MATH 2551 Guided Notes

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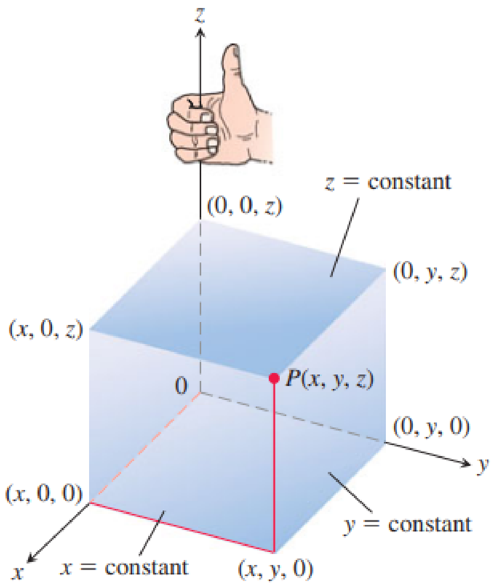
Fall 2025

Day 1 - Course Introduction and Cross Products

Pre-Lecture

Section 12.1: Three-Dimensional Coordinate Systems



Day 1 - Lecture

Daily Announcements & Reminders:

Goals for Today:

Sections 12.1, 12.3, 12.4

- Set classroom norms
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Icebreaker on PollEverywhere

Introduction to the Course

Purposes:

- Pre-Lecture: Get in math headspace, first exposure to the day's topic
- Lecture: Fill in the rest of the new ideas for the week
- Studio: Guided group practice with new ideas under supervision, develop independence
- WeBWorK: Basic skills practice for the topics
- LT Practice: Extra practice problems for assessments
- Quizzes, Checkpoints, & Exams: Demonstrate learning & get feedback

Feedback loops: Lecture -> practice -> studio -> practice -> assessment -> practice -> assessment -> ...

Important Canvas Items: Back to Canvas!

Big Idea: Extend differential & integral calculus.

What are some key ideas from these two courses?

Differential Calculus	Integral Calculus

Poll

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

Example 1. What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation

$$x^2 + y^2 = 1?$$

Section 12.3/4: Dot & Cross Products

Definition 2. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \dots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \dots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = \rule{8cm}{0.4pt}$$

This product tells us about ____.

In particular, two vectors are **orthogonal** if and only if their dot product is _______.

Example 3. Are $\mathbf{u} = \langle 1, 1, 4 \rangle$ and $\mathbf{v} = \langle -3, -1, 1 \rangle$ orthogonal?

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.

1.

2.

Definition 4. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ <u>in $\mathbb{R}^3$</u> is

$$\mathbf{u} \times \mathbf{v} = \underline{\hspace{8cm}}$$

Example 5. Find $\langle 1, 2, 1\rangle \times \langle 3, -1, 0\rangle$.

Day 2 - Lines, Planes, and Quadrics

Pre-Lecture

12.5: Lines

Lines in $\mathbb{R}^2$, a new perspective:

Example 6. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.

Day 2 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **G1: Lines and Planes.** I can describe lines using the vector equation of a line. I can describe planes using the general equation of a plane. I can find the equations of planes using a point and a normal vector. I can find the intersections of lines and planes. I can describe the relationships of lines and planes to each other. I can solve problems with lines and planes.
- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.

Goals for Today:

Sections 12.5-12.6

- Apply the cross product to solve problems
- Learn the equations that describe lines, planes, and quadric surfaces in $\mathbb{R}^3$
- Solve problems involving the equations of lines and planes
- Sketch quadric surfaces in $\mathbb{R}^3$

Example 7. Find a set of parametric equations for the line through the point $(1, 10, 100)$ which is parallel to the line with vector equation

$$\mathbf{r}(t) = \langle 1, 4, -3 \rangle t + \langle 0, -1, 1 \rangle$$

Section 12.5 Planes

Planes in $\mathbb{R}^3$

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction $\mathbf{n}$, called the *normal vector*.

Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

In $\mathbb{R}^3$, a pair of lines can be related in three ways:

parallel **skew** **intersecting**

On the other hand, a pair of planes can be related in just two ways:

parallel **intersecting**

Example 8. [Poll] The lines

$$\ell_1(t) = \langle 1, 1, 1\rangle t + \langle 0, 0, 1\rangle$$

and

$$\ell_2(t) = \langle 2, 2, 2\rangle t + \langle 0, 0, 1\rangle$$

are related in what way?

Example 9. [Poll] The lines

$$\ell_1(t) = \langle 1, 1, 1\rangle t + \langle 0, 0, 1\rangle$$

and

$$\ell_2(t) = 2t\mathbf{i} - t\mathbf{j} - t\mathbf{k}$$

are related in what way?

Example 10. Consider the planes $y - z = -2$ and $x - y = 0$. Show that the planes intersect and find an equation for the line of intersection of the planes.

Day 3 - Vector-Valued Functions & Calculus

Pre-Lecture

Section 13.1: Vector-Valued Functions

Last week, we used functions like

$$\ell(t) = \langle 2t+1, 3-t, t-1 \rangle, \quad -\infty \leq t \leq \infty$$

to produce lines in $\mathbb{R}^2$ and $\mathbb{R}^3$.

This is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

What happens when we change the component functions to be non-linear?

Given a fixed curve C in space, producing a vector-valued function $\mathbf{r}$ whose graph is C is called ________________ the curve C, and $\mathbf{r}$ is called a ____________________ of C.

Day 3 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.
- **G2: Calculus of Curves.** I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.
- **G5: Parameterization.** I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.

Goals for Today:

Sections 12.6, 13.1, 13.2

- Learn the equations that define quadric surfaces in $\mathbb{R}^3$
- Use technology to plot quadric surfaces
- Introduce vector-valued functions
- Plot vector-valued functions and construct them from a graph
- Compute limits, derivatives, and tangent lines for vector-valued functions

Section 12.6 Quadric Surfaces

Definition 11. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

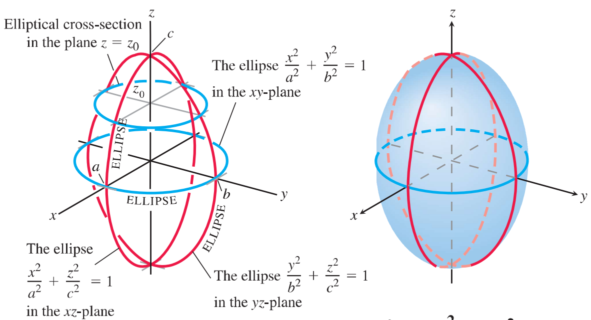
The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections. We'll also make heavy use of 3d graphing technology to get comfortable with these new objects.

Example 12. Use a 3d graphing utility to plot the quadric surface

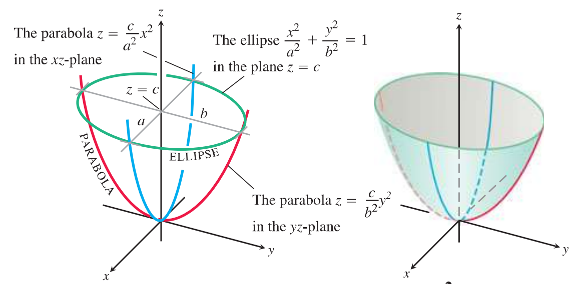
$$z = x^2 + y^2.$$

This surface is called a _________________________, because it has two coordinate directions with cross sections that are __________________ and one with cross sections that are ____________.

