MATH 2551 Guided Notes

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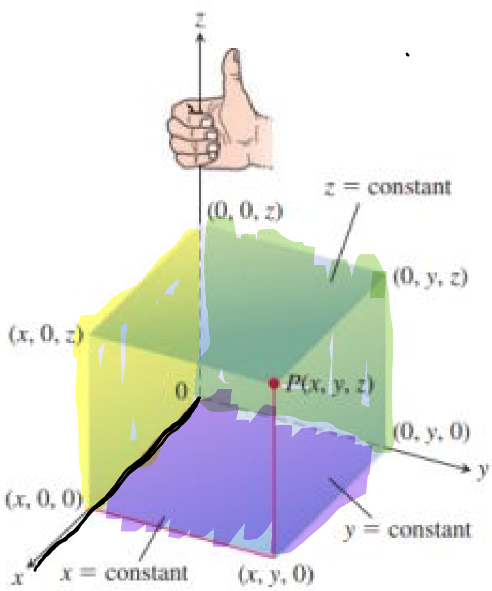
Fall 2025

Day 1 - Course Introduction and Cross Products

Pre-Lecture

12.1: Three-Dimensional Coordinates



- $\mathbb{R}^3$: (x, y, z)
- Right-handed system
- Coordinate planes
  - $x=0$ (the yz-plane)
  - $y=0$ (the xz-plane)
  - $z=0$ (the xy-plane)
- Eight octants
  - 1st octant has $x, y, z \geq 0$

Day 1 - Lecture

Daily Announcements & Reminders:

- Welcome to MATH 2551 G
- Use the QR code or visit pollev.com/drhgt to log into PollEverywhere
  - if you did not have an account before now, use your GT email & GTID to log in

Goals for Today:

Sections 12.1, 12.3, 12.4

- Set classroom norms
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Icebreaker on PollEverywhere

Introduction to the Course

Purposes:

- Pre-Lecture: Get in math headspace, first exposure to the day's topic
- Lecture: Fill in the rest of the new ideas for the week
- Studio: Guided group practice with new ideas under supervision, develop independence
- WeBWorK: Basic skills practice for the topics
- LT Practice: Extra practice problems for assessments
- Quizzes, Checkpoints, & Exams: Demonstrate learning & get feedback

Feedback loops: Lecture -> practice -> studio -> practice -> assessment -> practice -> assessment -> ...

From prereqS

- Computing deriv/integrals
  - polys, exp, log, trig
  - Deriv. Rules
  - u-sub, int. by parts
- matrix & vector operations
- linear maps as matrices

Important Canvas Items: Back to Canvas!

Big Idea: Extend differential & integral calculus.

What are some key ideas from these two courses?

Differential Calculus | Integral Calculus

Poll

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

e.g. $f(x,y) = x^2 + e^y + \sin(xy)$ $\quad f : \mathbb{R}^2 \to \mathbb{R}$

$f(t) = \begin{bmatrix} 2t \\ 3t-1 \\ 4-t \end{bmatrix}$ $\quad f : \mathbb{R} \to \mathbb{R}^3$

$f(s,t) = \begin{bmatrix} s\cos(t) & s\sin(t) \\ s^2 & t^2 \end{bmatrix}$ $\quad f : \mathbb{R}^2 \to \mathbb{R}^2$

• derivatives are linear maps and so we get matrices

Example 1. What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation

$$x^2 + y^2 = 1?$$

These solutions form a

circle radius 1] answer in $\mathbb{R}^2$

cylinder radius 1

really really infinitely long straw

$x^2 + y^2 + z^2 = 1$

sphere of radius 1 at origin

$x^2 + y^2 + z = 1$

paraboloid

$x^2 + y^2 = 1$; $\quad z = 4$

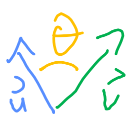
Section 12.3/4: Dot & Cross Products

Definition 2. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \dots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \dots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

This product tells us about angle between two vectors.

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$



or normal or perpendicular

In particular, two vectors are **orthogonal** if and only if their dot product is 0.

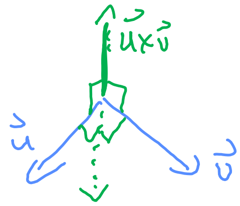
Example 3. Are $\mathbf{u} = \langle 1, 1, 4 \rangle$ and $\mathbf{v} = \langle -3, -1, 1 \rangle$ orthogonal?

$$\vec{u} \cdot \vec{v} = 1(-3) + 1(-1) + 4(1) = -3 - 1 + 4 = 0$$

so $\vec{u}$ & $\vec{v}$ are orthogonal.

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.

1. Right-handed:



2. Algebraically nice:
   - $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$ & $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
   - $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$

Definition 4. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ <u>in $\mathbb{R}^3$</u> is

$\vec{i}, \vec{j}, \vec{k}$ are standard basis vectors

$\vec{i} = \vec{e_1} = \langle 1, 0, 0 \rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ u & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

$$= (u_2 v_3 - v_2 u_3)\vec{i} - (u_1 v_3 - v_1 u_3)\vec{j} + (u_1 v_2 - v_1 u_2)\vec{k}$$

Example 5. Find $\langle 1, 2, 1\rangle \times \langle 3, -1, 0\rangle$.

$$\langle 1,2,1\rangle \times \langle 3,-1,0\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 3 & -1 & 0 \end{vmatrix} \leftarrow \quad \det\left(\quad \right)$$

$$= \left\langle \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \right\rangle$$

$$= \langle 0-(-1), -(0-3), -1-6\rangle$$

$$= \langle 1, 3, -7\rangle$$

$$\langle 3,-1,0\rangle \times \langle 1,2,1\rangle = -\left(\langle 1,2,1\rangle \times \langle 3,-1,0\rangle\right)$$

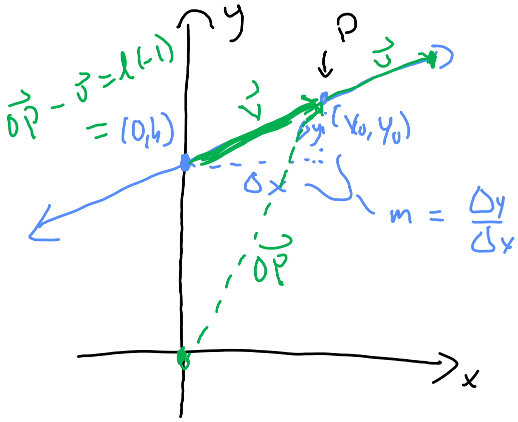
- antisymmetric

Day 2 - Lines, Planes, and Quadrics

Pre-Lecture

12.5: Lines

Lines in $\mathbb{R}^2$, a new perspective: $y = mx + b$ $\qquad y - y_0 = m(x - x_0)$



point $(0, b)$ $\qquad (x_0, y_0)$

Direction : $m = \frac{\Delta y}{\Delta x}$

In $\mathbb{R}^3$: $\frac{\Delta y}{\Delta x}$ & $\frac{\Delta z}{\Delta x}$ & $\frac{\Delta z}{\Delta y}$ & ...

A line is all points of the form

$$\ell(t) = \vec{OP} + \vec{v} \cdot t \quad \text{where}$$

$\vec{P} = (x_0, y_0, z_0)$ on the line & $\vec{v}$ is a direction for the line.

Example 7. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.

Need:
- point on the line : $P = (1, 0, 2)$
- direction vector :

$$\vec{v} = \vec{PQ} = \langle -2-1, 1-0, 1-2 \rangle$$
$$= \langle -3, 1, -1 \rangle$$

So a vector equation of the line is

$$\ell(t) = \langle -3, 1, -1 \rangle t + \langle 1, 0, 2 \rangle \qquad -\infty \le t \le \infty$$

or

$$\ell(t) = \langle -3t + 1, t, -t + 2 \rangle$$

Day 2 Lecture

Daily Announcements & Reminders:

- WW Fundamentals due F at 10pm
- Log in to PollEverywhere with your GT email
  - password is GTID until you update it
  - after today need to login for credit
- Do warmup at pollev.com/drhgt or

Learning Targets:

- **G1: Lines and Planes.** I can describe lines using the vector equation of a line. I can describe planes using the general equation of a plane. I can find the equations of planes using a point and a normal vector. I can find the intersections of lines and planes. I can describe the relationships of lines and planes to each other. I can solve problems with lines and planes.

- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.

Goals for Today: Sections 12.5-12.6

- Apply the cross product to solve problems
- Learn the equations that describe lines, planes, and quadric surfaces in $\mathbb{R}^3$
- Solve problems involving the equations of lines and planes
- Sketch quadric surfaces in $\mathbb{R}^3$

Example 7. Find a set of parametric equations for the line through the point $(1, 10, 100)$ which is parallel to the line with vector equation

$$\mathbf{r}(t) = \underbrace{\langle 1, 4, -3\rangle}_{\text{direction vector}} t + \underbrace{\langle 0, -1, 1\rangle}_{\text{reference point}}$$

Goal: $x(t) = 1 + t$

$y(t) = 10 + 4t$

$z(t) = 100 - 3t$

or $x(t) = 1 + 2t$

$y(t) = 10 + 8t$

$z(t) = 100 - 6t$

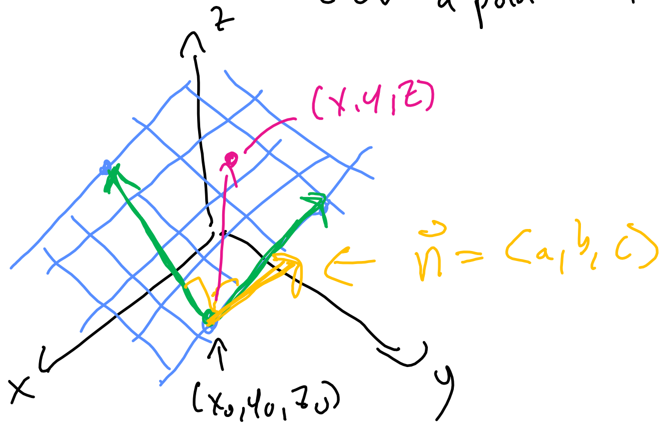
$t \in \mathbb{R}$

Section 12.5 Planes

Planes in $\mathbb{R}^3$

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction **n**, called the *normal vector*.

{ or a point & two directions



Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

(1) $ax + by + cz = d$

$\vec{v} = \langle x - x_0, y - y_0, z - z_0 \rangle$ is in plane so is $\perp$ to $\vec{n}$ $\Rightarrow$ $\vec{v} \cdot \vec{n} = 0$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

$\vec{n} = \langle a, b, c \rangle$
(x_0, y_0, z_0) is on plane

In (1), $\boxed{d = \vec{n} \cdot (x_0, y_0, z_0)}$

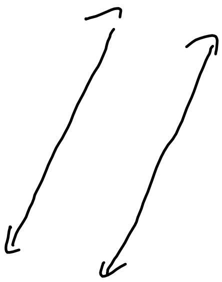
Ex: The plane w/ normal vector $\langle 1, -2, e \rangle$ containing $(\pi, \pi^2, 5)$

is $\boxed{(x - \pi) - 2(y - \pi^2) + e(z - 5) = 0}$

$d = \langle 1, -2, e \rangle \cdot \langle \pi, \pi^2, 5 \rangle = \boxed{\pi - 2\pi^2 + 5e}$

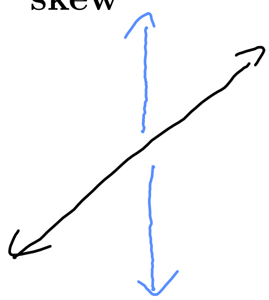
In $\mathbb{R}^3$, a pair of lines can be related in three ways:

parallel



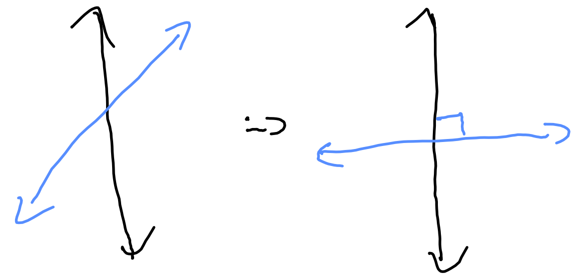
- if their direction vectors are parallel

skew



- direction vectors are not parallel & no point in common
  ↳ set equations equal (using diff. vars) & get no solution

intersecting



not orthogonal

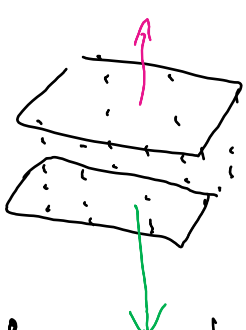
orthogonal

- dot prod. of dir vectors is 0 & have point in common

or get a solution

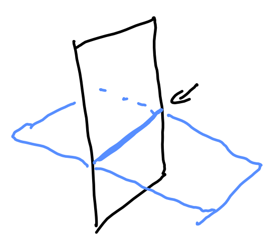
On the other hand, a pair of planes can be related in just two ways:

parallel



- normal vectors are parallel

intersecting



- sometimes orthogonal $\iff \vec{n}_1 \cdot \vec{n}_2 = 0$
- normal vectors are not parallel

Example 8. [Poll] The lines

$$\ell_1(t) = \langle 1, 1, 1 \rangle t + \langle 0, 0, 1 \rangle$$

and

$$\ell_2(t) = \langle 2, 2, 2 \rangle t + \langle 0, 0, 1 \rangle$$

are related in what way?

