Tropical Polynomials - Math Circle

November 24, 2019

1 Tropical Arithmetic

In tropical arithmetic, we define new addition and multiplication operations on the real numbers. The **tropical sum** of two numbers is their minimum:

$$x \oplus y = \min(x, y)$$

and the **tropical product** of two numbers is ther sum:

$$x \otimes y = x + y.$$

1. Which of the following properties hold in tropical arithmetic?

- **Addition is commutative:** $x \oplus y = y \oplus x$.

- **Addition is associative:** $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

- **An additive identity exists:** There is a real number r such that $x \oplus r = x$ for all real numbers x.

2. Let's expand our number set to include a tropical additive identity. What would be an appropriate name for this new "number"? Give appropriate definitions for the tropical sum and tropical product of this new number with a general real number x and with itself.

3. Which of the following properties hold in our completed tropical arithmetic?

- **Additive inverses exist:** For each number x, there is a number y such that $x \oplus y = r$, where r is the additive identity.

- **Multiplication is associative:** $(x \otimes y) \otimes z = x \otimes (y \otimes z)$.

- **Multiplication is commutative:** $x \otimes y = y \otimes x$.

- **A multiplicative identity exists:** There is a number i such that $x \otimes i = x$ for all numbers x.

- **Multiplicative inverses exist:** For each number x not equal to the additive identity r, there exists as number y such that $x \otimes y = i$, where i is the multiplicative identity.

- **Multiplication distributes over addition:** $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$.

4. Complete the tropical addition and multiplication tables below.

$\oplus$	1	2	3	4	∞
1					
2					
3					
4					
∞					

$\otimes$	0	1	2	3	4
0					
1					
2					
3					
4					

5. Expand and simplify $f(x) = (x \oplus 2)(x \oplus 3)$, where juxtaposition represents tropical multiplication. Then use your expression to compute $f(1)$ and $f(4)$.

2 Tropical Polynomials

First, let's review some facts about standard polynomials. A **polynomial** is an expression formed by adding and/or multiplying together numbers and copies of a variable x. Every polynomial can be written in the form

$$a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$$

for some nonnegative integer n (the **degree**) and **coefficients** $a_n, \ldots, a_2, a_1, a_0$.

It follows from the **Fundamental Theorem of Algebra** that any non-constant polynomial with real coefficients can be written as a product of polynomials of degree 1 or 2 with **real coefficients**. For example

$$x^5 + 8x^4 + 17x^3 - 2x^2 - 64x - 160 = (x^2 + 2x + 5)(x - 2)(x + 4)^2.$$

Over the complex numbers, every polynomial factors completely into polynomials of degree 1 with **complex coefficients**. For the example above,

$$x^5 + 8x^4 + 17x^3 - 2x^2 - 64x - 160 = (x + 1 - 2i)(x + 1 + 2i)(x - 2)(x + 4)^2.$$

The factors can be determined by computing the **roots** (or the "zeros") of the polynomial. The polynomial above has roots

$$-1 + 2i, -1 - 2i, 2, -4, -4.$$

We say that the root -4 has **multiplicity** 2.

There is a quadratic formula for determining the roots of a polynomial of degree 2, along with cubic and quartic formulas for degrees 3 and 4. However, there is no similar formula for finding the roots of a polynomial of degree 5 or higher. For these polynomials, we usually have to settle for approximate roots, found by a computer.

A **tropical polynomial** is an expression formed by tropically adding and/or tropically multiplying tropical numbers (i.e. real numbers or ∞) and copies of a variable x. Every tropical polynomial can be written in the form

$$(a_n \otimes x^n) \oplus \ldots \oplus (a_2 \otimes x^2) \oplus (a_1 \otimes x) \oplus (a_0)$$

for some nonnegative integer n and **coefficients** $a_n, \ldots, a_2, a_1, a_0$. Note that the exponents here represent repeated *tropical* multiplication! This is a bit unwieldy to write, so usually we represent tropical multiplication by juxtaposition in the usual manner. So the expression above is often written

$$a_n x^n \oplus \ldots \oplus a_2 x^2 \oplus a_1 x \oplus a_0.$$

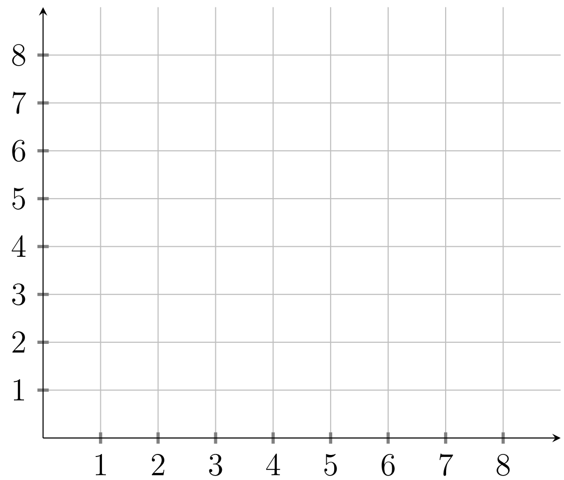
Be careful which type of operations you are using!

Questions:

- *Can tropical polynomials always be factored completely into polynomials of degree 1?*
- *Is there a tropical quadratic formula for finding the roots of quadratic polynomials?*

2.1 Tropical Quadratic Polynomials

6. Draw a precise graph of the tropical polynomial $f(x) = x^2 \oplus 1x \oplus 4$. You may find it helpful to first rewrite the tropical polynomial as an expression involving standard operations using the definitions of $\oplus$ and $\otimes$.