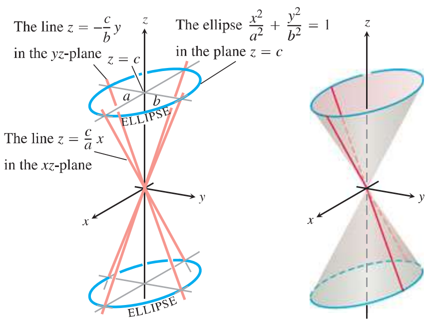
TABLE 12.1 Graphs of Quadric Surfaces



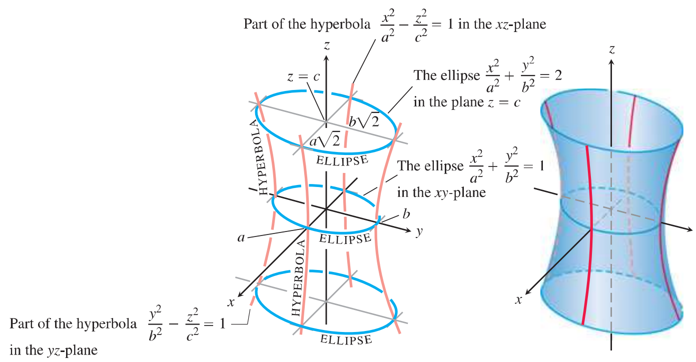
ELLIPSOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



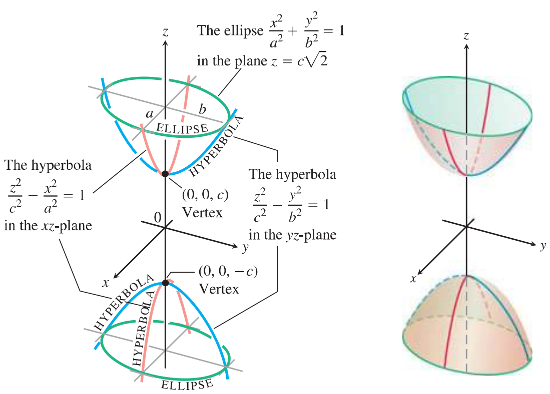
ELLIPTICAL PARABOLOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



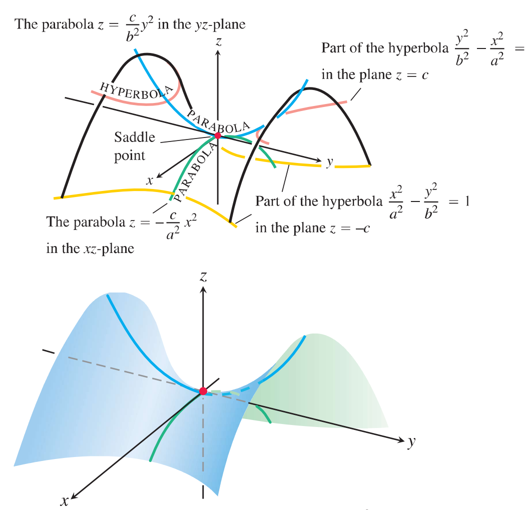
ELLIPTICAL CONE $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\quad \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

Example 13. [Poll] Which of the following are quadric surfaces?

1. A line
2. A sphere
3. A circle
4. An ellipse
5. The set of points (x, y, z) which solve $x^2 + y^2 - 3 = 0$.
6. The set of points (x, y, z) which solve $x^2 - y^2 - z^2 = 4$.

Example 14. [Poll] Classify the quadric surface $x + y^2 - z^2 - 3 = 0$.

Example 15. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?

Check your intuition

Section 13.1: Calculus of Vector-Valued Functions

Unifying theme: Do what you already know, componentwise.

This works with limits:

Example 16. Compute $\lim_{t\to e}\langle t^2, 2, \ln(t)\rangle$.

And with derivatives:

Example 17. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$, find $\mathbf{r}'(t)$.

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives ________________________________
- $\|\mathbf{r}'(t)\|$ gives ________________________________
- $\mathbf{r}''(t)$ gives ________________________________

Let's see this graphically

Example 18. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

Day 4 - Geometry of Curves

Pre-Lecture

Section 13.3: Arc Length

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Our next goal is to be able to measure distance traveled or arc length.

Motivating problem: Suppose the position of a fly at time t is

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle,$$

where $0 \leq t \leq 2\pi$.

How far does the fly travel from $t = 0$ to $t = \pi$?

Definition 19. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ from __________ to __________ that is traced out exactly once is

$$L = \underline{\hspace{6cm}}$$

Day 4 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **G2: Calculus of Curves.** I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.
- **G3: Geometry of Curves.** I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.

Goals for Today:

Sections 13.3, 13.4

- Compute integrals of vector-valued functions and solve initial value problems
- Solve initial value problems
- Compute arc lengths of curves using parameterizations
- Define and compute arc-length parameterizations

Example 20 (Poll)**. Warmup.** Compute the tangent line to

$$\mathbf{r}(t) = \langle 3t^3, \sin(t), t^2 + 1 \rangle, t \in \mathbb{R}$$

at the point $(0, 0, 1)$.

Working componentwise also works with integrals:

Example 21. Find $\int_0^1 \langle t, e^{2t}, \sec^2(t) \rangle \, dt$.

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 22. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t} \rangle \ m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, use calculus to reconstruct his flight path.

Example 23 (Poll). **T/F:** The function

$$\mathbf{r}(t) = \langle \cos(t), 2t^2, 4t + 2\rangle, t \in \mathbb{R}$$

is a solution of the IVP

$$\mathbf{r}'(t) = \langle -\sin(t), 4t, 4\rangle, \quad \mathbf{r}(0) = \langle 0, 0, 2\rangle$$

Example 24. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(2, 4, 8)$.

Example 25. Find the distance traveled by a particle moving along the path

$$\mathbf{r}(t) = \langle \ln(t), \sqrt{2}t, \frac{1}{2}t^2\rangle, \qquad t > 0$$

from $t = 1$ to $t = 2$.

Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

$$s(t) = \underline{\hspace{6cm}}$$

We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.

Example 26. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \leq t \leq 2\pi$.

Example 27 (Poll)**. T/F:** The parameterization $\mathbf{r}(t) = \langle t, t^2\rangle, t \in \mathbb{R}$ is an arc length parameterization of the parabola $y = x^2$.

Day 5 - Geometry of Curves Part II

Pre-Lecture

Section 13.4 - Curvature

The next idea we are going to explore is the <u>curvature</u> of a curve in space along with two vectors that orient the curve.

First, we need the **unit tangent vector**, denoted $\mathbf{T}(s)$: ________________

This lets us define the **curvature**, $\kappa(s) =$ ________________

Question: In which direction is $\mathbf{T}$ changing?

This is the direction of the **principal unit normal**, $\mathbf{N}(s) =$________________

Day 5 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **G3: Geometry of Curves.** I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.
- **G5: Parameterization.** I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.

Goals for Today: Section 13.4

- Define, interpret, and compute the curvature of a curve
- Compute the unit tangent and principal unit normal vectors of a curve
- Extend the set of curves that we can parameterize

Section 13.4: Computing Curvature, Tangent, and Normal Vectors

Example 28. Last time, we found an arc length parameterization of the circle of radius 4 centered at $(0, 0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \leq s \leq 8\pi.$$

Use this to find $\mathbf{T}(s)$, $\kappa(s)$, and $\mathbf{N}(s)$.

We said that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) =$ ________________
- $\mathbf{N}(t) =$ ________________
- $\kappa(t) =$ ________________ or ________________________

Example 29. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1 \rangle$.

Parameterizing Curves

Let's return to parameterizing curves. We have seen a few examples so far and want to solidify our understanding of a few more classes of curves that we will work with.

- **Line segments:** If P and Q are points in $\mathbb{R}^3$, then a parameterization of the line segment from P to Q is given by

- **Graphs of functions of one variable:** If $y = f(x)$ is a function of one variable, then a parameterization of its graph is given by

- **Circles:** A circle of radius r centered at (h, k) in $\mathbb{R}^2$ can be parameterized by

  .

- **Ellipses:** An ellipse with radius in the x-direction a and radius in the y-direction b centered at (h, k) in $\mathbb{R}^2$ can be parameterized by

Intersections of Surfaces:

If a curve is specified as the intersection of two surfaces, our parameterization will depend on the equations of the surfaces. Often we can use this by eliminating variables until we can use one of the basic forms above.

Example 30. Find a parameterization of the curve of intersection of the surfaces $z = x^2 + y^2$ and $z = 4 - y^2$.