These lines are the same

because they have a point in common & are parallel

Example 9. [Poll] The lines

$$\ell_1(t) = \langle 1, 1, 1 \rangle t + \langle 0, 0, 1 \rangle$$

and

$$\ell_2(t) = 2t\mathbf{i} - t\mathbf{j} - t\mathbf{k}$$

are related in what way?

1) $\langle 1,1,1 \rangle \cdot \langle 2,-1,-1 \rangle = 2-1-1 = 0$

2) Check intersection:

$x_1(t)$: $t = 2s$ $x_2(s)$

$y_1(t)$: $t = -s$ $y_2(s)$

$z_1(t)$: $t+1 = -s$ $z_2(s)$

inconsistent, so skew

Example 10. Consider the planes $\overset{①}{y - z = -2}$ and $\overset{②}{x - y = 0}$. Show that the planes intersect and find an equation for the line of intersection of the planes.

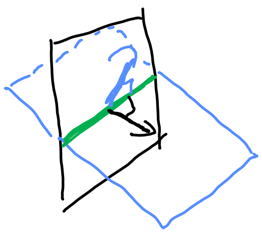
① Planes intersect: $\vec{n}_1 = \langle 0, 1, -1 \rangle$ & $\vec{n}_2 = \langle 1, -1, 0 \rangle$ are not parallel

OR $(0, 0, 2)$ is on both planes

② Find line of intersection of these planes

Need: • point (any point on both planes, e.g. $(0,0,2)$)

• direction $(\vec{n}_1 \times \vec{n}_2)$



$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \langle 0-1, -(0+1), (0-1) \rangle$$

$$= \langle -1, -1, -1 \rangle$$

$$\boxed{\text{line: } \ell(t) = \langle -1, -1, -1 \rangle t + \langle 0, 0, 2 \rangle, \quad t \in \mathbb{R}}$$

$y - z = 2 \rightarrow z = 2 - y$

$x - y = 0 \qquad x = y$

Section 12.6 Quadric Surfaces

On Tue.

Definition 11. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

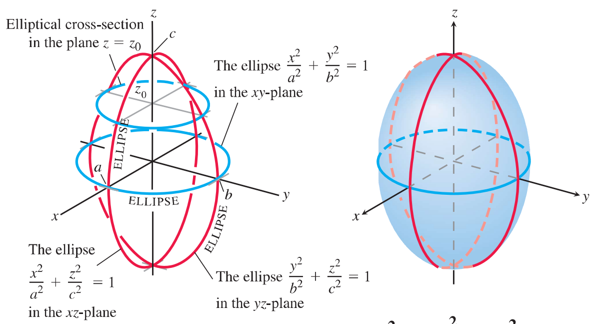
The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections. We'll also make heavy use of 3d graphing technology to get comfortable with these new objects.

Example 12. Use a 3d graphing utility to plot the quadric surface

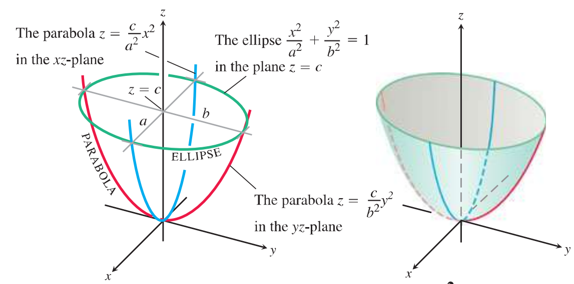
$$z = x^2 + y^2.$$

This surface is called a ____________________, because it has two coordinate directions with cross sections that are ______________ and one with cross sections that are __________.

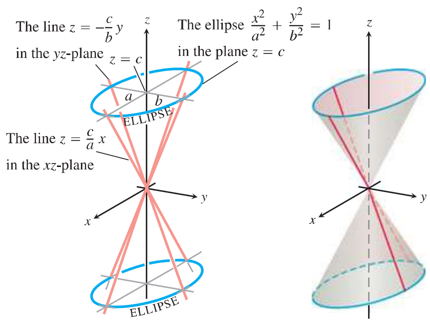
TABLE 12.1 Graphs of Quadric Surfaces



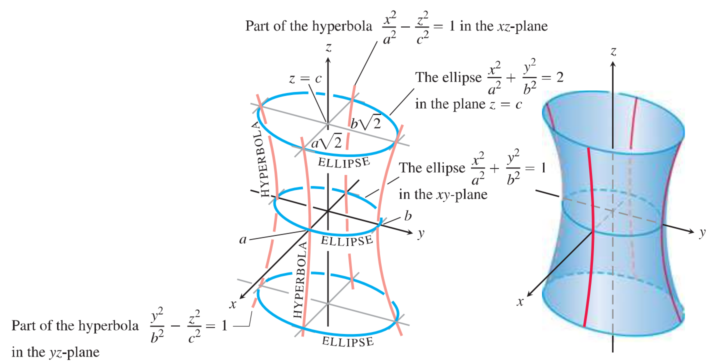
ELLIPSOID $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



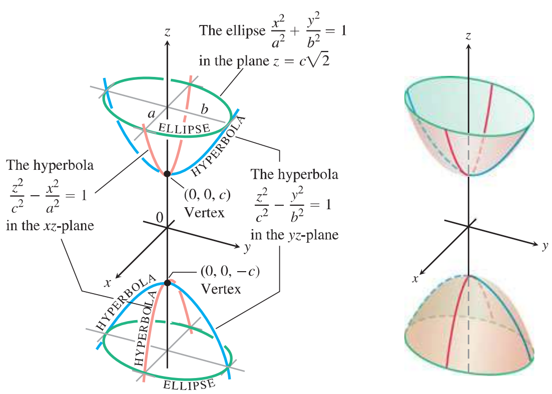
ELLIPTICAL PARABOLOID $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$



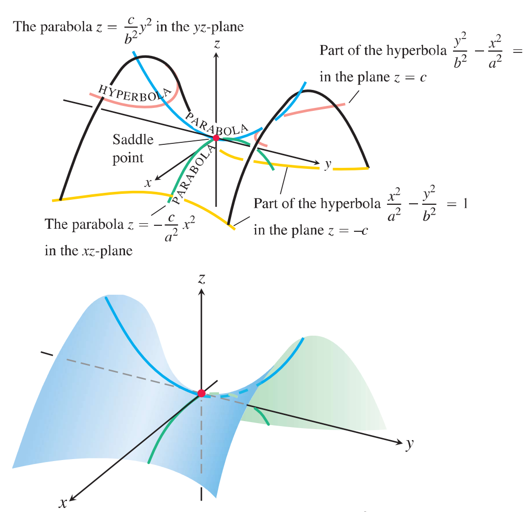
ELLIPTICAL CONE $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$



HYPERBOLOID OF ONE SHEET $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



HYPERBOLOID OF TWO SHEETS $$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



HYPERBOLIC PARABOLOID $$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$$

Example 13. [Poll] Which of the following are quadric surfaces?

1. A line
2. A sphere
3. A circle
4. An ellipse
5. The set of points (x, y, z) which solve $x^2 + y^2 - 3 = 0$.
6. The set of points (x, y, z) which solve $x^2 - y^2 - z^2 = 4$.

Example 14. [Poll] Classify the quadric surface $x + y^2 - z^2 - 3 = 0$.

Day 3 - Vector-Valued Functions & Calculus

Pre-Lecture

Section 13.1: Vector-Valued Functions

Last week, we used functions like

$$\ell(t) = \langle 2t+1, 3-t, t-1\rangle, \quad -\infty \le t \le \infty$$

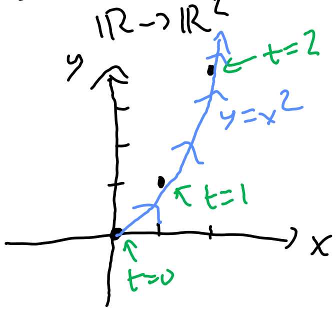
to produce lines in $\mathbb{R}^2$ and $\mathbb{R}^3$.

This is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

What happens when we change the component functions to be non-linear?

1) Let $\vec{r}(t) = \langle t, t^2\rangle$, $t \ge 0$.

$\mathbb{R} \to \mathbb{R}^2$



$\vec{r}(0) = \langle 0,0\rangle$

$\vec{r}(1) = \langle 1,1\rangle$

$\vec{r}(2) = \langle 2,4\rangle$

$y(t) = [x(t)]^2$

with $x \ge 0$

3) In 3D: $\vec{r}(t) = \langle \cos(t), \sin(t), 1-\cos(t)-\sin(t)\rangle$

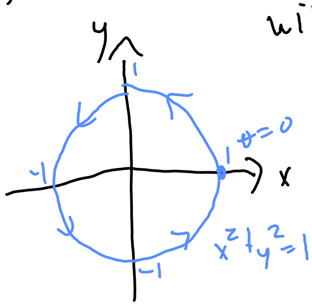
$0 \le t \le 2\pi$.

Same relation for x, y as #2!

$x(t)^2 + y(t)^2 = 1$

Also: $z(t) = 1 - x(t) - y(t)$

2) Find $\vec{r}(t) = \langle x(t), y(t)\rangle$ with this graph?

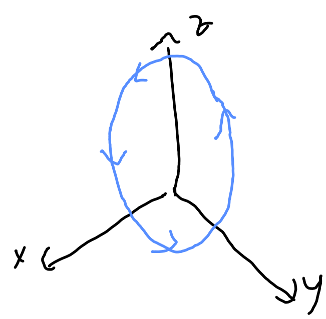


$\cos^2(\theta) + \sin^2(\theta) = 1$

so

let $\vec{r}(\theta) = \langle \cos(\theta), \sin(\theta)\rangle$

$0 \le \theta \le 2\pi$



Given a fixed curve C in space, producing a vector-valued function $\mathbf{r}$ whose graph is C is called parameterizing the curve C, and $\mathbf{r}$ is called a parametrization of C.

Day 3 Lecture

Daily Announcements & Reminders:

- HW G1 & G5 (curves) due F
- Quiz 1 on G1 on W
  - one set of 4 T/F, one short response
    need 3+ of these for Success
- Login required for PollEverywhere as of today
  - GT email, GTID initial password
- Office Hour update on Canvas: M 10-11 on Zoom; R 2-3 in Skiles
- Answer warm up poll at pollev.com/drhgt

Learning Targets:

- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.
- **G2: Calculus of Curves.** I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.
- **G5: Parameterization.** I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.

Goals for Today: Sections 12.6, 13.1, 13.2

- Learn the equations that define quadric surfaces in $\mathbb{R}^3$
- Use technology to plot quadric surfaces
- Introduce vector-valued functions
- Plot vector-valued functions and construct them from a graph
- Compute limits, derivatives, and tangent lines for vector-valued functions

Section 12.6 Quadric Surfaces

Definition 11. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

ex: $x^2+y^2=4$ (cylinder)
$x^2+y^2+z^2=4$ (sphere)

non-ex: $x+2y-z=4$
$x^3-y^2+\cos(z)=0$

The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections. We'll also make heavy use of 3d graphing technology to get comfortable with these new objects.

Example 12. Use a 3d graphing utility to plot the quadric surface

$$z = x^2 + y^2.$$

This surface is called a elliptical paraboloid, because it has two coordinate directions with cross sections that are parabolas and one with cross sections that are ellipses (here circles).

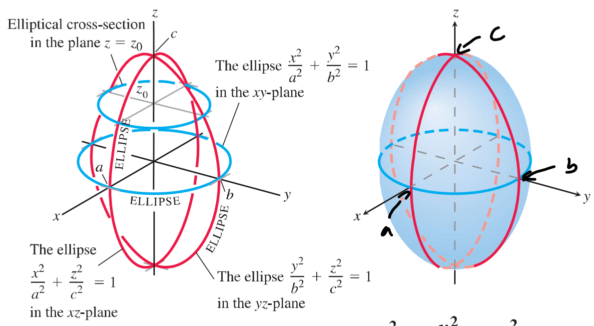
fix $x=k$ or $y=k$, k constant

To do this algebraically: set a variable to a constant
then analyze resulting 2-variable equation

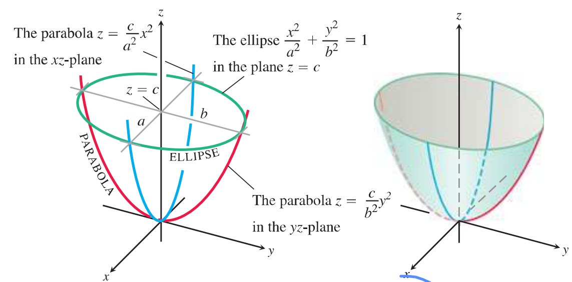
$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0$$

