' = y(1-y)$

eq. solutions: $y' = 0 = y(1-y)$

$y = 0,\ y = 1$

$y > 1$: $y' < 0$

$0 < y < 1$: $y' > 0$

$y < 0$: $y' < 0$

$y = 0$ is unstable $y = 1$ is stable

13

$\frac{dP}{dt} = -0.001P(P-400)$

a) carrying capacity is the stable equilibrium at $\boxed{P = 400}$

b) P' at $t=0$ is P' at $P=50$ b/c $P(0)=50$, so

$P'(0) = -0.001(50)(50-400)$

$= \boxed{17.5}$

c) Separate Vars:

$\frac{1}{P(P-400)} = -0.001\, dt$

PFD: $\frac{A}{P} + \frac{B}{P-400} = \frac{1}{P(P-400)}$

$A(P-400) + BP = 1$

$P=0$: $A = -\frac{1}{400}$

$P=400$: $B = \frac{1}{400}$

$\left(-\frac{1}{400}\frac{1}{P} + \frac{1}{400}\frac{1}{P-400}\right)dP = -\frac{1}{1000}\, dt$

$\int\left(\frac{1}{P} - \frac{1}{P-400}\right)dP = \int \frac{2}{5}\, dt$

$\ln|P| - \ln|P-400| = 0.4t + C$

$\ln\left|\frac{P}{P-400}\right| = 0.4t + C$: ~~t=0, P=50~~

$\frac{P}{P-400} = Ae^{0.4t}$

$t=0,\ P=50$: $\frac{50}{-350} = A = -\frac{1}{7}$.

13c cont.

We have: $\frac{P}{P-400} = -\frac{1}{7}e^{0.4t}$

$$P = -\frac{1}{7}e^{0.4t}P + \frac{400}{7}e^{0.4t}$$

$$P(1+\frac{1}{7}e^{0.4t}) = \frac{400}{7}e^{0.4t}$$

$$\boxed{P(t) = \frac{\frac{400}{7}e^{0.4t}}{1+\frac{1}{7}e^{0.4t}}}$$

50% of the carrying capacity is 200!

$$\frac{200}{200-400} = -\frac{1}{7}e^{0.4t}$$

$$-1 = -\frac{1}{7}e^{0.4t}$$

$$7 = e^{0.4t}$$

$$\boxed{\frac{10}{4}\ln 7 = t}$$

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14) $\frac{dy}{dx} = \frac{x\sin x}{y}$

Separate vars:

$$\int y\,dy = \int x\sin x\,dx \qquad \begin{array}{ll} u=x & dv=\sin x\,dx \\ du=dx & v=-\cos x \end{array}$$

$$\frac{1}{2}y^2 = -x\cos(x) + \int \cos(x)\,dx$$

$$= -x\cos(x) + \sin(x) + C$$

$$y^2 = -2x\cos(x) + 2\sin(x) + C$$

Apply IC:

$$(-1)^2 = -2\cdot 0 + 2\cancel{\sin(0)}^{0} + C$$

$$C = 1$$

$$y^2 = -2x\cos(x) + 2\sin(x) + 1$$

$$\boxed{y = -\sqrt{-2x\cos(x)+2\sin(x)+1}}$$

- negative square root b/c $y(0)=-1$.

15b) $t\frac{dy}{dt} = 2y+y^2$

$$\frac{dy}{2y+y^2} = \frac{1}{t}dt$$

PFD:

$$\frac{A}{y} + \frac{B}{2+y} = \frac{1}{(2+y)y}$$

$$A(2+y) + By = 1$$

$y=0$: $A=\frac{1}{2}$

$y=-2$: $B=-\frac{1}{2}$

$$\left(\frac{1}{2}\frac{1}{y} - \frac{1}{2}\frac{1}{2+y}\right)dy = \frac{1}{t}dt$$

$$\int\left(\frac{1}{y} - \frac{1}{2+y}\right)dy = \int \frac{2}{t}dt$$

$$\ln|y| - \ln|2+y| = 2\ln|t| + C$$

$$\ln\left|\frac{y}{2+y}\right| = \ln t^2 + C$$

$$\left|\frac{y}{2+y}\right| = e^C t^2$$

$$\frac{y}{2+y} = At^2$$

$$y = 2At^2 + yAt^2$$

$$y(1-At^2) = 2At^2$$

$$\boxed{y = \frac{2At^2}{1-At^2}}$$

16) From #6, $\frac{dy}{dx} = -\frac{\sin t}{\cos t}$ $\quad x = \cos^3 t,\ y = \sin^3 t$

a) $\frac{dy}{dx} = 0 = -\frac{\sin t}{\cos t}$ if $\sin(t) = 0 \iff t = 0, \pi, \ldots$

$t=0$: $\quad x = 1^3,\ y = 0^3$: $\boxed{(1,0)}$

$t=\pi$: $\quad x = (-1)^3,\ y = 0^3$: $\boxed{(-1,0)}$

b) $\frac{dy}{dx}$ is undefined if $\cos(t) = 0 \iff t = \frac{\pi}{2}, \frac{3\pi}{2}$

$t = \frac{\pi}{2}$: $x = 0^3,\ y = 1^3$: $\boxed{(0,1)}$

$t = \frac{3\pi}{2}$: $x = 0^3,\ y = (-1)^3$: $\boxed{(0,-1)}$

c) $\frac{dy}{dx} = -\tan t = -1 \iff \tan t = 1 \iff t = \frac{\pi}{4}, \frac{5\pi}{4}$

$t = \frac{\pi}{4}$: $x = \left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{2\sqrt{2}} = y$ $\quad \boxed{\left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)}$

$t = \frac{5\pi}{4}$: $x = \left(-\frac{1}{\sqrt{2}}\right)^3 = -\frac{1}{2\sqrt{2}} = y$ $\quad \boxed{\left(\frac{-1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)}$

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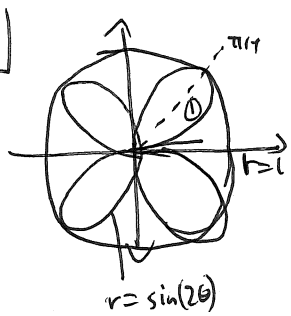
17)



$$A_{petal} = \int_0^{\pi/2} \frac{1}{2}(\sin(2\theta))^2\, d\theta$$
$$= \int_0^{\pi/2} \frac{1}{2} \sin^2(2\theta)\, d\theta$$
$$= \frac{1}{4}\int_0^{\pi/2} 1 - \cos(4\theta)\, d\theta$$
$$= \frac{1}{4}\left(\theta - \frac{1}{4}\sin(4\theta)\right)\Big|_0^{\pi/2}$$
$$= \frac{1}{4}\left(\frac{\pi}{2} - 0 + 0 - 0\right)$$
$$= \frac{\pi}{8}$$

$$A_{rose} = 4 \cdot A_{petal}$$
$$= \frac{\pi}{2}.$$

$$A_{circle} = \pi(1)^2 = \pi = 2 \cdot A_{rose}.$$