Orientation:

The orientation of a curve is determined by the direction in which it is traced as the parameter increases. For example, if a curve is parameterized by $\mathbf{r}(t)$ for $a \le t \le b$, then the orientation is from $\mathbf{r}(a)$ to $\mathbf{r}(b)$.

Example 31 (Poll)**.** Let C be a curve parameterized by $\mathbf{r}(t)$ from $a \le t \le b$. Select all of the true statements below.

(a) $\mathbf{r}(t+4)$ for $a \le t \le b$ is also a parameterization of C with the same orientation

(b) $\mathbf{r}(2t)$ for $a/2 \le t \le b/2$ is also a parameterization of C with the same orientation

(c) $\mathbf{r}(-t)$ for $a \le t \le b$ is also a parameterization of C with the opposite orientation

(d) $\mathbf{r}(-t)$ for $-b \le t \le -a$ is also a parameterization of C with the opposite orientation

(e) $\mathbf{r}(b-t)$ for $0 \le t \le b-a$ is also a parameterization of C with the opposite orientation

Day 6 - Functions of Multiple Variables

Pre-Lecture

Section 14.1: Functions of Multiple Variables

Definition 32. A __________________________________ is a rule that assigns to each __________ of real numbers (x, y) in a set D a __________________ denoted by $f(x, y)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^2$$

Example 33. Three examples are

$$f(x, y) = x^2 + y^2, \qquad g(x, y) = \ln(x + y), \qquad h(x, y) = h(x, y) = \sqrt{4 - x^2 - y^2}.$$

Example 34. Find the largest possible domains of f, g, and h.

Day 6 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.

Goals for Today:

Section 14.1

- Give examples of functions of multiple variables
- Find the domain of functions of two variables
- Introduce and sketch traces and contours of functions of two variables
- Use technology to graph functions of two variables
- Find level surfaces of functions of three variables

In the pre-lecture video, we discussed the domains of the functions $f(x, y) = x^2 + y^2, g(x, y) = \ln(x + y)$, and $h(x, y) = \sqrt{4 - x^2 - y^2}$.

Definition 35. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Here are the graphs of the three functions above.

Example 36. Suppose a small hill has height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x, y). How could we draw a picture that represents the hill in 2D?

In 3D, it looks like this.

Definition 37. The ____________ (also called ____________________) of a function f of two variables are the curves with equations ________________, where k is a constant (in the range of f). A plot of ____________ for various values of z is a ____________________(or ____________________________).

Some common examples of these are:

-
-
-

Example 38. Use technology to create a contour diagram of $f(x, y) = x^2 - y^2$.

What do we notice about the contours?

Example 39. Student work: Use technology to create a contour diagram of $g(x, y) = \sqrt{16 - 4x^2 - y^2}$.

CalcPlot3D

What do you notice about your contours?

Definition 40. The ________________ of a surface are the curves of ________________________ of the surface with planes parallel to the ____________________________.

Example 41. Find the traces of the surface $z = x^2 - y^2$. Can you see these in the graph produced by CalcPlot3D?

Example 42. Use the graph of the portion of $z = f(x, y) = 4 - 2x - y^2$ in the first quadrant to identify and understand all of the traces and contours.

Definition 43. A __________________________________ is a rule that assigns to each __________ of real numbers (x, y, z) in a set D a _________________ denoted by $f(x, y, z)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 44. Describe the domain of the function $f(x, y, z) = \dfrac{1}{4 - x^2 - y^2 - z^2}$.

Example 45. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.

Day 7 - Derivatives & Linear Approximation

Pre-Lecture

General Partial Derivatives

Definition 46. If f is a function of two variables x and y, its ____________________ are the functions f_x and f_y defined by

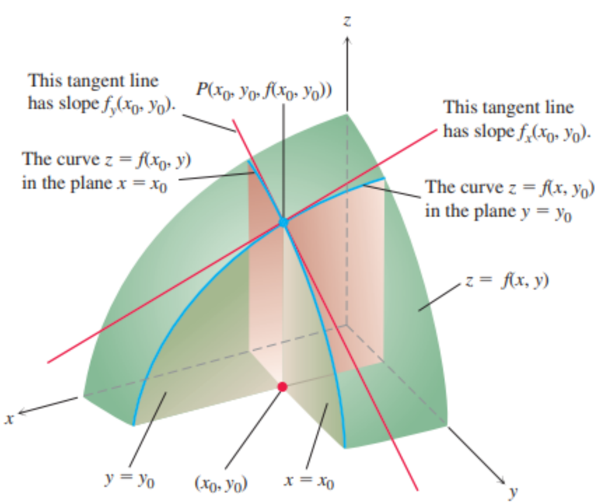
$$f_x(x,y) = \lim_{h\to 0} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \qquad f_y(x,y) = \lim_{h\to 0} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

Notations:

Example 47. Find the partial derivatives f_x and f_y for

$$f(x,y) = 5x^2 + 2xy + 3y^3.$$

Interpretations:



Day 7 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **D1: Computing Derivatives.** I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.
- **A2: Interpreting Derivatives.** I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.
- **D2: Tangent Planes and Linear Approximations.** I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.

Goals for Today:

Section 14.3, 14.6

- Learn how to compute partial derivatives of functions of multiple variables
- Learn how to compute higher-order partial derivatives
- Understand Clairaut's theorem
- Define the total derivative
- Learn how to find a linear approximation of a differentiable function of multiple variables

Example 48. Find the partial derivatives of the functions below.

Poll

(a) $f(x, y) = 3x^2y + x - 2y$

(b) $g(x, y) = \sqrt{5x - y}$

Question: How would you define the second partial derivatives?

Example 49. Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} of the function $f(x, y) = \sqrt{5x - y}$

What do you notice about f_{xy} and f_{yx} in the previous example?

Theorem 50 (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions f, f_x, f_y, f_{xy}, f_{yx} are all continuous on D, then*

Example 51. What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

Example 52. How many rates of change should the function $f(s, t) = \begin{bmatrix} s^2 + t \\ 2s - t \\ st \end{bmatrix}$ have? Compute them.

Day 8 - Chain Rule and Directional Derivatives

Pre-Lecture

Section 14.4 - Chain Rule

Recall the Chain Rule from single variable calculus:

Similarly, the **Chain Rule** for functions of multiple variables says that if $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable functions then

$$D(f(g(\mathbf{x}))) = Df(g(\mathbf{x}))Dg(\mathbf{x}).$$

Example 53. Suppose we are walking on our hill with height $h(x, y) = 4-\frac{1}{4}x^2-\frac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t+1, 2-t^2\rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

Day 8 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **D1: Computing Derivatives.** I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.
- **D2: Tangent Planes and Linear Approximations.** I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.

Goals for Today:

Sections 14.4-14.5

- Learn the Chain Rule for derivatives of functions of multiple variables
- Be able to compute implicit partial derivatives
- Introduce the directional derivative of a function of multiple variables

Total Derivatives

How might we **organize** derivative information?

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1, \dots, x_n) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{bmatrix}$,

we have ________ inputs, ________ output, and ________ partial derivatives, which we can use to form the **total derivative**.

This is a ____________ map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an ____________________, with one column per input and one row per output.

It has the formula $Df_{ij} =$

Example 54. Find the total derivatives of each function:

(a) $f(x) = x^2 + 1$

(b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

(c) $f(x, y) = \sqrt{5x - y}$

(d) $f(x, y, z) = 2xyz - z^2y$

(e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$

Example 55. Use the Chain Rule to compute the total derivative of the composite function $h(x, y) = f(g(x, y))$ at $(1, 0)$ when

$$g(x, y) = x^3 + 2xy - y^2 \qquad f(t) = \begin{bmatrix} t^3 + 2t \\ t - 4 \end{bmatrix}.$$

Example 56. Use the Chain Rule to compute the rate of change of the composite function $h(t) = g(f(t))$ at $t = 2$, where

$$f(t) = \begin{bmatrix} \frac{1}{4}t^2 + 2t \\ \sin(\pi t) - t \\ t + 1 \end{bmatrix} \qquad g(x, y, z) = x^2yz - y - z.$$

Poll

Example 57. Suppose that $W(s,t) = F(u(s,t), v(s,t))$, where F, u, v are differentiable functions and we know the following information.

$$\begin{aligned} u(1,0) &= 2 & v(1,0) &= 3 \\ u_s(1,0) &= -2 & v_s(1,0) &= 5 \\ u_t(1,0) &= 6 & v_t(1,0) &= 4 \\ F_u(2,3) &= -1 & F_v(2,3) &= 10 \end{aligned}$$

Find $W_s(1,0)$ and $W_t(1,0)$.