$\Downarrow$ lots of algebra

TABLE 12.1 Graphs of Quadric Surfaces

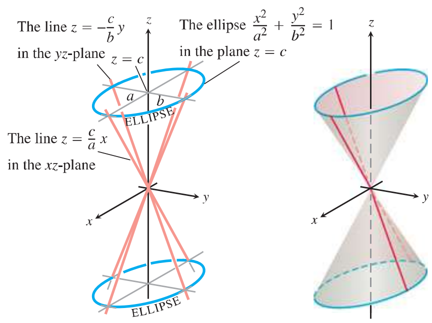


ELLIPSOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



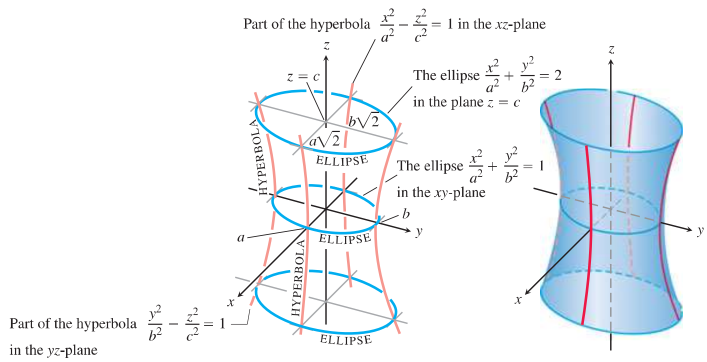
ELLIPTICAL PARABOLOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$

linear term present

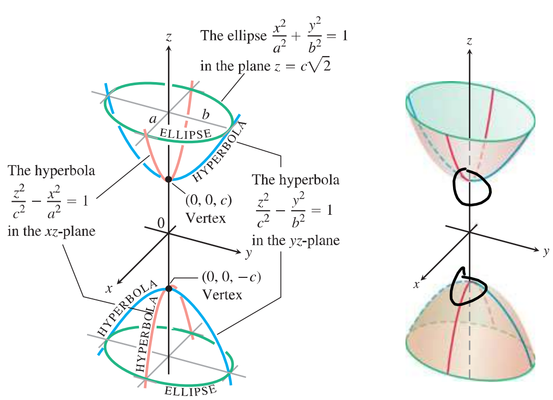


ELLIPTICAL CONE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

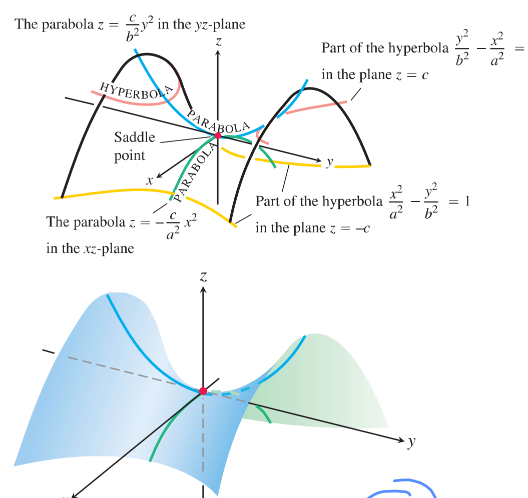
no constant



HYPERBOLOID OF ONE SHEET $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

Example 13. [Poll] Which of the following are quadric surfaces?

1. A line
2. A sphere
3. A circle
4. An ellipse
5. The set of points (x, y, z) which solve $x^2 + y^2 - 3 = 0$.
6. The set of points (x, y, z) which solve $x^2 - y^2 - z^2 = 4$.

Example 14. [Poll] Classify the quadric surface $x + y^2 - z^2 - 3 = 0$.

Section 13.1

Example 15. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?

Graphs: • spring/spiral/helix ✓ • same helix, but graph two goes higher ✓

#1: 1 turn

#2: 2 turns & taller

• graph 2 will 2x as many rotations/unit height

$\geq$ $r_3(t) = \langle \cos(2t), \sin(2t), t \rangle$, $0 \leq t \leq 2\pi$

• If $t \in \mathbb{R}$ instead, these two are both parameterizations of the same curve; i.e. they have the same graph

Check your intuition

Section 13.1: Calculus of Vector-Valued Functions

Unifying theme: Do what you already know, componentwise.

This works with limits:

Example 16. Compute $\lim_{t \to e} \langle t^2, 2, \ln(t) \rangle$.

$= \langle \lim_{t\to e} t^2, \lim_{t\to e} 2, \lim_{t\to e} \ln(t) \rangle$

$= \langle e^2, 2, 1 \rangle$

And with derivatives:

Example 17. If $\mathbf{r}(t) = \langle \underbrace{2t - \frac{1}{2}t^2 + 1}_{x(t)}, \underbrace{t - 1}_{y(t)} \rangle$, find $\mathbf{r}'(t)$.

$\vec{r}'(t) = \langle x'(t), y'(t) \rangle$

$= \langle 2 - t, 1 \rangle$

$\vec{r}''(t) = \langle -1, 0 \rangle$

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives velocity
- $\|\mathbf{r}'(t)\|$ gives speed
- $\mathbf{r}''(t)$ gives acceleration

- $\vec{r}'(t_0)$ is tangent to the curve $r(t)$ at $t=t_0$

Let's see this graphically

Example 18. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

tangent line to $\vec{r}(t)$ at $t=t_0$:

$$\ell(t) = \underbrace{\vec{r}(t_0)}_{\text{point}} + \underbrace{\vec{r}'(t_0)}_{\text{tangent direction}} \cdot t$$

$\vec{r}'(t) = \langle 2-t, 1\rangle$ so $\vec{r}(2) = \langle 3, 1\rangle$
& $\vec{r}'(2) = \langle 0, 1\rangle$

So tangent line: $\ell(t) = \langle 3, 1\rangle + \langle 0, 1\rangle t$, $t \in \mathbb{R}$

or $\langle 3, 1\rangle + \langle 0, 1\rangle(t-2)$ to make both curve & tangent pass through $(3,1)$ at same time

Day 4 - Geometry of Curves

Pre-Lecture

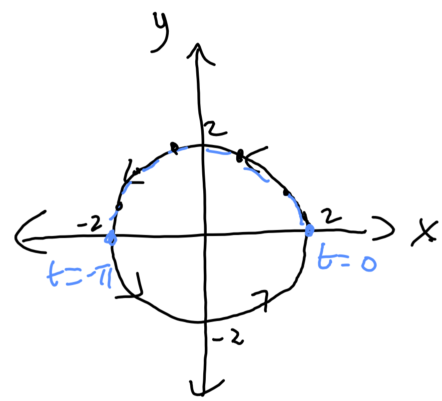
Section 13.3: Arc Length

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

Our next goal is to be able to measure distance traveled or arc length.

Motivating problem: Suppose the position of a fly at time t is

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle,$$



where $0 \le t \le 2\pi$.

How far does the fly travel from $t = 0$ to $t = \pi$?

Algebra: dist = rate · time

might change!

rate = $\|\vec{r}\,'(t)\|$ = speed

$$\text{dist} \approx \sum_{i=1}^{n} \|\vec{r}\,'(t_i)\|\,\Delta t \longrightarrow \text{dist} = \int_a^b \|\vec{r}\,'(t)\|\,dt$$

Here:
$$\text{dist} = \int_0^{\pi} \|\langle -2\sin(t), 2\cos(t)\rangle\|\,dt = \int_0^{\pi} \sqrt{4\sin^2(t) + 4\cos^2(t)}\,dt = \int_0^{\pi} 2\,dt = \boxed{2\pi}$$

length of a semicircle of radius 2 is $\frac{1}{2}(2\pi(2)) = 2\pi$ ✓

$\vec{r}\,'(t) \neq \vec{0}$ or undefined

Definition 19. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ from $t=a$ to $t=b$ that is traced out exactly once is

$$L = \int_a^b \|\vec{r}\,'(t)\|\,dt$$

Day 4 Lecture

Daily Announcements & Reminders:

- HW G1, G5 pt 1 due F
- No studio M - Labor Day
- No office hour today
- Do warmup on PollEv

Learning Targets:

- **G2: Calculus of Curves.** I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.
- **G3: Geometry of Curves.** I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.

Goals for Today: Sections 13.3, 13.4

- Compute integrals of vector-valued functions and solve initial value problems
- Solve initial value problems
- Compute arc lengths of curves using parameterizations
- Define and compute arc-length parameterizations

B) $\underbrace{\ell(t) = \overbrace{\langle 9t^2, \cos(t), 2t\rangle}^{\vec{r}'(t)} t + \langle 0,0,1\rangle}_{\text{not a line!}}$

Example 20 (Poll)**. Warmup.** Compute the tangent line to

$$\mathbf{r}(t) = \langle 3t^3, \sin(t), t^2+1\rangle, t \in \mathbb{R}$$

at the point $(0,0,1)$.

$\vec{r}'(t) = \langle 9t^2, \cos(t), 2t\rangle$

C) $\ell(t) = \underbrace{\langle 0, 1, 2\rangle}_{\text{plug in } x'(0),\ y'(0),\ z'(1)} t + \langle 0,0,1\rangle$

t has to be the same in all coords

To get t_0: solve: $\vec{r}(t_0) = \langle 0,0,1\rangle$

$3t_0^3 = 0$

$\sin(t_0) = 0$

$t_0^2 + 1 = 1$

$\Rightarrow t_0 = 0$

tangent: $\ell(t) = \overbrace{\boxed{\vec{r}'(t_0)}}^{\text{curve}} t + \overbrace{\boxed{\vec{r}(t_0)}}^{\text{point}}$

$= \vec{r}'(0) t + \langle 0,0,1\rangle$

$= \langle 0,1,0\rangle t + \langle 0,0,1\rangle, \quad t \in \mathbb{R}$

Working componentwise also works with integrals:

• $\int_a^b \vec{r}\,'(t)\,dt$ or $\int_a^b \vec{v}(t)\,dt$

is $\vec{r}(b)-\vec{r}(a)$

= displacement btwn $t=a$ & $t=b$

Example 21. Find $\displaystyle\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle\,dt.$

$$= \left\langle \int_0^1 t\,dt,\ \int_0^1 e^{2t}\,dt,\ \int_0^1 \sec^2(t)\,dt \right\rangle$$

$$= \left\langle \tfrac{1}{2}t^2\Big|_0^1,\ \tfrac{1}{2}e^{2t}\Big|_0^1,\ \tan(t)\Big|_0^1 \right\rangle$$

$$= \underbrace{\left\langle \tfrac{1}{2},\ \tfrac{1}{2}(e^2-1),\ \tan(1) \right\rangle}_{\text{integral of vector-valued fn is a vector}}$$

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 22. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t}\rangle\ m/s.$$

If he also knows that he started at the point $\boxed{\mathbf{r}(0) = \langle 0,0,0\rangle,}$ I.C.
use calculus to reconstruct his flight path.

1) Find antiderivative:

$400\left(1-\frac{1}{1+t}\right)$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \left\langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t} \right\rangle dt$$

$$= \langle 100\cos(2t), 200\sin(t), 400(t-\ln|1+t|)\rangle + \boxed{\vec{C}}$$

$$= \langle 100\cos(2t)+C_1, 200\sin(t)+C_2, 400(t-\ln(1+t))+C_3\rangle\ \text{m}$$

2) Apply I.C.:

$$\langle 0,0,0\rangle = \vec{r}(0) = \langle 100, 0, \overset{0}{\cancel{400(0-\ln(1))}}\rangle + \vec{C}$$

$$\langle -100,0,0\rangle = \vec{C}$$

$$\boxed{\text{Flight path: } \vec{r}(t) = \langle 100(\cos(2t)-1), 200\sin(t), 400(t-\ln(1+t))\rangle}$$

Example 23 (Poll). **T/F:** The function $\vec{r}(0)=\langle 1,0,2\rangle$

$$\mathbf{r}(t) = \langle \cos(t), 2t^2, 4t+2\rangle, t \in \mathbb{R}$$

is a solution of the IVP ← initial value problem

$$\mathbf{r}'(t) = \langle -\sin(t), 4t, 4\rangle, \quad \mathbf{r}(0) = \langle 0, 0, 2\rangle$$

Section 13.3 Arc Length

Example 24. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1,1,1)$ to the point $(2,4,8)$.

$$L = \int_a^b \|\vec{r}'(t)\|\,dt$$

1) Find times corresponding to points:

$\vec{r}(a) = \langle a, a^2, a^3\rangle = \langle 1,1,1\rangle$ so $a=1$

$\vec{r}(b) = \langle b, b^2, b^3\rangle = \langle 2,4,8\rangle$ so $b=2$

2) Find speed: $\vec{r}'(t) = \langle 1, 2t, 3t^2\rangle$

$\|\vec{r}'(t)\| = \sqrt{1+(2t)^2+(3t^2)^2} = \sqrt{1+4t^2+9t^4}$

3) Plug in: $\boxed{L = \int_1^2 \sqrt{1+4t^2+9t^4}\,dt}$

Example 25. Find the distance traveled by a particle moving along the path

$$\mathbf{r}(t) = \langle \ln(t), \sqrt{2}t, \frac{1}{2}t^2\rangle, \qquad t > 0$$

from $t=1$ to $t=2$.

1) Find times: ✓

2) Find speed: $\vec{r}'(t) = \langle \frac{1}{t}, \sqrt{2}, t\rangle$

$\|\vec{r}'(t)\| = \sqrt{\frac{1}{t^2}+2+t^2}$

3) Plug in: $L = \int_1^2 \sqrt{\frac{1}{t^2}+2+t^2}\,dt$

$= \int_1^2 \sqrt{\frac{1+2t^2+t^4}{t^2}}\,dt$

$= \int_1^2 \frac{1}{t}\sqrt{(t^2+1)^2}\,dt = \int_1^2 \frac{1}{t}(t^2+1)\,dt = \int_1^2 t+\frac{1}{t}\,dt$

$= \frac{1}{2}t^2 + \ln(t)\Big|_1^2$

$= \boxed{2+\ln(2) - \frac{1}{2} - 0}$

Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

$$s(t) = \int_{t_0}^{t} \|\vec{r}\,'(T)\|\, dT$$

variable time

constant time (starting time)

We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.

- Idea: like mile markers
- arc-length parameterizations or unit-speed parameterizations b/c $\|\vec{r}\,'(s)\| = 1$

Example 26. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \le t \le 2\pi$.

