MathExcel Supplemental Problems #12: The Fundamental Theorem of Calculus

FTC Part 1. *If f is a continuous function on an interval I containing the real number a, and*

$$F(x) = \int_a^x f(t)\ dt,$$

then $F'(x) = f(x)$ for every x in I.

That is, F is an antiderivative of f on I.

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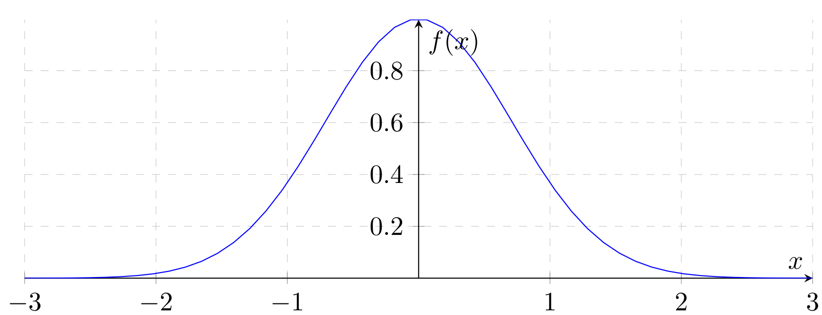
1. Let $F(x) = \int_1^x \sqrt{9+t^2}\ dt$. Find the slope of the tangent line to $F(x)$ at $x = 4$.
2. Let $g(x) = \int_2^x \sqrt{t}\ dt$. Find an equation of the tangent line to $g(x)$ at $x = 1$.
3. Differentiate each of the following functions.
   (a) $f(x) = \int_{-x}^x \sin(\theta)\ d\theta$
   (b) $\varphi(x) = \int_0^\pi 3t^2\ dt$
   (c) $\psi(x) = \int_{\pi/2}^{\sin(x)} \sqrt{1-z^2}\ dz$
   (d) $\tau(x) = \int_1^{\sqrt{x}} t\ dt$
4. Let $f(x) = \int_2^x t^2 - t - 2\ dt$.
   (a) Find $f'(x)$ and $f''(x)$.
   (b) Locate and classify all local extrema of $f(x)$.
   (c) Identify the absolute extrema of $f(x)$ on $[-2, 4]$.
   (d) Find the intervals of concavity of $f(x)$, and identify any inflection points.
5. Use FTC Part 1 to write down an antiderivative of $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. This is the standard normal curve (or bell curve) from statistics. It is a very important probability distribution that occurs frequently in nature. Since it is impossible to express the antiderivative as anything other than an integral, we denote it as $\mathrm{erf}(x)$, otherwise known as the *error function*.



FTC Part 2. *Let f be a continuous function on $[a, b]$. If F is any antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

Recall that previously, we evaluated this definite integral by taking a limit of a Riemann sum,

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\Delta x\right)\Delta x,$$

where $\Delta x = \dfrac{b-a}{n}$.

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6. Last Monday, we evaluated $\int_1^3 x^2 + 2x\ dx$ by evaluating the limit of the Riemann sum

$$\int_1^3 x^2 + 2x\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(1 + \frac{2i}{n}\right)^2 + 2\left(1 + \frac{2i}{n}\right)\right) \cdot \frac{2}{n}.$$

   Now evaluate this definite integral using FTC Part 2.

7. Sketch a graph to explain why we cannot use FTC Part 2 to evaluate the definite integral $\displaystyle\int_{-1}^{1} \frac{1}{x}\ dx$.

8. Evaluate each of the following definite integrals.

   (a) $\displaystyle\int_1^5 \frac{4-x}{x^2}\ dx$

   (b) $\displaystyle\int_0^\pi \cos(\theta)\ d\theta$

   (c) $\displaystyle\int_0^r 2\pi t\ dt$

   (d) $\displaystyle\int_a^a 4x^3 + 2\sqrt{x}\ dx$

   (e) $\displaystyle\int_0^2 3e^{5x}\ dx$

9. (a) Differentiate $F(x) = x\ln(x) - x$.

   (b) Evaluate $\displaystyle\int_1^e \ln(x)\ dx$.

10. Sketch a triangle in the coordinate plane, with vertices at $(0, 0)$, $(b, 0)$, and $(0, h)$. We can visualize the area of the triangle as the area under the line $f(x) = -\dfrac{h}{b}x + h$. Use a definite integral to recover the area of the triangle.

11. FTC Part 2 shows us that $\displaystyle\int_a^b f'(x)\ dx = f(b) - f(a)$. That means we can integrate the derivative f' to find the net change in f on $[a, b]$. Use this idea to determine how far a car travels from $t = 0$ to $t = 5$, if its velocity (in meters per second) during this time frame is given by $v(t) = t^2 - 3\sqrt{t}$.