Application to Implicit Differentiation: If $F(x, y, z) = c$ is used to *implicitly* define z as a function of x and y, then the chain rule says:

Example 58. Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the sphere $x^2 + y^2 + z^2 = 4$.

Day 9 - Directional Derivatives, Gradients, Tangent Planes

Pre-Lecture

Section 14.5 - Directional Derivatives

Definition 59. The ______________________________ of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(\mathbf{p}) =$$

if this limit exists.

Note that $D_{\mathbf{i}} f =$ $\qquad D_{\mathbf{j}} f =$ $\qquad D_{\mathbf{k}} f =$

In practice, we want to avoid using this limit definition!

Note: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

$$D_{\mathbf{u}} f(\mathbf{p}) =$$

Example 60. Compute the rate of change of $f(x, y) = e^{xy}$ at the point $(1, 2)$ in the direction $\mathbf{u} = \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle$.

Day 9 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **D1: Computing Derivatives.** I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.
- **D2: Tangent Planes and Linear Approximations.** I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.
- **A2: Interpreting Derivatives.** I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.

Goals for Today:

Sections 14.5-14.6

- Learn to compute the rate of change of a multivariable function in any direction
- Investigate the connection between the gradient vector and level curves/surfaces
- Discuss tangent planes to surfaces, how to find them, and when they exist

Section 14.5: Gradients

Definition 61. If $f : \mathbb{R}^n \to \mathbb{R}$, then the ________________ of f at $\mathbf{p} \in \mathbb{R}^n$ is the vector function ____ (or ________) defined by

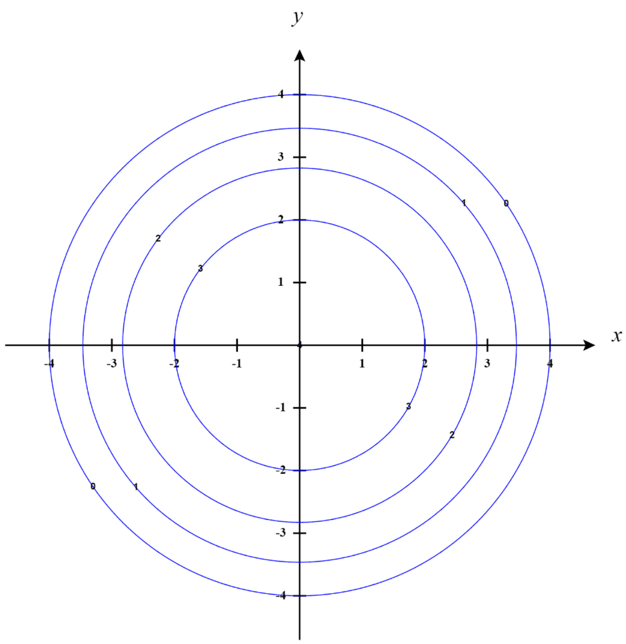
$$\nabla f(\mathbf{p}) =$$

Example 62. Find the gradient vector and the directional derivative of each function at the given point $\mathbf{p}$ in the direction of the given vector $\mathbf{u}$.

(a) $f(x, y) = \ln(x^2 + y^2), \mathbf{p} = (-1, 1), \mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

(b) $g(x, y, z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = (1, 2, 1)$, $\mathbf{u}$ the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

Example 63. If $h(x,y) = 4 - \dfrac{1}{4}x^2 - \dfrac{1}{4}y^2$, the contour map is given below. Find and draw ∇h on the diagram at the points $(2,0), (0,4)$, and $(-\sqrt{2}, -\sqrt{2})$. At the point $(2,0)$, compute $D_{\mathbf{u}}h$ for the vectors $\mathbf{u}_1 = \mathbf{i}, \mathbf{u}_2 = \mathbf{j}, \mathbf{u}_3 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.



Note that the gradient vector is ________________ to level curves.

Similarly, for $f(x,y,z)$, $\nabla f(a,b,c)$ is ____

Section 14.6: Linear Approximation

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \dots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $f(x) = f(a) + f'(a)(x - a)$.

Definition 64. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \dots, a_n)$ is

$$L(\mathbf{x}) =$$

Example 65. Find the linearization of the function $f(x,y) = \sqrt{5x-y}$ at the point $(1,1)$. Use it to approximate $f(1.1, 1.1)$.

Question: What do you notice about the equation of the linearization?

Day 10 - Optimization: Local & Global

Pre-Lecture

Section 14.7 - Local Extreme Values

Last time: If $f(x,y)$ is a function of two variables, we said $\nabla f(a,b)$ points in the direction of greatest change of f.

What does it mean if $\nabla f(a,b) = \langle 0,0 \rangle$?

Definition 66. Let $f(x,y)$ be defined on a region containing the point (a,b). We say

- $f(a,b)$ is a ________________________ value of f if $f(a,b)$ ____ $f(x,y)$ for all domain points (x,y) in a disk centered at (a,b)
- $f(a,b)$ is a ________________________ value of f if $f(a,b)$ ____ $f(x,y)$ for all domain points (x,y) in a disk centered at (a,b)

In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $z = x^2 - y^2$ (a hyperbolic paraboloid!) near $(0,0)$.

This is called a ________________________

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e.

Definition 67. If $f(x, y)$ is a function of two variables, a point (a, b) in the domain of f with $Df(a, b) =$ ______________ or where $Df(a, b)$ ______________________ is called a ______________________ of f.

Example 68. Find the critical points of the function $f(x, y) = x^3 + y^3 - 3xy$.

Day 10 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **D3: Optimization.** I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.

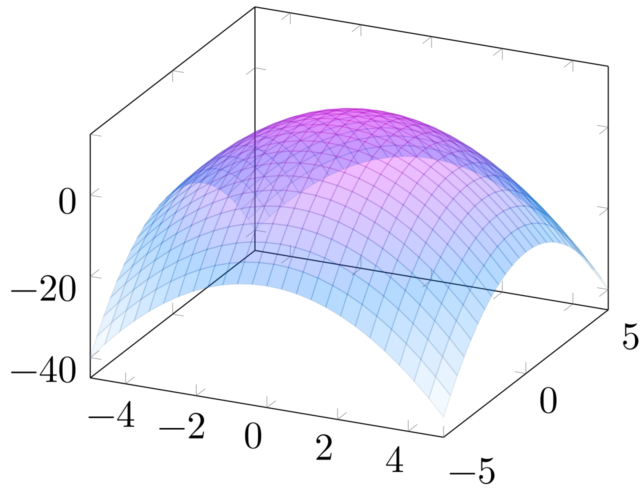
Goals for Today:

Section 14.7

- Discuss tangent planes to surfaces, how to find them, and when they exist
- Define local & global extreme values for functions of two variables
- Learn how to find local extreme values for functions of two variables
- Learn how to classify critical points for functions of two variables

Section 14.6 - Tangent Planes

Suppose S is a surface with equation $F(x, y, z) = k$. How can we find an equation of the tangent plane of S at $P(x_0, y_0, z_0)$?