1) Find arc-length function:

$$\vec{r}\,'(t) = \langle -4\sin(t), 4\cos(t)\rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{16\sin^2(t) + 16\cos^2(t)} = \sqrt{16(\underbrace{\sin^2(t) + \cos^2(t)}_{1})} = 4$$

$$s(t) = \int_0^t \|\vec{r}\,'(T)\|\, dT = \int_0^t 4\, dT = 4t$$

integral could be hard

2) Solve $s = s(t)$ for t

$$s = 4t \Rightarrow \boxed{t = \frac{s}{4}} = f(s)$$

solving here could hard

3) Replace t with $f(s)$

$$\vec{r}_2(s) = \vec{r}(f(s)) = \vec{r}\left(\frac{s}{4}\right) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \quad 0 \le \frac{s}{4} \le 2\pi, \quad 0 \le s \le 8\pi$$

Example 27 (Poll). **T/F:** The parameterization $\mathbf{r}(t) = \langle t, t^2\rangle, t \in \mathbb{R}$ is an arc length parameterization of the parabola $y = x^2$.

- $\vec{r}(t)$ is a parameterization of $y = x^2$; $t^2 = (t)^2$
- $\vec{r}\,'(t) = \langle 1, 2t\rangle \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{1+4t^2} \neq 1$

Section 13.4 - Curvature, Tangent, Normal

The next idea we are going to explore is the curvature of a curve in space along with two vectors that orient the curve.

First, we need the **unit tangent vector**, denoted $\mathbf{T}(s)$: ________________

This lets us define the **curvature**, $\kappa(s) =$ ________________

Question: In which direction is $\mathbf{T}$ changing?

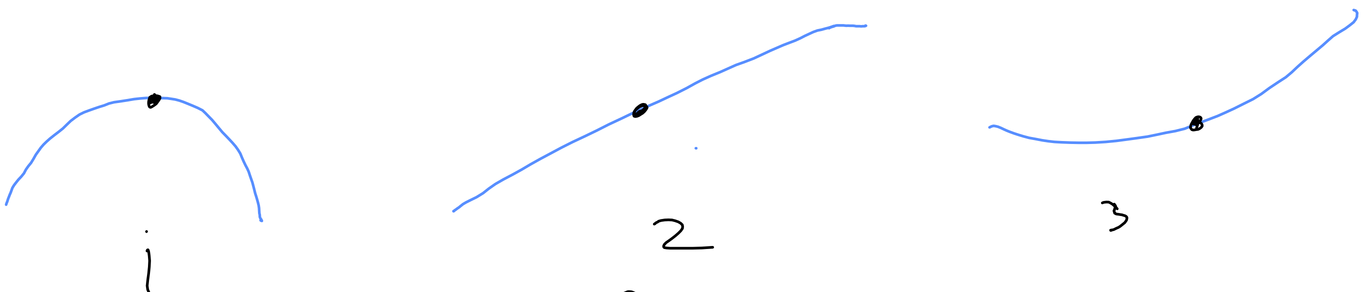
This is the direction of the **principal unit normal**, $\mathbf{N}(s) =$________________

Day 5 - Geometry of Curves Part II

Pre-Lecture

Section 13.4 - Curvature

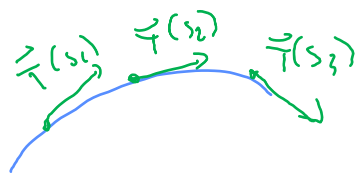
The next idea we are going to explore is the curvature of a curve in space along with two vectors that orient the curve.



curvature: 1 > 3 > 2

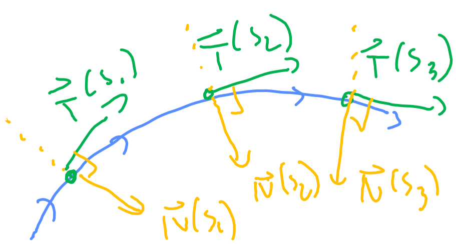
- geometric property of C; independent of parametrization
- To accomplish ↑, use arc-length parametrization $\vec{r}(s)$

First, we need the **unit tangent vector**, denoted $\mathbf{T}(s)$: $\vec{r}\,'(s)$



$\|\vec{T}(s_i)\| = 1$

This lets us define the **curvature**, $\kappa(s) = \left\| \frac{d}{ds}(\vec{T}(s)) \right\|$



- if $\|\vec{r}\,'(t)\|$ is constant then all of its acceleration is orthogonal to curve: $\vec{r}\,'(t) \cdot \vec{r}\,''(t) = 0$

normal

Question: In which direction is **T** changing? (rather than - - - -)

This is the direction of the **principal unit normal**, $\mathbf{N}(s) = \frac{d}{ds}(\vec{T}(s)) \Big/ \left\| \frac{d}{ds}(\vec{T}(s)) \right\|$

unit

Day 5 Lecture

Daily Announcements & Reminders:

- HW G2 & G3 due F
- Quiz 2 tomorrow on G2, similar format to Q1
  - Quiz 1 returned tomorrow
- Do warmup on PollEv →

Learning Targets:

- **G3: Geometry of Curves.** I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.
- **G5: Parameterization.** I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.

Goals for Today: Section 13.4

- Define, interpret, and compute the curvature of a curve
- Compute the unit tangent and principal unit normal vectors of a curve
- Extend the set of curves that we can parameterize

Warmup: The curvature of a line is $\underline{0}$.

Section 13.4: Computing Curvature, Tangent, and Normal Vectors

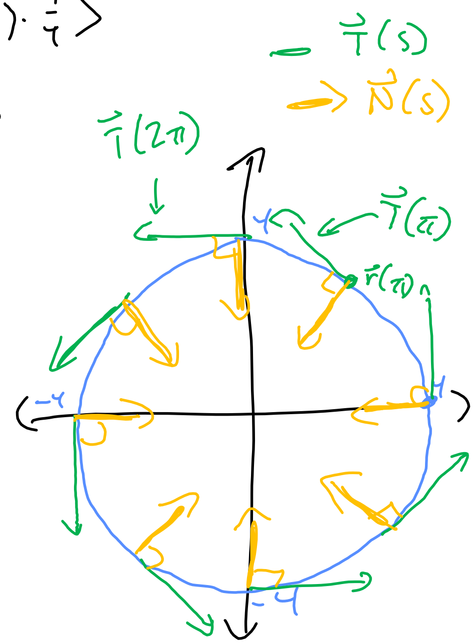
Example 28. Last time, we found an arc length parameterization of the circle of radius 4 centered at $(0,0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \le s \le 8\pi.$$

Use this to find $\underline{\mathbf{T}(s)}$, $\underline{\kappa(s)}$, and $\underline{\mathbf{N}(s)}$.

- $\|\vec{r}\,'(s)\| = 1$

- $\vec{T}(s) = \vec{r}\,'(s) = \left\langle -4\sin\left(\frac{s}{4}\right)\cdot\frac{1}{4}, \; 4\cos\left(\frac{s}{4}\right)\cdot\frac{1}{4}\right\rangle$
  $= \left\langle -\sin\left(\frac{s}{4}\right), \cos\left(\frac{s}{4}\right)\right\rangle$



- $\vec{T}\,'(s) = \left\langle -\frac{1}{4}\cos\left(\frac{s}{4}\right), -\frac{1}{4}\sin\left(\frac{s}{4}\right)\right\rangle$

$\kappa(s) = \|\vec{T}\,'(s)\| = \sqrt{\frac{1}{16}\cos^2\left(\frac{s}{4}\right) + \frac{1}{16}\sin^2\left(\frac{s}{4}\right)}$

$= \boxed{\frac{1}{4}}$

$\vec{N}(s) = \dfrac{\vec{T}\,'(s)}{\|\vec{T}\,'(s)\|} = \left\langle -\cos\left(\frac{s}{4}\right), -\sin\left(\frac{s}{4}\right)\right\rangle$

$\kappa(s)$ for circle of radius $R = \frac{1}{R}$

We said that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) =$ $\vec{r}'(t) / \|\vec{r}'(t)\|$
- $\mathbf{N}(t) =$ $\vec{T}'(t) / \|\vec{T}'(t)\|$
- $\kappa(t) =$ $\|\vec{T}'(t)\| / \|\vec{r}'(t)\|$ any $\mathbb{R}^n$ or $\dfrac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$ ← only in $\mathbb{R}^3$

Example 29. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1\rangle$. $t \in \mathbb{R}$

$\vec{T}(t)$: $\vec{r}'(t) = \langle -2\sin(t), 2\cos(t), 1\rangle$

$\|\vec{r}'(t)\| = \sqrt{4\sin^2(t) + 4\cos^2(t) + 1} = \sqrt{5}$

$$\boxed{\vec{T}(t) = \frac{1}{\sqrt{5}}\langle -2\sin(t), 2\cos(t), 1\rangle}$$

$\vec{N}(t)$: $\vec{T}'(t) = \frac{1}{\sqrt{5}}\langle -2\cos(t), -2\sin(t), 0\rangle$

$\|\vec{T}'(t)\| = \frac{1}{\sqrt{5}}\sqrt{4\cos^2(t) + 4\sin^2(t)} = \frac{2}{\sqrt{5}}$

$\vec{N}(t) = \frac{1}{2}\langle -2\cos(t), -2\sin(t), 0\rangle$

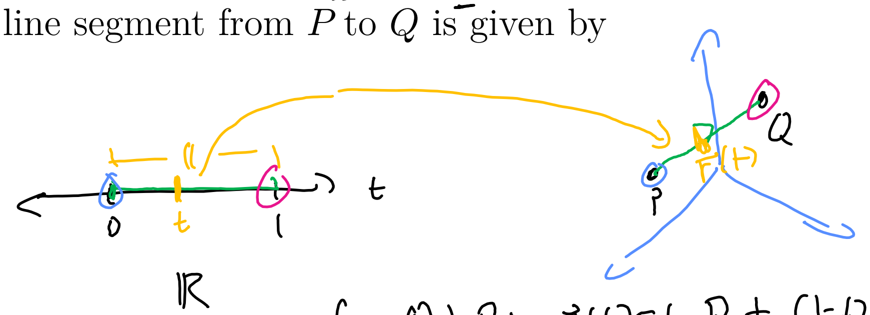
$$\boxed{\vec{N}(t) = \langle -\cos(t), -\sin(t), 0\rangle}$$

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{2/\sqrt{5}}{\sqrt{5}} = \boxed{\frac{2}{5}}$$

Parameterizing Curves

Let's return to parameterizing curves. We have seen a few examples so far and want to solidify our understanding of a few more classes of curves that we will work with.

- **Line segments:** If P and Q are points in $\mathbb{R}^{\cancel{3}\,n}$, then a parameterization of the line segment from P to Q is given by



$\vec{r}(t) = t\cdot \underset{\text{end}}{Q} + \underset{\text{start}}{(1-t)\cdot P}$, $0 \le t \le 1$

$\vec{r}(0) = P$ $\quad \vec{r}(1) = Q$

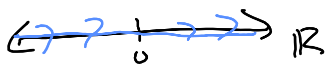
$= \boxed{P + t(Q-P),\ 0 \le t \le 1}$

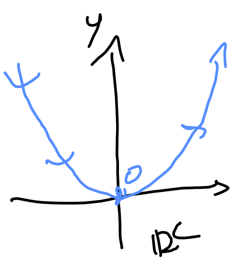
from Q to P: $\vec{r}(t) = t\cdot P + (1-t)\cdot Q$, $0 \le t \le 1$

- **Graphs of functions of one variable:** If $y = f(x)$ is a function of one variable, then a parameterization of its graph is given by

e.g. $y = x^2$

$\vec{r}(t) = \langle t, t^2\rangle$, $t \in \mathbb{R}$

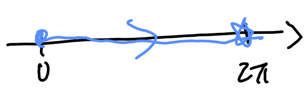


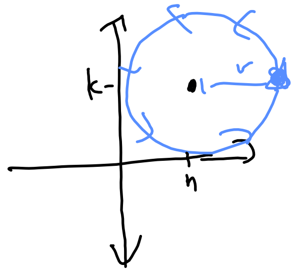


for $y = f(x)$: $\vec{r}(t) = \langle t, f(t)\rangle$, t in domain of f

for $x = g(y)$: $\vec{r}(t) = \langle g(t), t\rangle$, t in domain of g

- **Circles:** A circle of radius r centered at (h, k) in $\mathbb{R}^2$ can be parameterized by





$\vec{r}(t) = \langle r\cos(t) + h, r\sin(t) + k\rangle$

$t \in [0, 2\pi)$

- **Ellipses:** An ellipse with radius in the x-direction $\overset{2}{a}$ and radius in the y-direction b $(\sqrt{2})$ centered at (h, k) $(0,0)$ in $\mathbb{R}^2$ can be parameterized by

$\vec{r}(t) = \langle a\cos(t) + h, b\sin(t) + k\rangle$, $0 \le t \le 2\pi$

Intersections of Surfaces:

If a curve is specified as the intersection of two surfaces, our parameterization will depend on the equations of the surfaces. Often we can use this by eliminating variables until we can use one of the basic forms above.

Example 30. Find a parameterization of the curve of intersection of the surfaces $z = x^2 + y^2$ and $z = 4 - y^2$.