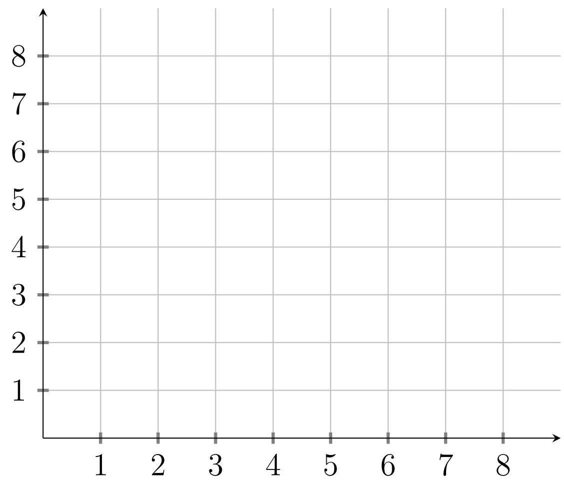
Now, try to factor the tropical polynomial $x^2 \oplus 1x \oplus 4$ into linear factors. In other words, find numbers r and s such that

$$x^2 \oplus 1x \oplus 4 = (x \oplus r) \otimes (x \oplus s).$$

These numbers r and s are called the **roots** of the tropical polynomial. (Note that we use $x \oplus r$ and $x \oplus s$ because we do not have a tropical subtraction.)

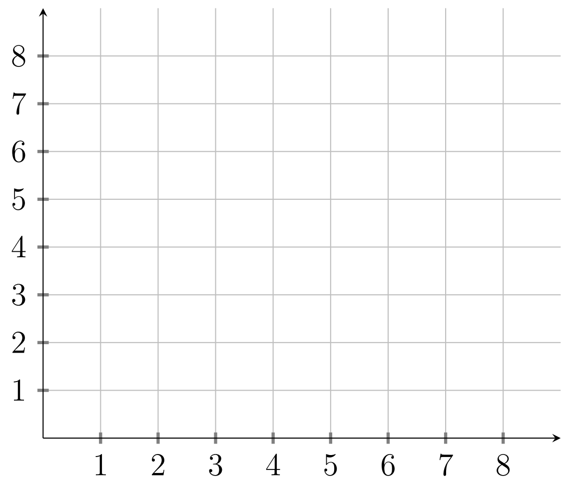
Do you notice any relationship between the graph and the factorization? Can you see the roots in the graph?

7. Graph $f(x) = -2x^2 \oplus x \oplus 8$, and then find a factorization of $f(x)$ in the form $a(x \oplus r)(x \oplus s)$. Can you see the roots r and s in the graph? How are the roots related to the coefficients of $f(x)$?

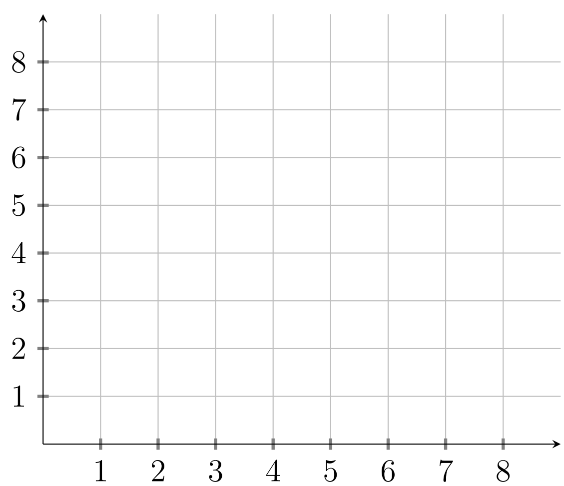


8. Find a tropical polynomial $f(x)$ with a value of 7 for all sufficiently large x and with roots 4 and 5.

9. Graph $f(x) = 1x^2 \oplus 3x \oplus 5$, and then find a factorization in the form $f(x) = a(x \oplus r)(x \oplus s)$. How is this graph different from the previous ones? How is this factorization different from the others? How are the roots related to the coefficients of $f(x)$?



10. Graph $f(x) = 2x^2 \oplus 4x \oplus 4$. Find a factorization in the form $f(x) = a(x \oplus r)(x \oplus s)$ or argue that one does not exist.



11. Can you find a tropical polynomial which has the same graph as $f(x) = 2x^2 \oplus 4x \oplus 4$, but which can be factored into linear tropical polynomials?

The ***Tropical Fundamental Theorem of Algebra*** *says that, for every tropical polynomial $f(x)$, there is a unique tropical polynomial $\bar{f}(x)$ with the same graph (and hence determining the same function) which can be factored into linear factors. We sometimes say "the roots of $f(x)$" when we really mean "the roots of $\bar{f}(x)$."*

12. If $f(x) = ax^2 \oplus bx \oplus c$, then $\bar{f}(x) = ax^2 \oplus Bx \oplus c$ for some B. Find a formula for B in terms of a, b, and c. There are two different cases to consider.

13. State a tropical quadratic formula in terms of a, b, c for the roots r, s of a tropical polynomial $f(x) = ax^2 \oplus bx \oplus c$ (that is, the roots of the corresponding $\bar{f}$). There are again two separate cases to consider.