$x^2 + y^2 + z = 10, P = (-1, 3, 0)$

Example 69. Find the equation of the tangent plane at the point $(-2, 1, -1)$ to the surface given by

$$z = 4 - x^2 - y$$

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

This should look familiar: it's ________________________________

Example 70. [Poll] Which of the following functions have a critical point at $(0,0)$?

$$f(x,y) = 3x+y^3+2y^2 \quad g(x,y) = \cos(x)+\sin(y) \quad h(x,y) = \frac{4}{x^2+y^2} \quad k(x,y) = x^2+y^2$$

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix** of $f(x,y)$ at (a,b):

$$Hf(a,b) =$$

Theorem 71 (2nd Derivative Test)**.** *Suppose (a,b) is a critical point of $f(x,y)$ and f has continuous second partial derivatives. Then we have:*

- *If* $\det(Hf(a,b)) > 0$ *and* $f_{xx}(a,b) > 0$, $f(a,b)$ *is a local minimum*
- *If* $\det(Hf(a,b)) > 0$ *and* $f_{xx}(a,b) < 0$, $f(a,b)$ *is a local maximum*
- *If* $\det(Hf(a,b)) < 0$, f *has a saddle point at* (a,b)
- *If* $\det(Hf(a,b)) = 0$, *the test is inconclusive.*

[Advanced] More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at $\mathbf{p}$ then

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, f is concave up in every direction from $\mathbf{p}$ and so has a local minimum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, f is concave down in every direction from $\mathbf{p}$ and so has a local maximum at $\mathbf{p}$.
- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, f is concave up in some directions from $\mathbf{p}$ and concave down in others, so has neither a local minimum or maximum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, f may have either a local minimum or neither at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, f may have either a local maximum or neither at $\mathbf{p}$.

Example 72. Classify the critical points of $f(x, y) = x^3 + y^3 - 3xy$ from Example 68.

Example 73. Find and classify the critical points of $f(x, y) = x^2y + y^2 + xy$.

Two Local Maxima, No Local Minimum: The function $g(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2 + 2$ has two critical points, at $(-1,0)$ and $(1,2)$. Both are local maxima, and the function never has a local minimum!

Day 11 - Optimization: Global & Constrained

Pre-Lecture

Section 14.8 - Constrained Optimization

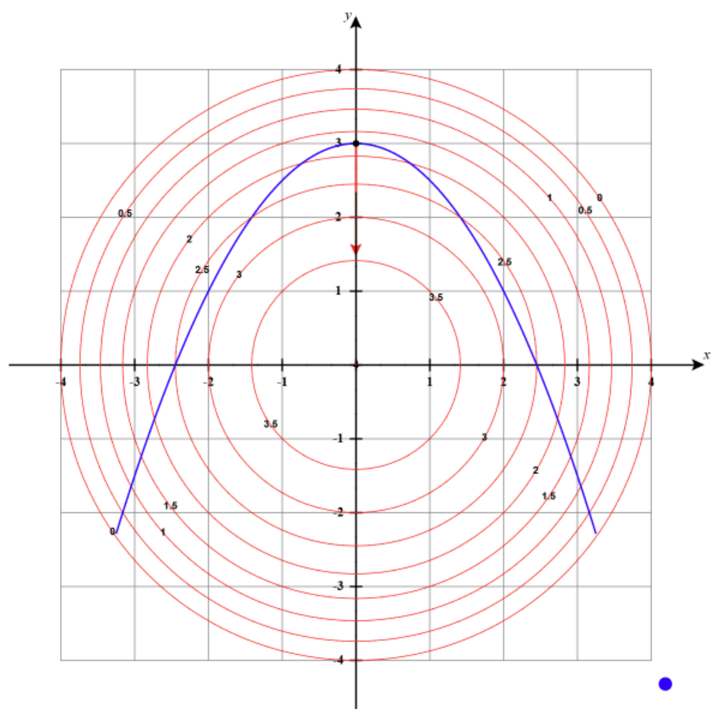
Goal: Maximize or minimize $f(x, y)$ subject to a *constraint*, $g(x, y) = c$.

Example 74. A new hiking trail has been constructed on the hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the xy-plane. What is the highest point on the hill on this path?

Objective function:

Constraint equation:

Goal: Locate points with no rate of change along the constraint curve



Day 11 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **D3: Optimization.** I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.

Goals for Today:

Sections 14.7, 14.8

- Learn how to find global extreme values on a closed & bounded domain
- Find global extreme values of continuous functions of two variables on closed & bounded domains
- Apply the method of Lagrange multipliers to find extreme values of functions of two or more variables subject to one or more constraints

A global maximum of $f(x, y)$ is like a local maximum, except we must have $f(a, b) \geq f(x, y)$ for **all** (x, y) in the domain of f. A global minimum is defined similarly.

14.7 - Applying Extreme Value Theorem

Theorem 75 (Extreme Value Theorem). *On a closed & bounded domain, any continuous function $f(x, y)$ attains a global minimum & maximum.*

Closed:

Bounded:

Example 76. [Poll] Now you try! Which of the following domains are closed? Which of the following are bounded?

Strategy for finding global min/max of continuous $f(x, y)$ on a closed & bounded domain R

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

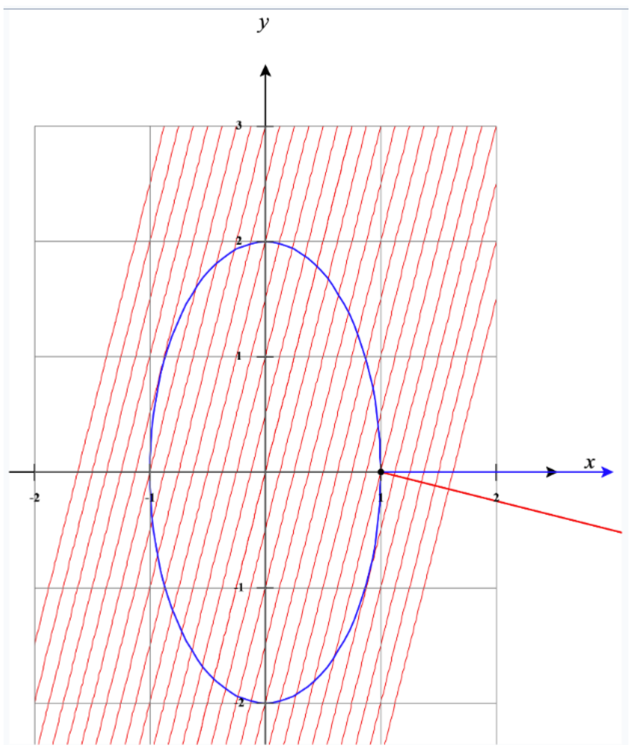
Example 77. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R consisting of those points with $x^2 \leq y \leq 4$.

14.8 - Lagrange Multipliers

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

Example 78 (Poll)**.** Find the points where the value of the function of f with contours given in red (the lines) may take a minimum or maximum value subject to the constraint $4x^2 + y^2 = 1$ (the ellipse).



Example 79. Find the constrained maxima and minima of $f(x, y) = 2x + y$ given that $x^2 + y^2 = 4$

Example 80. Set up a system of equations to find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

Example 81. Set up the system of equations from the method of Lagrange multipliers to find the extreme values of the function $f(x, y, z) = x^2 + y^2 + z^2$ on the curve of intersection of the surfaces $x^2 + y^2 - z = 3$ and $x + 2y - 2z = 2$.

Day 12 - Double & Iterated Integrals

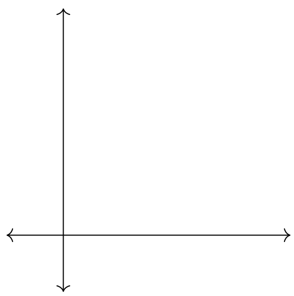
Pre-Lecture

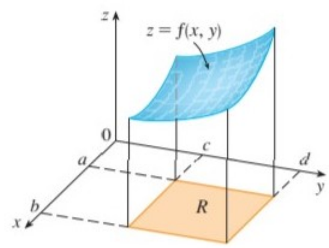
Section 15.1: Introduction to Double Integrals

Volumes and Double integrals. Let R be the closed rectangle defined below:

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

Let $f(x, y)$ be a function defined on R such that $f(x, y) \geq 0$. Let S be the solid that lies above R and under the graph f.





Question: How can we estimate the volume of S?

