Separable Equations

Let's solve the IVP we drew slope field for yesterday,

$y' = -ty, \quad y(0) = 3.$

Now, $\frac{dy}{dt} = -ty$, so let's rewrite by grouping y's on the left and t's on the right:

So, $\frac{dy}{y} = -t\,dt$. Now integrate both sides

$$\int \frac{dy}{y} = \int -t\,dt$$

$$\ln|y| = -\tfrac{1}{2}t^2 + C \qquad \text{solve for } y.$$

$$e^{\ln|y|} = e^{-\frac{1}{2}t^2 + C}$$

$$|y| = e^{-\frac{1}{2}t^2} e^{C}$$

$y = Ae^{-\frac{1}{2}t^2}$ ← This is the general solution to the diff. eq. $y' = -ty$.

To find the solution that satisfies $y(0) = 3$, we substitute:

$3 = Ae^0 \Rightarrow A = 3.$

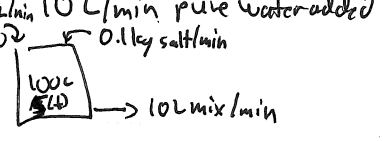
The solution to the IVP $y' = -ty, y(0) = 3$ is $\boxed{y(t) = 3e^{-\frac{1}{2}t^2}}$ ← particular solution

This method works on any separable first order diff. eq: $y' = f(x)\cdot g(y)$.

Ex: Find the general solution to $y\frac{dy}{dx} - x = 0$.

$$\frac{dy}{dx} = xy^{-1} \qquad y\,dy = x\,dx \qquad \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C \qquad y = \pm\sqrt{x^2 + 2C} \qquad y = \pm\sqrt{x^2 + C_2}$$

Ex: We can show that the salt in a 100L tank with 0.1kg salt/min and 10L/min pure water added and 10L/min mixed salt water out obeys the diff eq.



$$\frac{ds}{dt} = 0.1 - \frac{s}{10} = 0.1(1 - s) = -0.1(s - 1)$$

so, $\frac{ds}{s-1} = -0.1\,dt \Rightarrow \ln|s-1| = -0.1t + C, \quad |s-1| = e^{-0.1t}e^{C}, \quad s - 1 = Ae^{-0.1t}$ $\boxed{s(t) = 1 + Ae^{-0.1t}}$

Ex: Use separation of variables to find the solution to the IVP:

$\frac{dy}{dt} = te^{-y}, y(1) = 0 \qquad e^{y} = \frac{1}{2}t^2 + C \qquad e^{y} = \frac{1}{2}(t^2 + 1) \qquad \frac{t}{\frac{1}{2}(t^2+1)}$

$e^0 = \frac{1}{2} + C \qquad C = \frac{1}{2} \qquad \boxed{y = \ln\left(\frac{1}{2}(t^2+1)\right)}$

~~Ex: Find the general solution to $y' + 4xy^2 = 0$~~