Idea: eliminate z to get "shadow curve" in xy-plane, parameterize x & y, plug in to z

$$x^2+y^2 = 4-y^2 \Rightarrow x^2+2y^2=4$$

$$\Rightarrow \frac{x^2}{4}+\frac{y^2}{2}=1 \quad \text{ellipse at } (0,0) \text{ w/ radii } 2 \text{ \& } \sqrt{2}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \begin{array}{l} x\text{-radius} = a \\ y\text{-radius} = b \end{array}$$

So $\vec{r}(t) = \langle 2\cos(t), \sqrt{2}\sin(t), \_\_\_\_ \rangle$

To get z: Plug into surface eqn:

$$z = x^2+y^2 = 4\cos^2(t)+2\sin^2(t)$$

So $\vec{r}(t) = \langle 2\cos(t), \sqrt{2}\sin(t), 4\cos^2(t)+2\sin^2(t)\rangle$

$0 \le t \le 2\pi$

Eliminate y first: $2z = x^2+4$

$$z = \frac{x^2}{2}+2$$

$\vec{r}(t) = \langle t, \_\_\_\_, \frac{t^2}{2}+2\rangle$, $-2 \le t \le 2$

$$\frac{t^2}{2}+2 = 4-y^2 \Rightarrow y^2 = 2-\frac{t^2}{2}$$

$\vec{r}_1(t) = \langle t, \sqrt{2-t^2/2}, \frac{t^2}{2}+2\rangle$ $\quad$ $\vec{r}_2(t) = \langle t, -\sqrt{2-t^2/2}, \frac{t^2}{2}+2\rangle$, $t \in [-2,2]$

Orientation: We will cover this topic in detail later in the semester.

The orientation of a curve is determined by the direction in which it is traced as the parameter increases. For example, if a curve is parameterized by $\mathbf{r}(t)$ for $a \leq t \leq b$, then the orientation is from $\mathbf{r}(a)$ to $\mathbf{r}(b)$.

Example 31 (Poll)**.** Let C be a curve parameterized by $\mathbf{r}(t)$ from $a \leq t \leq b$. Select all of the true statements below.

(a) $\mathbf{r}(t+4)$ for $a \leq t \leq b$ is also a parameterization of C with the same orientation

(b) $\mathbf{r}(2t)$ for $a/2 \leq t \leq b/2$ is also a parameterization of C with the same orientation

(c) $\mathbf{r}(-t)$ for $a \leq t \leq b$ is also a parameterization of C with the opposite orientation

(d) $\mathbf{r}(-t)$ for $-b \leq t \leq -a$ is also a parameterization of C with the opposite orientation

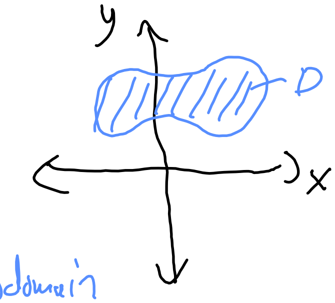
(e) $\mathbf{r}(b-t)$ for $0 \leq t \leq b-a$ is also a parameterization of C with the opposite orientation

Day 6 - Functions of Multiple Variables

Pre-Lecture

Section 14.1: Functions of Multiple Variables

Definition 25. A function of two variables is a rule that assigns to each pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$.



name —> $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^2$

domain: the set of allowable inputs to f

codomain

• the range: subset of codomain that are actually outputs

CAUTION: Domain cannot be an interval. [0,1] is not a domain for any fn. of 2 vars.

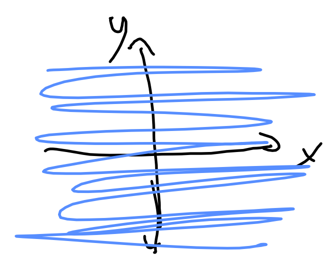
Example 26. Three examples are

$$f(x, y) = x^2 + y^2, \qquad g(x, y) = \ln(x + y), \qquad h(x, y) = \cancel{h(x,y)} = \sqrt{4 - x^2 - y^2}.$$

Example 27. Find the largest possible domains of f, g, and h.

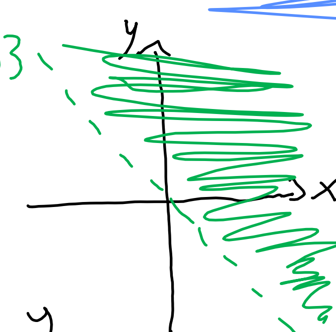
$f(x,y) = x^2+y^2$, -> All of $\mathbb{R}^2 = \{(x,y) \in \mathbb{R}^2 \mid x, y \in \mathbb{R}\}$.

$f(1,2) = 1^2 + 2^2 = 5$



$g(x,y) = \ln(x+y)$. Domain is $\{(x,y) \in \mathbb{R}^2 \mid x+y > 0\}$,

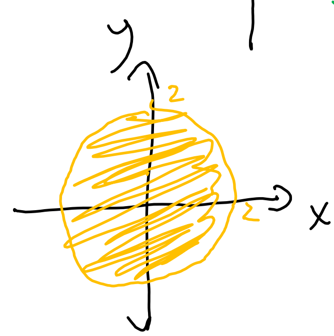
or $\{(x,y) \in \mathbb{R}^2 \mid y > -x\}$



$h(x,y) = \sqrt{4-x^2-y^2}$ Domain is

$\{(x,y) \in \mathbb{R}^2 \mid 4-x^2-y^2 \geq 0\}$

$\iff \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4\}$



Day 6 Lecture

Daily Announcements & Reminders:

- G2, G3 HW due F
- Q1 grades back on Gradescope
- Next week: Checkpoint 1 in studio W, 25 min, G3, G4, G5
- Today only: office hours are M 10-11 on Zoom
  Th 2-3 in Skiles 218C
- Do warmup on PollEv

Learning Targets:

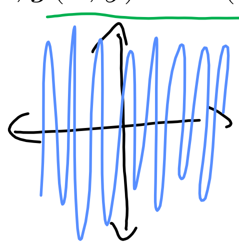
- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.

Goals for Today: Section 14.1

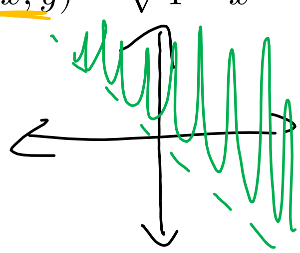
- Give examples of functions of multiple variables
- Find the domain of functions of two variables
- Introduce and sketch traces and contours of functions of two variables
- Use technology to graph functions of two variables
- Find level surfaces of functions of three variables

In the pre-lecture video, we discussed the domains of the functions $f(x,y) = x^2 + y^2$, $g(x,y) = \ln(x+y)$, and $h(x,y) = \sqrt{4-x^2-y^2}$.

f:

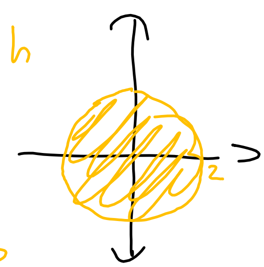


g



$x+y>0 \Rightarrow y>-x$

h



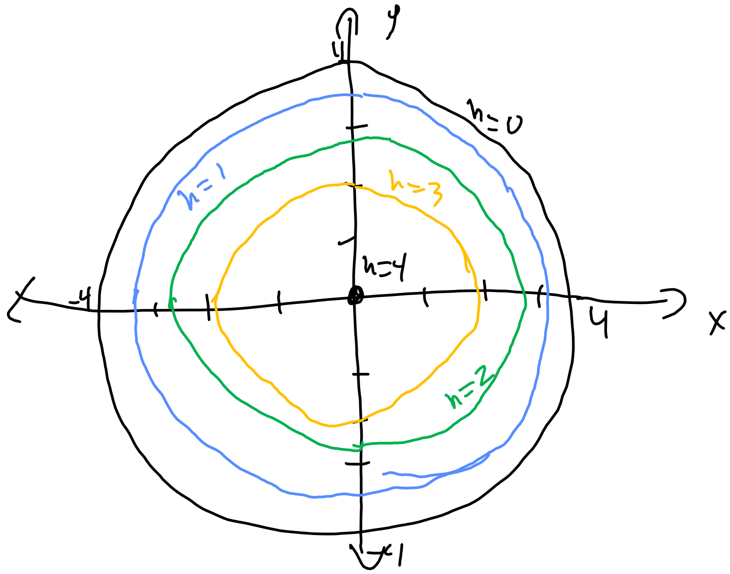
? $(1,1,\sqrt{2})$ is on graph of h

Definition 35. If f is a function of two variables with domain D, then the graph of f is the set of all points (x,y,z) in $\mathbb{R}^3$ such that $z = f(x,y)$ and (x,y) is in D.

but $(3,0,\text{—})$ not in graph

Here are the graphs of the three functions above.

Example 36. Suppose a small hill has height $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x,y). How could we draw a picture that represents the hill in 2D?



Use $\underbrace{\text{level curves}/\text{level sets}}_{\text{2D}}/\underbrace{\text{contours}}_{\text{general n-dim.}}$

↳ Set output z = constant & plot

$z=0: \quad 0 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$x^2 + y^2 = 16$

$z=1: \quad 1 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$x^2 + y^2 = 12$

$z=2: \quad 2 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

$x^2 + y^2 = 8$

In 3D, it looks like this.

Definition 37. The <u>contours</u> (also called <u>level curves</u>) of a function f of two variables are the curves with equations <u>$f(x,y)=k$</u>, where k is a constant (in the range of f). A plot of <u>contours</u> for various values of z is a <u>contour plot</u> (or <u>level curve plot</u>).

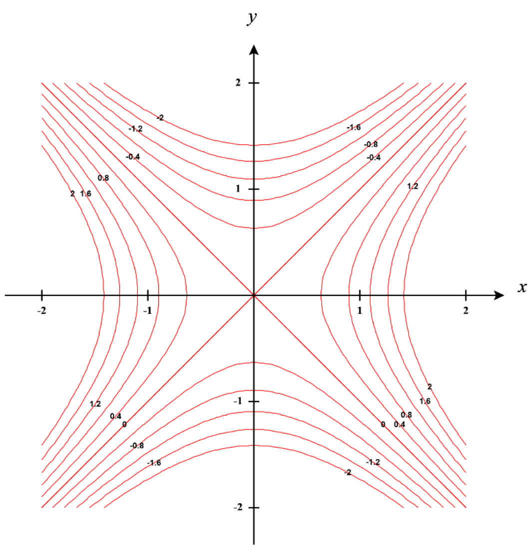
in $\mathbb{R}^2$

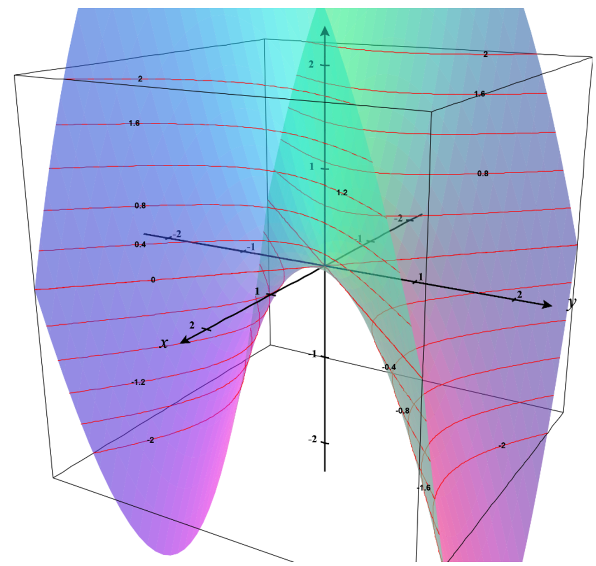
Some common examples of these are:

- topographical maps
- weather maps
- thermal imaging
- equipotential surfaces (sort of)
- radar scanner output
- MRI

Example 38. Use technology to create a contour diagram of $f(x, y) = x^2 - y^2$.

What do we notice about the contours?

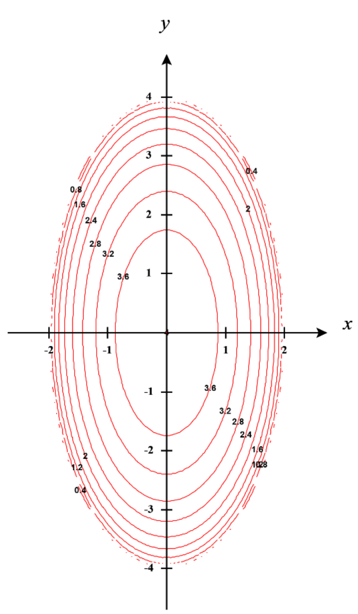


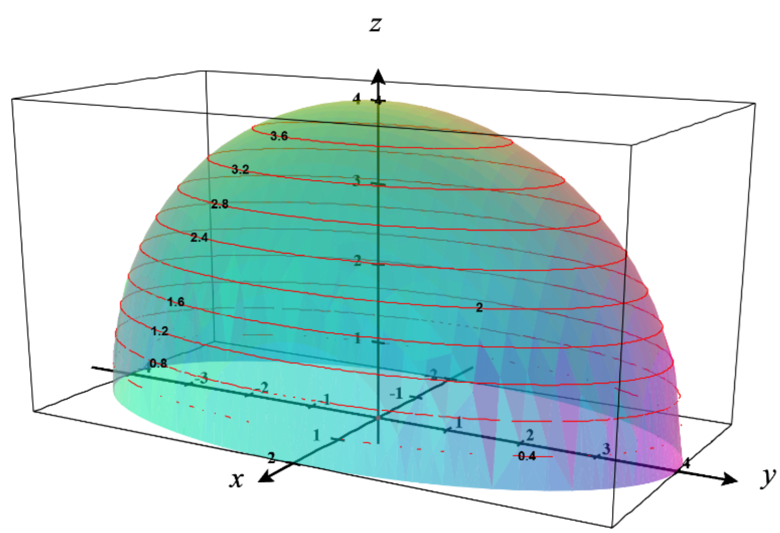


Example 39. Student work: Use technology to create a contour diagram of $g(x, y) = \sqrt{16 - 4x^2 - y^2}$.