Definition 82. The ____________________ of $f(x, y)$ over a rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{|P| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta x_k\, \Delta y_k$$

if this limit exists.

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Day 12 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **I2: Iterated Integrals.** I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.

Goals for Today:

Sections 15.1, 15.2

- Introduce double and iterated integrals for functions of two variables on rectangles
- Use Fubini's Theorem to change the order of integration of a iterated integral
- Be able to set up & evaluate a double integral over a general region
- Change the order of integration for general regions

Question: How can we compute a double integral?

Answer:

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

What does $\int_c^d f(2, y)dy$ represent?

What about $\int_c^d f(x,y)dy$?

Let $A(x) = \int_c^d f(x,y)dy$. Then,

$$= \int_a^b A(x)dx =$$

This is called an ____________________.

Example 83. Evaluate $\displaystyle\int_1^2 \int_3^4 6x^2y \; dy \; dx$.

Theorem 84 (Fubini's Theorem)**.** *If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then*

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 85. Compute $\iint_R xe^{e^{e^y}}\,dA$, where R is the rectangle $[-1,1]\times[0,4]$.

Question: What if the region R we wish to integrate over is not a rectangle?

Answer: Repeat same procedure - it will work if the boundary of R is smooth and f is continuous.

Example 86. Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the xy-plane and whose top is $z = 2 - x - y$.

Vertically simple:

Horizontally simple:

Day 13 - General Regions, Area, Average Value

Pre-Lecture

Section 15.3: Area and Average Value

Two other applications of double integrals are computing the area of a region in the plane and finding the average value of a function over some domain.

Area: If R is a region bounded by smooth curves, then

$$\text{Area}(R) = \underline{\hspace{5cm}}$$

Example 87. Find the area of the region R bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.

Average Value: The average value of $f(x, y)$ on a region R contained in $\mathbb{R}^2$ is

$$f_{avg} = \underline{\hspace{6cm}}$$

Example 88. Find the average temperature on the region R in the previous example if the temperature at each point is given by $T(x, y) = 4xy^2$.

Day 13 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **I2: Iterated Integrals.** I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.
- **A2: Integral Applications.** I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.

Goals for Today:

Sections 15.2, 15.3

- Be able to set up & evaluate a double integral over a general region
- Compute areas of general regions in the plane
- Compute the average value of a function of two variables

Vertically simple:

Horizontally simple:

Example 89. Set up an iterated integral to evaluate the double integral

$$\iint_R \cos(x)\sin(y)\; dA,$$

where R is the triangle with vertices $(0,0), (\pi,0)$, and (π,π).

Example 90. Set up an iterated integral to compute the average value of

$$f(x,y) = e^{1-x^2-y^2}$$

over the triangle with vertices $(0,0), (1,0)$, and $(0,1)$, treating the region as horizontally simple.

Example 91. [Poll] Set up an iterated integral to evaluate the double integral $\iint_R 6x^2y \, dA$, where R is the region bounded by $x = 0, x = 1, y = 2$, and $y = x$.

Example 92. Sketch the region of integration for the integral expression

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\ dy\ dx + \int_1^4 \int_{x-2}^{\sqrt{x}} f(x,y)\ dy\ dx.$$

Then write an equivalent iterated integral expression in the order $dx\ dy$.

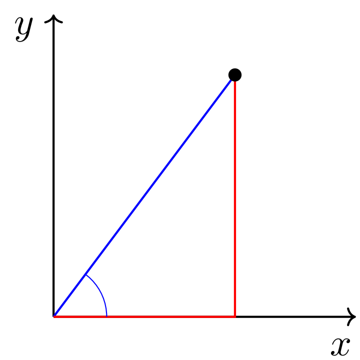
Example 93. [Poll] Sketch the region with $x^3 \leq y \leq 4x$ and compute its area.

Day 14 - Polar Coordinates & Integration

Pre-Lecture

Section 15.4: Polar Coordinates

Polar Coordinates:



Cartesian coordinates: Give the distances in ________ and ________ directions from ________

Polar coordinates:

- r = distance from ________ to ________
- θ = angle between the ray ________ and the positive ____________

Polar to Cartesian:

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Cartesian to Polar:

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

Example 94. Find a set of polar coordinates for the point $(x, y) = (-1, 1)$. Graph the set of points (x, y) that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the** xy**-plane.**

Day 14 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **I3: Change of Variables.** I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.

Goals for Today:

Sections 15.4

- Convert double integrals to iterated polar integrals
- Compute iterated polar integrals

Example 95.

(a) Write the function $f(x, y) = \sqrt{x^2 + y^2}$ in polar coordinates.

(b) [Poll] Write a Cartesian equation describing the points that satisfy $r = 2\sin(\theta)$.

15.4: Double Integrals in Polar Coordinates

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r, \theta)$ on R, compute $\iint_R f(r, \theta)\, dA$.

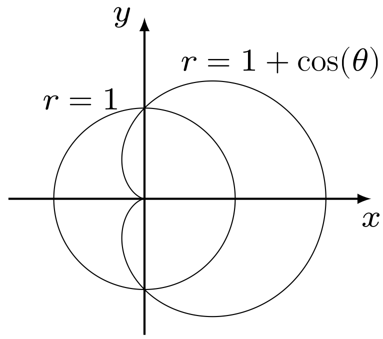
Example 96. Compute the area of the disk of radius 5 centered at $(0, 0)$.

Remember: In polar coordinates, the area form $dA =$ ________________

Example 97. Compute $\iint_D e^{-(x^2+y^2)}\, dA$ on the washer-shaped region $1 \leq x^2+y^2 \leq 4$.

Example 98. Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$.

Example 99 (Poll)**.** Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



Day 15 - Triple Integrals & More Applications

Pre-Lecture

Section 15.5/6: Triple Integrals and Spatial Applications

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k.

Taking the limit as the volume of the prisms goes to zero gives a

$$\rule{6cm}{0.4pt} : \iiint_D f(x, y, z)\ dV$$

Important special case:

$$\iiint_D 1\ dV = \rule{5cm}{0.4pt}$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

Example 100. Compute $\int_0^1 \int_0^2 \int_0^3 \, dz \, dy \, dx$ and interpret your answer.

Other important spatial applications

THREE-DIMENSIONAL SOLID

- **Mass:** If $\delta(x, y, z)$ is the mass density of a solid D, then $M = \iiint_D \delta(x, y, z) \, dV$
- **First moments about the coordinate planes:**

$$M_{yz} = \iiint_D x \; \delta \; dV, \quad M_{xz} = \iiint_D y \; \delta \; dV, \quad M_{xy} = \iiint_D z \; \delta \; dV$$

- **Center of mass:**

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

- **Mass:** If $\delta(x, y)$ is the mass density of a plate R, then $M = \iint_R \delta(x, y) \, dA$
- **First moments about the coordinate planes:**

$$M_y = \iint_R x \; \delta(x, y) \; dA, \quad M_x = \iint_R y \; \delta(x, y) \; dA$$

- **Center of mass:**

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}.$$

Day 15 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **A3: Integral Applications.** I can use multiple integrals to solve physical problems, such as finding area, average value, volume, mass of a lamina, or solid.

Goals for Today:

Sections 15.5, 15.6

- Learn how to write triple integrals as iterated integrals.
- Compute triple iterated integrals
- Change the order of integration in a triple iterated integral.
- Apply our work to find the mass and center of mass of objects in $\mathbb{R}^2$ and $\mathbb{R}^3$

Example 101. Let's practice with triple integrals.

1. **Mechanics:** Compute $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} \, dz \, dy \, dx$.

2. **Interpretation:** What shape is this the volume of?

3. **Rearrange:** Write an equivalent iterated integral in the order $dy \, dz \, dx$.

