MA 114 Worksheet #16: Average Value of a Function, Volumes I

1. Write down the equation for the average value of an integrable function $f(x)$ on $[a, b]$.

2. Find the average value of the following functions over the given interval.

   (a) $f(x) = x^3,\ [0, 4]$
   (b) $f(x) = x^3,\ [-1, 1]$
   (c) $f(x) = \cos(x),\ \left[0, \frac{\pi}{6}\right]$
   (d) $f(x) = \frac{1}{x^2+1},\ [-1, 1]$
   (e) $f(x) = \frac{\sin \pi/x}{x^2},\ [1, 2]$
   (f) $f(x) = e^{-nx},\ [-1, 1]$
   (g) $f(x) = 2x^3 - 6x^2,\ [-1, 3]$
   (h) $f(x) = x^n$ for $n \geq 0$, $[0, 1]$

3. If a solid has a cross-sectional area given by the function $A(x)$, what integral should be evaluated to find the volume of the solid?

4. Calculate the volume of the following solid. The base is a square, one of whose sides is the interval $[0, l]$ along the x-axis. The cross sections perpendicular to the x-axis are rectangles of height $f(x) = x^2$.

5. Calculate the volume of the following solid. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the y-axis are squares.

6. The base of a certain solid is the triangle with vertices at $(-10, 5)$, $(5, 5)$, and the origin. Cross-sections perpendicular to the y-axis are squares. Find the volume of the solid.

7. Calculate the volume of the following solid. The base is a circle of radius r centered at the origin. The cross sections perpendicular to the x-axis are squares.

8. Calculate the volume of the following solid. The base is the parabolic region $\{(x, y) \mid x^2 \leq y \leq 4\}$. The cross sections perpendicular to the y-axis are right isosceles triangles whose hypotenuse lies in the region.

9. Sketch the solid given by the integral

$$\pi \int_0^1 (y^2 + 1)^2 - 1 \, dy.$$

10. For each of the following, use disks or washers to find the an integral expression for the volume of the region. Evaluate the integrals for parts (a) and (d).

    (a) R is region bounded by $y = 1 - x^2$ and $y = 0$; about the x-axis.
    (b) R is region bounded by $y = \frac{1}{x}$, $x = 1$, $x = 2$, and $y = 0$; about the x-axis.
    (c) R is region bounded by $x = 2\sqrt{y}$, $x = 0$, and $y = 9$; about the y-axis.
    (d) R is region bounded by $y = 1 - x^2$ and $y = 0$; about the line $y = -1$.

(e) Between the regions in part (a) and part (d), which volume is bigger? Why?

(f) R is region bounded by $y = e^{-x}$, $y = 1$, and $x = 2$; about the line $y = 2$.

(g) R is region bounded by $y = x$ and $y = \sqrt{x}$; about the line $x = 2$.

11. Find the volume of the cone obtained by rotating the region under the segment joining $(0, h)$ and $(r, 0)$ about the y-axis.

12. The torus is the solid obtained by rotating the circle $(x-a)^2+y^2 = b^2$ around the y-axis (assume that $a > b$). Show that it has volume $2\pi^2 ab^2$.
[Hint: Draw a picture, set up the problem and evaluate the integral by interpreting it as the area of a circle.]