CalcPlot3D

What do you notice about your contours?





Definition 40. The traces of a surface are the curves of intersection of the surface with planes parallel to the yz or xz planes.

Example 41. Find the traces of the surface $z = x^2 - y^2$. Can you see these in the graph produced by CalcPlot3D?

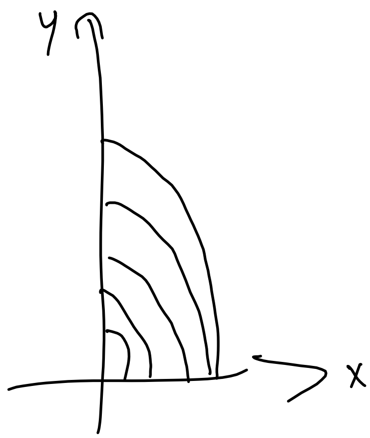
Yes!

Example 42. Use the graph of the portion of $z = f(x, y) = 4 - 2x - y^2$ in the first ~~quadrant~~ to identify and understand all of the traces and contours.

octant

↳ $x \geq 0, y \geq 0, z \geq 0$

contours:

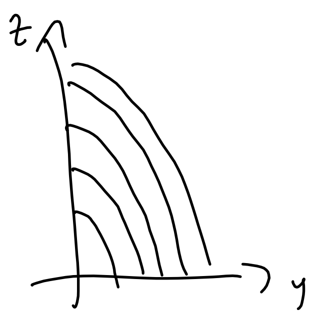


$k = 4 - 2x - y^2$

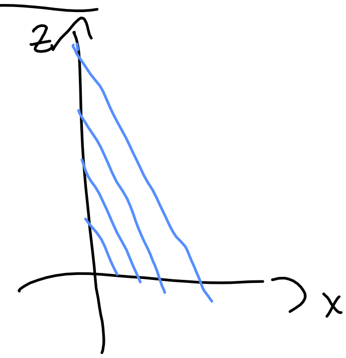
$y^2 = 4 - 2x - k$

$y = \sqrt{4 - 2x - k}$

x-traces: $z = 4 - 2k - y^2$



y-traces: $z = 4 - 2x - k^2$



Definition 43. A function of three variables is a rule that assigns to each triple of real numbers (x, y, z) in a set D a output (single real number) denoted by $f(x, y, z)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

$f(x,y,z) = 4$

Example 44. Describe the domain of the function $f(x, y, z) = \dfrac{1}{4 - x^2 - y^2 - z^2}$.

largest possible

Domain $= \{(x,y,z) \in \mathbb{R}^3 \mid 4 - x^2 - y^2 - z^2 \neq 0\}$

$\Updownarrow$

$x^2 + y^2 + z^2 \neq 4$

$\Updownarrow$

The domain of f is all of $\mathbb{R}^3$ except the sphere of radius 2 centered at the origin.

Example 45. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.

$0 = 2x^2 + y^2 + z^2 \Rightarrow$ point $(0,0,0)$

$1 = 2x^2 + y^2 + z^2 \Rightarrow$ ellipsoid

$k > 0 \quad k = 2x^2 + y^2 + z^2 \iff 1 = \dfrac{x^2}{\frac{k}{2}} + \dfrac{y^2}{k} + \dfrac{z^2}{k}$

Day 8 - Derivatives & Linear Approximation

Pre-Lecture

General Partial Derivatives

Definition 46. If f is a function of two variables x and y, its partial derivatives are the functions f_x and f_y defined by

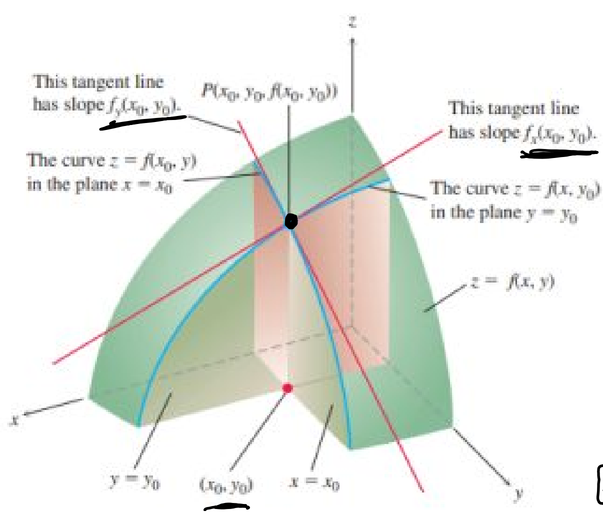
$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notations:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f)$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(f)$$

Interpretations:



Example 47. Find the partial derivatives f_x and f_y for

$$f(x,y) = 5x^2 + 2xy + 3y^3.$$

To find $f_x(x,y)$: treat y's as constants & derive w.r.t. x

$$f_x(x,y) = \frac{\partial}{\partial x}(5x^2) + \frac{\partial}{\partial x}(\underbrace{2xy}_{2y\cdot x}) + \frac{\partial}{\partial x}(3y^3)$$

$$= 10x + 2y + 0$$

$$f_y(x,y) = \frac{\partial}{\partial y}(5x^2) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial y}(3y^3)$$

$$= 0 + 2x + 9y^2$$

To get interpretation, plug in a point:

E.g. $f_x(1,1) = 10+2 = 12$, so rate of change of f at $(1,1)$ in the x-direction is 12.

The slope of the tangent line at $(1,1,10)$ to the curve $z = 5x^2 + 2x + 3$ in the plane $y=1$ is 12.

Day 7 Lecture

Daily Announcements & Reminders:

- Checkpoint 1 tomorrow: G3, G4, G5
  - formulas provided are on Ed Discussion
- Exam 1 is next T, 9/16, during lecture
  - G1-G5, D1
  - Bring 1 page handwritten notes
- Do warmup on Poll G

Learning Targets:

- **D1: Computing Derivatives.** I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.
- **A2: Interpreting Derivatives.** I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.
- **D2: Tangent Planes and Linear Approximations.** I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.

Goals for Today:

Section 14.3, 14.6

- Learn how to compute partial derivatives of functions of multiple variables
- Learn how to compute higher-order partial derivatives
- Understand Clairaut's theorem
- Define the total derivative
- Learn how to find a linear approximation of a differentiable function of multiple variables

Example 48. Find the partial derivatives of the functions below.

Poll

(a) $f(x,y) = 3x^2y + x - 2y$

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}(3x^2y) + \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial x}(2y)$$
$$= 3y\cdot 2x + 1 - 0$$
$$= 6xy + 1$$

← At $(1,2)$ the tangent line to the curve of intersection of $z=f(x,y)$ & $y=2$ ($z = 6x^2+x-4$ in $y=2$) has slope $f_x(1,2) = 13$

$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial y}(x) - \frac{\partial}{\partial y}(2y)$$
$$= 3x^2 + 0 - 2$$

(b) $g(x,y) = \sqrt{5x - y}$

$$g_x(x,y) = \frac{\partial}{\partial x}\left(\sqrt{5x-y}\right) = \frac{\partial}{\partial x}\left((5x-y)^{1/2}\right) = \frac{1}{2}(5x-y)^{-1/2}\cdot \frac{\partial}{\partial x}\underbrace{(5x-y)}_{}\ \overset{(5-0)}{}$$
$$= \frac{5}{2\sqrt{5x-y}}$$

$$g_y(x,y) = \frac{\partial}{\partial y}\left(\sqrt{5x-y}\right) = \frac{1}{2}(5x-y)^{-1/2}\cdot \frac{\partial}{\partial y}(5x-y)\ \overset{(0-1)}{}$$
$$= -\frac{1}{2\sqrt{5x-y}}$$

Question: How would you define the second partial derivatives?

Take partial derivatives of partial derivatives

∂

Notation:

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$
$$f_{xy} = \boxed{(f_x)_y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$
$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$
$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

ORDER IS INSIDE TO OUTSIDE

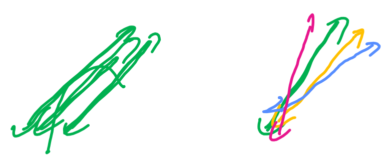
pure partials

mixed partials

Curvature of the other trace: $f_{xx}(a,b)$ measures concavity of the trace w/ $y=b$

"twist"

f_{xy}



$f_{xy} < 0$, $f_x < 0$

$f_{xy} > 0$, $f_x < 0$

Example 49. Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} of the function $f(x, y) = \sqrt{5x - y}$

$$f_x = \frac{5}{2}(5x-y)^{-1/2}$$
$$f_y = \frac{-1}{2}(5x-y)^{-1/2}$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{5}{2}(5x-y)^{-1/2}\right) = -\frac{5}{4}(5x-y)^{-3/2} \cdot 5$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{5}{2}(5x-y)^{-1/2}\right) = -\frac{5}{4}(5x-y)^{-3/2} \cdot (-1)$$

$$\parallel$$

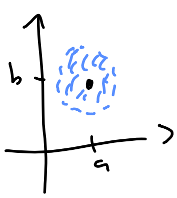
$$f_{yx} = \frac{\partial}{\partial x}\left(-\frac{1}{2}(5x-y)^{-1/2}\right) = \frac{1}{4}(5x-y)^{-3/2} \cdot (5)$$

$$\left(\frac{\partial}{\partial x}(5xy) = 5y \right.$$

$$f_{yy} = \frac{\partial}{\partial y}\left(-\frac{1}{2}(5x-y)^{-1/2}\right) = \frac{1}{4}(5x-y)^{-3/2} \cdot (-1)$$

What do you notice about f_{xy} and f_{yx} in the previous example? $f_{xy} = f_{yx}$

Theorem 50 (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions f, f_x, f_y, f_{xy}, f_{yx} are all continuous on D, then*



$$f_{xy}(a,b) = f_{yx}(a,b)$$

think "graphs don't have holes or asymptotes or weird jumps"

• true in higher dimensions & higher orders

$$f_{xyy} = f_{yxy} = f_{yyx}$$ if $f, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}, f_{xyy}, f_{yxy}, f_{yyx}$ are all cts

$$\neq f_{xxy}$$

Example 51. What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

$$f_x = \frac{\partial}{\partial x}(2x\underbrace{yz}_{y,z \text{ constant}}) - \frac{\partial}{\partial x}(z^2y) = 2yz - 0$$

$$f_y = \frac{\partial}{\partial y}(2xyz) - \frac{\partial}{\partial y}(z^2y) = 2xz - z^2$$

x, z constant

$$f_z = \frac{\partial}{\partial z}(2xyz) - \frac{\partial}{\partial z}(z^2y) = 2xy - 2yz$$

x, y constant

first derivatives

Example 52. How many rates of change should the function $\underbrace{f(s,t)}_{2 \text{ inputs}} = \begin{bmatrix} s^2 + t \\ 2s - t \\ st \end{bmatrix} \begin{matrix} \leftarrow x(s,t) \\ \leftarrow y(s,t) \\ \leftarrow z(s,t) \end{matrix}$ have? Compute them.

2 → deriv w.r.t. s, t

3 → look at rows

6 → Both!

$D x(s,t)$

$$Df(s,t) = \begin{bmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{bmatrix} = \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix}$$

total derivative of f

Total Derivatives

How might we **organize** this information?

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1, \dots, x_n) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{bmatrix}$,

we have ________ inputs, ________ output, and ________ partial derivatives, which we can use to form the **total derivative**.

This is a ____________ map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an ____________________, with one column per input and one row per output.

It has the formula $Df_{ij} =$

Example 53. Find the total derivatives of each function:

(a) $f(x) = x^2 + 1$

(b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

(c) $f(x, y) = \sqrt{5x - y}$

(d) $f(x, y, z) = 2xyz - z^2 y$

(e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$

Section 14.6: Linear Approximation

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \dots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $f(x) = f(a) + f'(a)(x - a)$.

Definition 54. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \dots, a_n)$ is

$$L(\mathbf{x}) =$$

Example 55. Find the linearization of the function $f(x, y) = \sqrt{5x - y}$ at the point $(1, 1)$. Use it to approximate $f(1.1, 1.1)$.

Question: What do you notice about the equation of the linearization?

Day 8 - Chain Rule and Directional Derivatives

Pre-Lecture

Section 14.4 - Chain Rule

Recall the Chain Rule from single variable calculus:

$$\frac{d}{dx}\big(f(g(x))\big) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x)$$

Similarly, the **Chain Rule** for functions of multiple variables says that if $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable functions then

$$\underbrace{D(f(g(\mathbf{x})))}_{m \times n} = \underbrace{\underbrace{Df(g(\mathbf{x}))}_{m \times p}\underbrace{Dg(\mathbf{x})}_{p \times n}}.$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R}^p \arrow[r, "f"] & \mathbb{R}^m \end{tikzcd}$$

Example 48. Suppose we are walking on our hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t + 1, 2 - t^2\rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

Want: $h_2'(1)$ (where $h_2(t) = h(\vec{r}(t))$)

$= Dh_2(1) \underset{\text{Chain Rule}}{=} Dh(\underbrace{\vec{r}(1)})\, D\vec{r}(1)$

Plug $\vec{r}(1)$ into Dh, not $t=1$

Need: $Dh(x,y)$, $D\vec{r}(t)$, $\vec{r}(1) = \langle x(1), y(1)\rangle$

$Dh(x,y) = \begin{bmatrix} -\frac{1}{2}x & -\frac{1}{2}y \end{bmatrix}$ $\quad D\vec{r}(t) = \begin{bmatrix} 1 \\ -2t \end{bmatrix}$ $\quad \vec{r}(1) = \langle 2, 1\rangle$

$h_2'(1) = Dh_2(1) = \begin{bmatrix} -\frac{1}{2}x & -\frac{1}{2}y \end{bmatrix}\Big|_{(2,1)} \cdot \begin{bmatrix} 1 \\ -2t \end{bmatrix}_{t=1}$

$= \begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -1 + 1 = \boxed{0 \text{ m/min}}$

Day 8 Lecture

Daily Announcements & Reminders:

- HW G4 due F
- Tech Demo 1 due tonight
- Checkpoint 1 grades back Monday evening
- Exam 1 on T, G1-G5 + D1
  - bring 1 handwritten page of notes
- Warmup on Poll Ev

Learning Targets:

- **D1: Computing Derivatives.** I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.
- ~~**D2: Tangent Planes and Linear Approximations.** I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.~~

Goals for Today:

- Be able to compute total derivatives

Sections 14.4-14.5

- Learn the Chain Rule for derivatives of functions of multiple variables
- Be able to compute implicit partial derivatives
- Introduce the directional derivative of a function of multiple variables

Total Derivatives

How might we **organize** derivative information?