Example 102. [Poll] Which of the following integrals is equal to

$$\int_0^3 \int_0^2 \int_0^y f(x, y, z)\ dz\ dy\ dx?$$

1. $\int_0^2 \int_0^3 \int_0^y f(x, y, z)\ dz\ dx\ dy$

2. $\int_0^2 \int_0^3 \int_0^y f(x, y, z)\ dz\ dy\ dx$

3. $\int_0^3 \int_0^2 \int_0^y f(x, y, z)\ dx\ dy\ dz$

4. $\int_0^3 \int_0^2 \int_0^z f(x, y, z)\ dy\ dz\ dx$

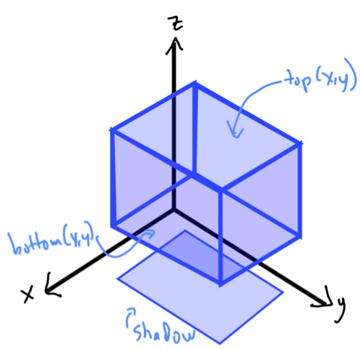
We can think about converting triple integrals to iterated integrals in three ways: two geometric and one algebraic. We'll start with the two geometric methods.

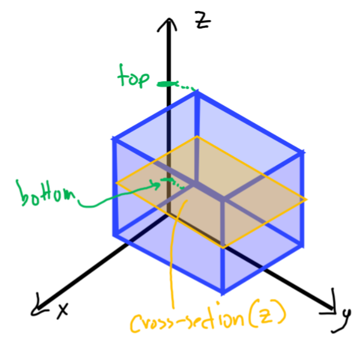
Method 1: Projection/Shadow

1. Choose the innnermost variable.
2. Find the bounds for the innermost integral.
3. Find the bounds for the outer double integral by looking at the shadow.

Here we have

$$\iiint_D f(x,y,z)\,dV = \iint_{shadow} \left(\int_{bottom(x,y)}^{top(x,y)} f(x,y,z)\,dz \right)\,dA$$





Method 2: Cross-Section

1. Choose the outermost variable.
2. Find the bounds for the outermost integral.
3. Find the bounds for the inner double integral by looking at the cross-sections.

Here we have

$$\iiint_D f(x,y,z)\,dV = \int_{bottom}^{top} \left(\iint_{crosssection(z)} f(x,y,z)\,dA \right)\,dz$$

Example 103. Write an integral for the mass of the solid D in the first octant bounded by the planes

$$x = 0, z = 0, x = y, \text{ and } x + y + z = 2$$

with density $\delta(x, y, z) = x^2 y$ using the shadow method and using the cross-section method. Which orders of integration work well for each method?

Example 103 (cont.)

Day 16 - Triple Integrals Algebraically & Applications

Pre-Lecture

Rules for Triple Integrals for the Sketching Impaired

Rules for Triple Integrals for the Sketching Impaired (credit to Wm. Douglas Withers)

Rule 1: Choose a variable appearing exactly twice for the next integral.

Rule 2: After setting up an integral, cross out any constraints involving the variable just used.

Rule 3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

Rule 4: A square variable counts twice.

Rule 5: The argument of a square root must be non-negative.

Rule 6: If you do not know which is the upper limit and which is the lower, take a guess - but be prepared to backtrack.

Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Example 104. Set up an integral for the volume of the region D defined by

$$x + y^2 \leq 8, \quad y^2 + 2z^2 \leq x, \quad y \geq 0$$

Day 16 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **A2: Integral Applications.** I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.

Goals for Today:

Section 15.5,15.6

- Learn how to write triple integrals as iterated integrals.
- Apply triple integrals to solve problems in $\mathbb{R}^3$

Example 105. Write an integral for the mass of the solid D in the first octant with $2y \leq z \leq 3 - x^2 - y^2$ with density $\delta(x, y, z) = 3x + y^{1.5} + 0.2$ using any method and order of integration. Which orders of integration work well?

Example 105 (cont.)

Example 106. Set up a triple iterated integral for the triple integral of $f(x, y, z) = x^3y$ over the region D bounded by

$$x^2 + y^2 = 1, \quad z = 0, \quad x + y + z = 2.$$

Example 107. Set up triple iterated integrals to compute the center of mass of an object that occupies the volume of the upper hemisphere of $x^2 + y^2 + z^2 \leq 4$ with density z at (x, y, z).

Another application of integration is to compute probabilities. A *joint density function* of two random variables X and Y (such as the height and weight of a randomly chosen person) is a function $f(x,y)$ such that the probability that (X,Y) lies in a region R of possible values is

$$P((X,Y) \in R) =$$

The function f must satisfy two properties:

- $f(x,y) \geq$
- $\iint_{\mathbb{R}^2} f(x,y)\, dA =$

Example 108. Suppose the joint density function for random variables X and Y is

$$f(x,y) = \begin{cases} C(x+2y) & 0 \leq x \leq 10, 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}.$$

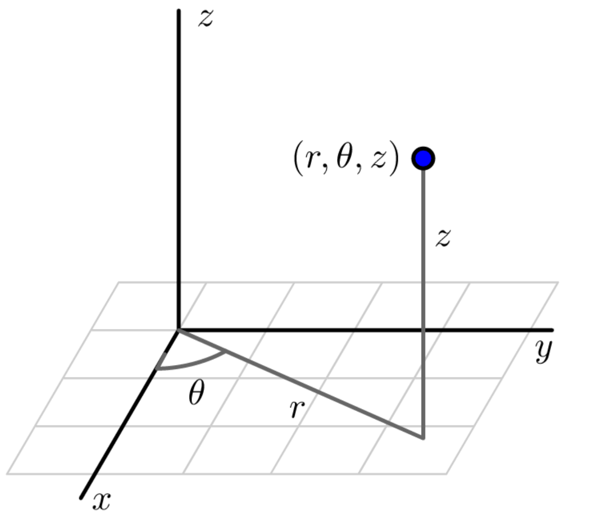
Find the value of C and the probability that $X + Y \leq 10$.

Day 17 - Triple Integrals in Cylindrical & Spherical Coordinates

Pre-Lecture

Section 15.7: Cylindrical Coordinates

Cylindrical Coordinate System



Cylindrical to Cartesian:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

Cartesian to Cylindrical:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$$

For uniqueness:

Example 109.

(a) Find cylindrical coordinates for the point with Cartesian coordinates $(-1, \sqrt{3}, 3)$.

(b) Find Cartesian coordinates for the point with cylindrical coordinates $(2, 5\pi/4, 1)$.

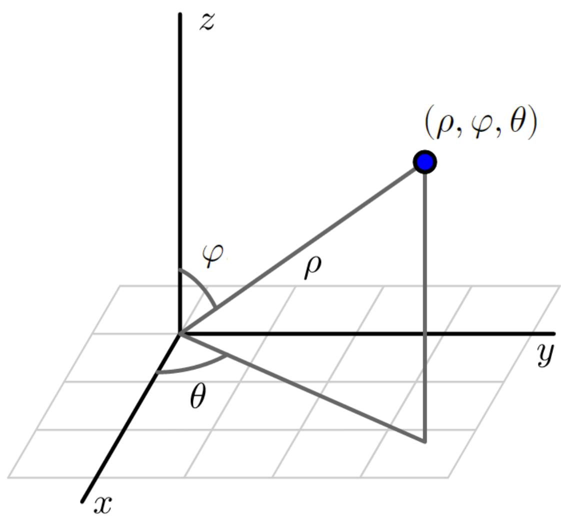
Example 110. In xyz-space sketch the *cylindrical box*

$$B = \{(r, \theta, z) \mid 1 \le r \le 2,\ \pi/6 \le \theta \le \pi/3,\ 0 \le z \le 2\}.$$

Pre-Lecture

Section 15.7: Spherical Coordinates

Spherical Coordinate System



For uniqueness:

Example 111.

(a) Find spherical coordinates for the point with Cartesian coordinates $(-2, 2, \sqrt{8})$.

(b) Find Cartesian coordinates for the point with spherical coordinates $(2, \pi/2, \pi/3)$.