$m \times 1$ col. vector

For any differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(\underbrace{x_1, \dots, x_n}_{1\times n \text{ vector input}}) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{bmatrix}$,

we have <u>n</u> inputs, <u>m</u> output, and <u>$n \cdot m$</u> partial derivatives, which we can use to form the **total derivative**.

This is a <u>linear</u> map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted $\boxed{Df}$, (or Jf or $Jac(f)$) and we can represent it with an <u>matrix</u>, with one column per input and one row per output.

It has the formula $Df_{ij} = \dfrac{\partial}{\partial x_j}\left(f_i(x_1, \dots, x_n)\right)$

$$Df_{23} = \frac{\partial}{\partial x_3}\left(f_2(x_1, \dots, x_n)\right)$$

Example 54. Find the total derivatives of each function:

(a) $f(x) = x^2 + 1$ $\quad f: \mathbb{R} \to \mathbb{R}$

$$Df(x) = [2x] = f'(x)$$

(b) $\mathbf{r}\boxed{(t)} = \langle \cos(t), \sin(t), t \rangle$ $\quad f: \mathbb{R} \to \mathbb{R}^3$

$$D\vec{r}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{bmatrix} = \vec{r}'(t)$$

$3 \times 1 \to$

(c) $f(x, y) = \sqrt{5x - y}$ $\quad f: \mathbb{R}^2 \to \mathbb{R}$

$$Df(x,y) = [f_x \quad f_y] = \left[\frac{5}{2\sqrt{5x-y}} \quad \frac{-1}{2\sqrt{5x-y}}\right]$$

1×2

(d) $f(x, y, z) = 2xyz - z^2 y$ $\quad f: \mathbb{R}^3 \to \mathbb{R}$

$$Df(x,y,z) = [f_x \quad f_y \quad f_z] = [2yz \quad 2xz - z^2 \quad 2xy - 2yz]$$

1×3

(e) $\mathbf{f}(s, t) = \langle \underbrace{s^2 + t}_{x(s,t)}, \underbrace{2s - t}_{y(s,t)}, \underbrace{st}_{z(s,t)} \rangle$ $\quad f: \mathbb{R}^2 \to \mathbb{R}^3$

$$Df(s,t) = \begin{bmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{bmatrix} = \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix}$$

3×2

At $(1,0)$:

$$Df(1,0) = \begin{bmatrix} 2 & 1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$$

Example 55. Use the Chain Rule to compute the total derivative of the composite function $h(x, y) = f(g(x, y))$ at $(1, 0)$ when

$$g(x, y) = x^3 + 2xy - y^2 \qquad f(t) = \begin{bmatrix} t^3 + 2t \\ t - 4 \end{bmatrix}.$$

Chain Rule: $D(f(g)) = Df(g)\, Dg$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "g"] \arrow[rr, bend right, "h"] & \mathbb{R} \arrow[r, "f"] & \mathbb{R}^2 \end{tikzcd}$$

$Dh(1,0) = Df(g(1,0))\, Dg(1,0)$

$Df(t) = \begin{bmatrix} 3t^2 + 2 \\ 1 \end{bmatrix}$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[rr, bend right, "\ell"] & \mathbb{R}^2 \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

$\ell = g(f(t))$

$\mathbb{R} \to \mathbb{R}$

$Dg(x,y) = [\, 3x^2 + 2y \quad 2x - 2y \,]$

DO NOT DO

$\cancel{Df(g(x,y)) = \begin{bmatrix} 3(3x^3 + 2xy - y^2)^2 + 2 \\ 1 \end{bmatrix}}$

$Dh(1,0) = Dh(x,y)\big|_{(1,0)}$

$g(1,0) = 1$

$$\underbrace{\begin{bmatrix} 5 \\ 1 \end{bmatrix}}_{Df(1)} \underbrace{[\, 3 \quad 2 \,]}_{Dg(1,0)} = \begin{bmatrix} 15 & 10 \\ 3 & 2 \end{bmatrix}$$

Example 56. Use the Chain Rule to compute the rate of change of the composite function $h(t) = g(f(t))$ at $t = 2$, where

$$f(t) = \begin{bmatrix} \frac{1}{4}t^2 + 2t \\ \sin(\pi t) - t \\ t + 1 \end{bmatrix} \qquad g(x, y, z) = x^2yz - y - z.$$

Poll

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[rr, bend right, "h"'] & \mathbb{R}^3 \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

$$Df(t) = \begin{bmatrix} \frac{1}{2}t + 2 \\ \pi\cos(\pi t) - 1 \\ 1 \end{bmatrix}$$

$$f(2) = \begin{array}{c|c} \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \end{array}$$

$$Dg(x,y,z) = \begin{array}{c} \begin{bmatrix} 2xyz & x^2z - 1 & x^2y - 1 \end{bmatrix} \\ \quad g_x \qquad\quad g_y \qquad\quad g_z \end{array}$$

Chain Rule

$$\boxed{Dh(t) = Dg(f(t)) \cdot Df(t)}$$

$$Dh(2) = Dg(f(2)) \cdot Df(2)$$

$$= \begin{bmatrix} -60 & 74 & -51 \end{bmatrix} \begin{bmatrix} 3 \\ \pi - 1 \\ 1 \end{bmatrix} = \boxed{74\pi - 305}$$

Example 57. Suppose that $W(s,t) = F(u(s,t), v(s,t))$, where F, u, v are differentiable functions and we know the following information.

$$\begin{array}{ll} u(1,0) = 2 & v(1,0) = 3 \\ u_s(1,0) = -2 & v_s(1,0) = 5 \\ u_t(1,0) = 6 & v_t(1,0) = 4 \\ F_u(2,3) = -1 & F_v(2,3) = 10 \end{array}$$

$$G(s,t) = \begin{bmatrix} u(s,t) \\ v(s,t) \end{bmatrix}$$

Find $\boxed{W_s(1,0) \text{ and } W_t(1,0)}$. Find $DW(1,0)$

$$\begin{tikzcd} \underset{(s,t)}{\mathbb{R}^2} \arrow[r, "G"] \arrow[rr, bend right, "W"'] & \underset{(u,v)}{\mathbb{R}^2} \arrow[r, "F"] & \mathbb{R} \end{tikzcd}$$

$$W(s,t) = F(G(s,t))$$

$$DW(s,t) = DF(G(s,t))\, DG(s,t)$$

$$\begin{bmatrix} W_s(1,0) & W_t(1,0) \end{bmatrix} = DW(1,0) = DF(G(1,0))\, DG(1,0)$$

$$= DF(2,3)\, DG(1,0)$$

$$= \begin{bmatrix} F_u(2,3) & F_v(2,3) \end{bmatrix} \begin{bmatrix} u_s(1,0) & u_t(1,0) \\ v_s(1,0) & v_t(1,0) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 10 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 52 & 34 \end{bmatrix}$$

Application to Implicit Differentiation: If $F(x, y, z) = c$ is used to *implicitly* define z as a function of x and y, then the chain rule says:

$$F(\underbrace{x, y, z(x,y)}_{G}) = c \qquad G(x,y) = \begin{bmatrix} x \\ y \\ z(x,y) \end{bmatrix}$$

$$\begin{bmatrix} F_x & F_y & F_z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ z_x & z_y \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$F_x + z_x F_z = 0$$

Example 58. Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the sphere $\boxed{x^2 + y^2 + z^2} = 4$.

$$F(x,y,z) = x^2 + y^2 + z^2$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$\frac{\partial z}{\partial x} = -\frac{2x}{2z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{2y}{2z} = -\frac{y}{z}$$

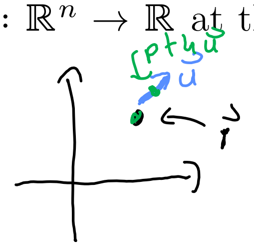
Day 10 - Directional Derivatives, Gradients, Tangent Planes

Pre-Lecture

Section 14.5 - Directional Derivatives

Definition 64. The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(\mathbf{p}) = \lim_{h \to 0} \frac{f(\vec{p} + h\vec{u}) - f(\vec{p})}{h}$$



if this limit exists.

Note that $D_{\mathbf{i}} f = f_x$ $\quad D_{\mathbf{j}} f = f_y$ $\quad D_{\mathbf{k}} f = f_z$

In practice, we want to avoid using this limit definition!

Note: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

- partial deriv's form a basis for all directional derivatives

$$D_{\mathbf{u}} f(\mathbf{p}) = \underbrace{Df(\vec{p})}\,\underbrace{\vec{u}}$$

matrix-vector product

Example 65. Compute the rate of change of $f(x, y) = e^{xy}$ at the point $(1, 2)$ in the direction $\mathbf{u} = \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle$.

↑ directional derivative

$$D_{\vec{u}} f(1,2) = \lim_{h \to 0} \frac{e^{\left(1 + \frac{h}{\sqrt{5}}\right)\left(2 + \frac{2h}{\sqrt{5}}\right)} - e^2}{h}$$

OR $Df = \begin{bmatrix} ye^{xy} & xe^{xy} \end{bmatrix}$

$Df(1,2) = \begin{bmatrix} 2e^2 & e^2 \end{bmatrix}$

$$D_{\vec{u}} f(1,2) = \begin{bmatrix} 2e^2 & e^2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$= \frac{2}{\sqrt{5}} e^2 + \frac{2}{\sqrt{5}} e^2 = \boxed{\frac{4}{\sqrt{5}} e^2}$$

Day 9 Lecture

Daily Announcements & Reminders:

- HW D1 due tomorrow
- Exam1 graded by W next week
- Do warmup on PollEv ⟶

Learning Targets:

- **D1: Computing Derivatives.** I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.
- **D2: Tangent Planes and Linear Approximations.** I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.
- **A2: Interpreting Derivatives.** I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.

Goals for Today:

Sections 14.4-14.6

- Learn to compute the rate of change of a multivariable function in any direction
- Investigate the connection between the gradient vector and level curves/surfaces
- Discuss tangent planes to surfaces, how to find them, and when they exist

$D_{\vec{u}} f(\vec{p})$ = rate of change of f in direction of $\vec{u}$ from point $\vec{p}$

$= Df(\vec{p}) \cdot \vec{u}$

Section 14.5: Gradients

Definition 61. If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient of f at $\mathbf{p} \in \mathbb{R}^n$ is the vector function ∇f (or grad(f)) defined by

$$\nabla f(\mathbf{p}) = \langle f_{x_1}(\vec{p}), f_{x_2}(\vec{p}), \ldots, f_{x_n}(\vec{p}) \rangle = Df(\vec{p})^T$$

Example 62. Find the gradient vector and the directional derivative of each function at the given point $\mathbf{p}$ in the direction of the given vector $\mathbf{u}$.