Spherical to Cartesian:

$$x = \rho \sin(\varphi)\cos(\theta)$$
$$y = \rho \sin(\varphi)\sin(\theta)$$
$$z = \rho \cos(\varphi)$$

Cartesian to Spherical:

$$\rho^2 = x^2 + y^2 + z^2$$
$$\tan(\theta) = \frac{y}{x}$$
$$\tan(\varphi) = \frac{\sqrt{x^2 + y^2}}{z}$$

Example 112. In xyz-space sketch the *spherical box*

$$B = \{(\rho, \varphi, \theta) \mid 1 \le \rho \le 2,\ 0 \le \varphi \le \pi/4,\ \pi/6 \le \theta \le \pi/3\}.$$

Day 17 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **I2: Iterated Integrals.** I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.
- **I3: Change of Variables.** I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.

Goals for Today: Section 15.7

- Be able to convert between Cartesian, cylindrical, and spherical coordinate systems in $\mathbb{R}^3$
- Compute triple integrals expressed in cylindrical coordinates
- Compute triple integrals expressed in spherical coordinates

Triple Integrals in Cylindrical Coordinates

We have $dV =$ ______________________

Example 113. Set up a iterated integral in cylindrical coordinates for the volume of the region D lying below $z = x+2$, above the xy-plane, and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Example 114 (Poll)**.** Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.

Triple Integrals in Spherical Coordinates

We have $dV =$ ________________________________

Example 115. Write an iterated integral for the volume of the "ice cream cone" D bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the cone $z = \sqrt{3}\sqrt{x^2 + y^2}$.

Example 116 (Poll)**.** Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

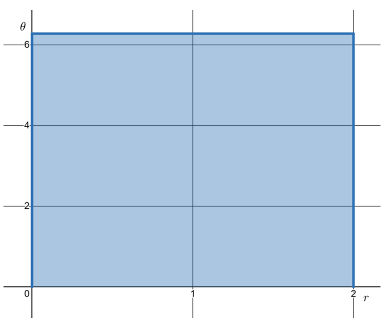
Day 18 - Change of Variables in Multiple Integrals

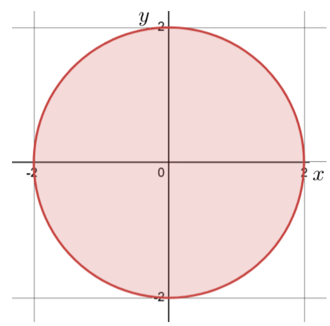
Pre-Lecture

Section 15.8: Coordinate Transformations

We have seen three examples of changing variables in multiple integrals so far: the polar, cylindrical, and spherical coordinate systems.

For example, the change of coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ transforms the rectangle $0 \leq r \leq 2, 0 \leq \theta \leq 2\pi$ to the disk $x^2 + y^2 \leq 2$:

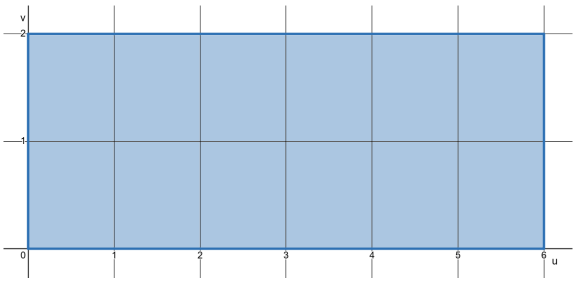


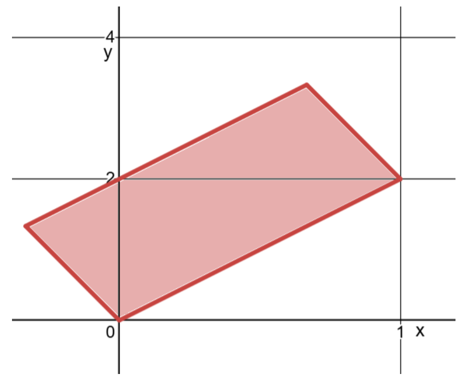


Goal: Generalize this idea.

Definition 117. A **coordinate transformation** of a region $G \subseteq \mathbb{R}^n$ is a map $\mathbf{T} : G \to \mathbb{R}^n$ which is invertible and differentiable in the interior of G.

Example 118. Find a coordinate transformation $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ that maps the rectangle $[0, 6] \times [0, 2]$ to the parallelogram bounded by $y = 2x$, $y = 2x + 2$, $y = -4x$, and $y = 6 - 4x$.





Day 18 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **I2: Iterated Integrals.** I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.
- **I3: Change of Variables.** I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.

Goals for Today:

Sections 15.7-8

- Be able to convert between Cartesian, cylindrical, and spherical coordinate systems in $\mathbb{R}^3$
- Change variables in multiple integrals
- Identify choices for changing variables in a given integration problem

Example 119. Consider the integral

$$\iiint_D 3x^2 + 3y^2 - 2z \, dV$$

where D is the region bounded by the plane $z = 97$ and the paraboloid $z = 4x^2 + 4y^2 - 3$. Which coordinate system is best suited for computing this integral? Convert the integral into that coordinate system.

Question: When should we use these coordinate systems?

- Cylindrical coordinates:

- Spherical coordinates:

Theorem 120 (Substitution Theorem)**.** *Suppose* $\mathbf{T}(u, v)$ *is a one-to-one, differentiable transformation that maps the region* G *in the* uv*-plane to the region* R *in the* xy*-plane. Then*

$$\iint_R f(x, y)\ dx\ dy = \iint_G f(\mathbf{T}(u, v)) |\det(D\mathbf{T}(u, v))|\ du\ dv.$$

Thinking about single variable calculus: Compute $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4 - x^2}}\ dx$

Example 121. Evaluate $\displaystyle\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\, dx\, dy$ via the transformation $x = u+v$, $y = 2v$.

1. **Find T:**

2. **Find G and sketch:**

3. **Find Jacobian:**

4. **Convert and use theorem:**

Example 122.

(a) [Poll] Find the Jacobian of the transformation

$$x = u + (1/2)v, \ y = v.$$

(b) [Poll] Which transformation(s) seem suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2} \, dx \, dy?$$

i. $u = x, v = y$

ii. $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$

iii. $u = 2x - y, \ v = y^3$

iv. $u = y, \ v = 2x - y$

v. $u = 2x - y, \ v = y$

vi. $u = e^{(2x-y)^2}, \ v = y^3$

(c) Change variables in the integral above to one which is easier to compute using the transformation from (b).

Day 19 - Change of Variables Cont.

Pre-Lecture

Section 15.8: Inverse Function Theorem

Theorem 123 (Inverse Function Theorem). *If* $\mathbf{T}(u,v)$ *is a one-to-one differentiable transformation that maps a region* G *in the uv-plane to a region* R *in the xy-plane and* $T(u_0,v_0)=(x_0,y_0)$*, then we have*

$$|\det(D\mathbf{T}(u_0,v_0))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x_0,y_0))|}$$

Example 124. Use the Inverse Function Theorem to rewrite the integrand (including Jacobian) for the integral

$$\iint_R xy\ dx\ dy$$

under the coordinate transformation $\mathbf{T}(u,v)$ whose inverse is given by

$$u(x,y)=xy \quad v(x,y)=y/x.$$

Day 19 Lecture

Daily Announcements & Reminders:

Learning Targets:

- **I3: Change of Variables.** I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.

Goals for Today:

Sections 15.7-8

- Change variables in multiple integrals
- Use the Inverse Function Theorem in change of variables problems

Example 125. Use the transformation

$$\mathbf{T}(u, v) = \begin{bmatrix} u^2 - v^2 \\ 2uv \end{bmatrix}$$

to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\; dy\; dx.$$

Example 126. Let's evaluate $\iint_R \frac{y(x+y)}{x^3}$ where R is the region in the xy-plane bounded by $y = x, y = 3x, y = 1 - x$, and $y = 2 - x$. Consider the coordinate transformation $u = x + y, v = y/x$.

1. **Find the rectangle G in the uv plane that is mapped to R**

2. **Evaluate $f(\mathbf{T}(u, v))|\det(D\mathbf{T}(u, v))|$ in terms of u and v without directly solving for $\mathbf{T}$ using the theorem above**

3. **Use the Substitution Theorem to compute the integral.**