$f: \mathbb{R}^2 \to \mathbb{R}$

$D_{\vec{u}} f(\vec{p}) = \nabla f(\vec{p}) \cdot \vec{u}$

(a) $f(x, y) = \ln(x^2 + y^2), \mathbf{p} = (-1, 1), \mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

$Df(-1,1) = \begin{bmatrix} -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 \\ 1 \end{bmatrix}^T = \nabla f(-1,1)^T$

$\nabla f(x,y) = \left\langle \frac{1}{x^2+y^2} \cdot 2x, \frac{2y}{x^2+y^2} \right\rangle$

$\nabla f(-1,1) = \left\langle \frac{-2}{2}, \frac{2}{2} \right\rangle = \langle -1, 1 \rangle = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

coincidence that this is $\vec{p}$

rate of change $= D_{\vec{u}} f(-1,1) = \langle -1, 1 \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

$= \boxed{-\frac{3}{\sqrt{5}}}$

(b) $g(x, y, z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = (1, 2, 1)$, $\mathbf{u}$ the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$\nabla g(x,y,z) = \begin{bmatrix} 2x + 4y^2 \\ 8xy \\ 2z \end{bmatrix}$

$\nabla g(1,2,1) = \begin{bmatrix} 18 \\ 16 \\ 2 \end{bmatrix}$

rate of change $= D_u g(1,2,1) \cancel{=} \begin{bmatrix} 18 \\ 16 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 48$

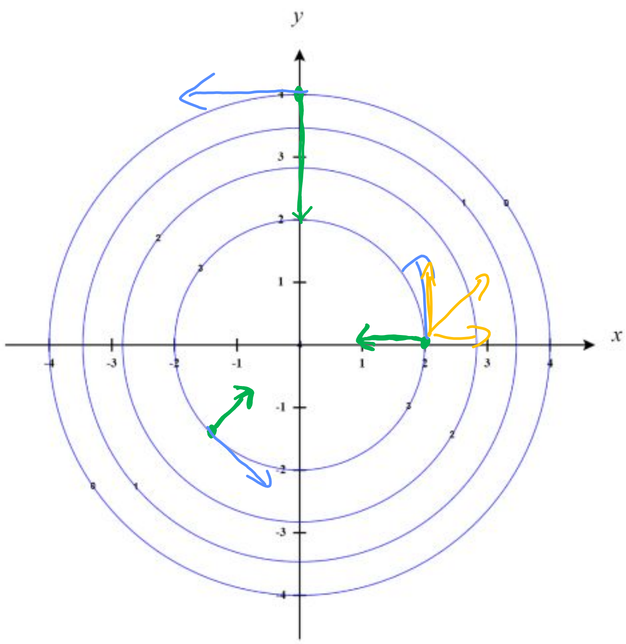
← speed $\sqrt{6}$ motion, so too big! need unit vector

$\to = \begin{bmatrix} 18 \\ 16 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{6}}$

$= 48/\sqrt{6}$

Example 63. If $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, the contour map is given below. Find and draw ∇h on the diagram at the points $(2,0), (0,4)$, and $(-\sqrt{2}, -\sqrt{2})$. At the point $(2,0)$, compute $D_{\mathbf{u}}h$ for the vectors $\mathbf{u}_1 = \mathbf{i}, \mathbf{u}_2 = \mathbf{j}, \mathbf{u}_3 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

$\nabla h = \langle -\frac{1}{2}x, -\frac{1}{2}y \rangle$



(a,b)	$\nabla h(a,b)$
$(2,0)$	$\langle -1, 0 \rangle$
$(0,4)$	$\langle 0, -2 \rangle$
$(-\sqrt{2}, -\sqrt{2})$	$\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$\vec{u}$	$D_u h(2,0)$
$\vec{i}$	$(-1)(1)+(0)(0) = -1$
$\vec{j}$	$(-1)(0)+(0)(1) = 0$
$\vec{u}_3$	$-\frac{1}{\sqrt{2}}$

What do we notice?
- point toward origin / increasing direction (b/c symmetry)
- vectors on same circle → same magnitude
- orthogonal to contours

Always true

• moving in tangent dir to a contour
$\Rightarrow$ 0 rate of change

• $D_{\vec{u}}h = \nabla h \cdot \vec{u} = \|\nabla h\| \, \|\vec{u}\| \cos\theta$ (with $\|\vec{u}\| = 1$)

maximized when $\theta = 0 \iff \vec{u} = \frac{\nabla h}{\|\nabla h\|}$

& then max rate of change $= \|\nabla h\|$

Note that the gradient vector is orthogonal to level curves.

Similarly, for $f(x,y,z)$, $\nabla f(a,b,c)$ is orthogonal to level surfaces

Section 14.6: Linear Approximation

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \dots, a_n)$, is the linear function that best approximates $\underbrace{f(\mathbf{x}) - f(\mathbf{a})}_{\text{change in } f}$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $\boxed{f(x) = f(a) + f'(a)(x - a).}$

y-value start

mult. by deriv.

change in input

Definition 64. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \dots, a_n)$ is

$$L(\mathbf{x}) = f(\vec{a}) + Df(\vec{a})\,(\vec{x} - \vec{a})$$

vector

if $f: \mathbb{R}^2 \to \mathbb{R}$ $(f(x,y)$ at $(a,b))$ $[f_x \;\; f_y]$

$$L(x,y) = f(a,b) + Df(a,b)\begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

$$= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Example 65. Find the linearization of the function $f(x,y) = \sqrt{5x - y}$ at the point $(1,1)$. Use it to approximate $f(1.1, 1.1)$.

Find $L(x,y)$ at $(1,1)$: $f(1,1) = \sqrt{4} = 2$

$$Df(x,y) = \begin{bmatrix} \frac{5}{2\sqrt{5x-y}} & \frac{-1}{2\sqrt{5x-y}} \end{bmatrix}$$

$$Df(1,1) = \begin{bmatrix} \frac{5}{4} & -\frac{1}{4} \end{bmatrix}$$

So $L(x,y) = f(1,1) + Df(1,1)\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$

$$= 2 + \begin{bmatrix} \frac{5}{4} & \frac{-1}{4} \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$= 2 + \frac{5}{4}(x-1) - \frac{1}{4}(y-1)$$

$$\sqrt{4.4} = f(1.1, 1.1) \approx L(1.1, 1.1) = 2 + \frac{5}{4}(1.1-1) - \frac{1}{4}(1.1-1)$$

$$= 2 + 0.125 - .025$$

$$= 2.1$$

$$\sqrt{40} = f(10,10) \overset{??}{\approx} L(10,10) = 2 + \frac{5}{4}\cdot 9 - \frac{1}{4}(9) = 11$$

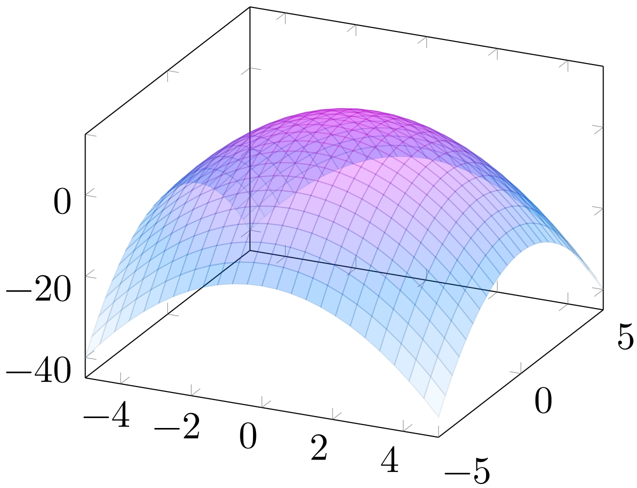
≈ 6.32

Question: What do you notice about the equation of the linearization?

This is a plane – the tangent plane to $z = f(x,y)$ at (a,b).

Section 14.6 - Tangent Planes

Suppose S is a surface with equation $F(x, y, z) = k$. How can we find an equation of the tangent plane of S at $P(x_0, y_0, z_0)$?



$x^2 + y^2 + z = 10, P = (-1, 3, 0)$

Example 66. Find the equation of the tangent plane at the point $(-2, 1, -1)$ to the surface given by

$$z = 4 - x^2 - y$$

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

This should look familiar: it's ______________________________

Day 11 - Optimization: Local & Global

Pre-Lecture

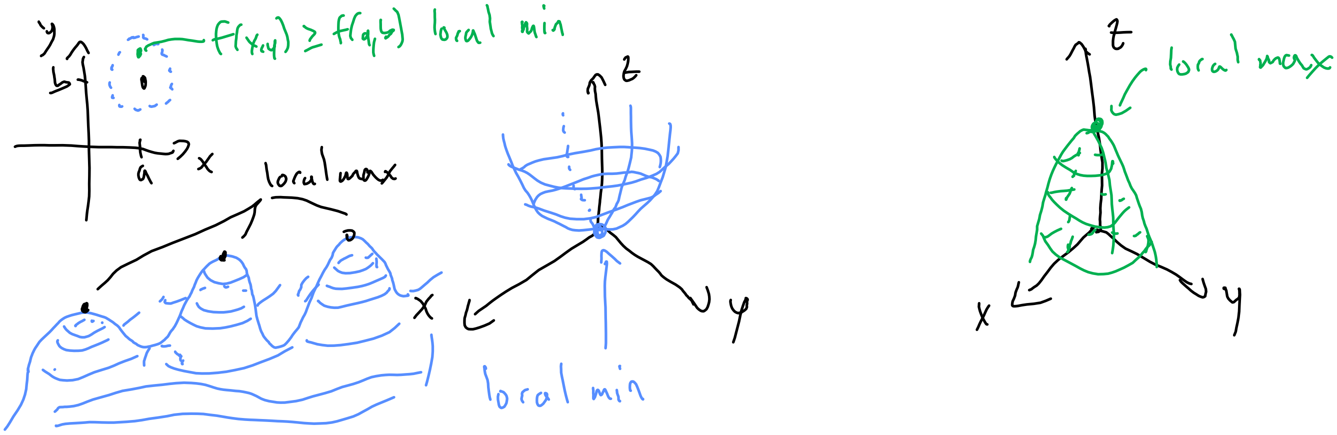
Section 14.7 - Local Extreme Values

Last time: If $f(x, y)$ is a function of two variables, we said $\nabla f(a, b)$ points in the direction of greatest change of f.

What does it mean if $\nabla f(a, b) = \langle 0, 0\rangle$?

Definition 70. Let $f(x, y)$ be defined on a region containing the point (a, b). We say

- $f(a, b)$ is a local minimum value of f if $f(a, b)$ $\leq$ $f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)
- $f(a, b)$ is a local maximum value of f if $f(a, b)$ $\geq$ $f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)



In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $z = x^2 - y^2$ (a hyperbolic paraboloid!) near $(0, 0)$.

This is called a saddle point : ≥ 1 direction of increase at point, ≥ 1 direction of decrease

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e. $Df = [0\ 0]$ $\nabla f = \langle 0, 0\rangle$

Definition 71. If $f(x,y)$ is a function of two variables, a point (a,b) in the domain of f with $Df(a,b) =$ $[0\ \ 0]$ or where $Df(a,b)$ undefined is called a critical points of f.

- If f has a local min or local max at (a,b) then (a,b) is a critical point.

Example 72. Find the critical points of the function $f(x,y) = x^3 + y^3 - 3xy$.

Solve $Df = [3x^2-3y \quad 3y^2-3x] = [0 \quad 0]$

$$\begin{cases} 3x^2-3y=0 \\ 3y^2-3x=0 \end{cases} \to \begin{cases} x^2=y \\ y^2=x \end{cases}$$

$\to x^4 = x$

$\to x(x^3-1)=0$

$x=0$ or $x=1$

$y=x^2$

so $y=0$ $y=1$

crit points: $(0,0)$ & $(1,1)$

Day 10 Lecture

Daily Announcements & Reminders:

- HW D2, A1 due Friday now (D2, D3, A1)
- Quiz 3 topic switch w/ Checkpoint 2
  - now on D1 (computing partial/total/directional derivatives & Chain Rule)
- Exam 1 graded & back tomorrow (solutions avail at 9am)
- Tech Demo 1 graded & back by next T
- Do warmup on PollEv

Learning Targets:

- **D3: Optimization.** I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.
- D2

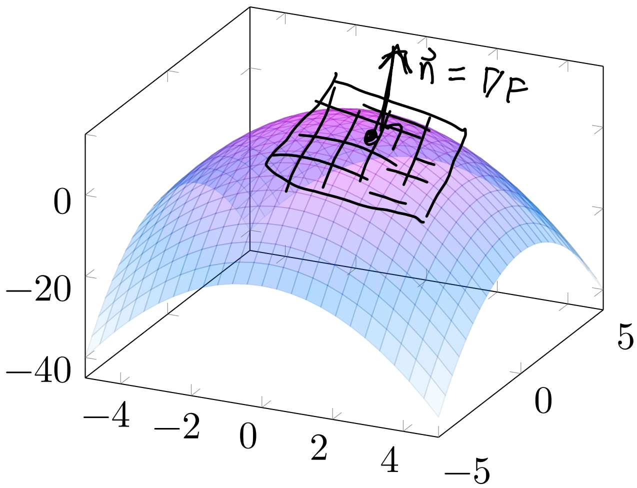
Goals for Today: Section 14.7

- Discuss tangent planes to surfaces, how to find them, and when they exist
- Define local & global extreme values for functions of two variables
- Learn how to find local extreme values for functions of two variables
- Learn how to classify critical points for functions of two variables

Section 14.6 - Tangent Planes

Suppose S is a surface with equation $F(x,y,z)=k$. How can we find an equation of the tangent plane of S at $P(x_0,y_0,z_0)$?

$z = L(x,y)$ for $f(x,y)$ at (a,b) is a tangent plane for surface $z=f(x,y)$ at $(a,b,f(a,b))$



$x^2+y^2+z=10, P=(-1,3,0)$

What about a sphere? $\Leftarrow F(x,y,z) = x^2+y^2+z^2$

To write down this tangent plane need:

1) a point $\Rightarrow$ point of tangency (x_0,y_0,z_0) or (a,b,c)

$(-1,3,0)$ in ex.

2) normal vector $\Rightarrow$ gradient of function F at (x_0,y_0,z_0)

not: $G = \cancel{z+z = 10-y^2}$

$F(x,y,z) = x^2+y^2+z-10$; $F(x,y,z)=0$

$$\nabla F = \langle \underset{F_x}{2x}, \underset{F_y}{2y}, \underset{F_z}{1} \rangle$$

At $(-1,3,0)$: $\nabla F = \langle -2, 6, 1 \rangle$

$\hookrightarrow$ tangent plane:

$$\nabla F(-1,3,0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

$$\boxed{-2(x+1)+6(y-3)+z=0}$$

$$-2x+6y+z=20$$

Why is ∇F orthogonal to level sets?

Take a level curve C of $f(x,y)$

$C: f(x,y)=4$

Parametrize $C: \vec{r}(t)$, $a \le t \le b$

Now, what is $f(\vec{r}(t)) = 4$

$$\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$$

Example 69. Find the equation of the tangent plane at the point $(-2, 1, -1)$ to the surface given by

$$\boxed{z = 4 - x^2 - y}$$

1) Rearrange to write as level set:

$$\underbrace{x^2 + y + z - 4}_{F(x,y,z)} = 0$$

$0 = \underbrace{4 - x^2 - y - z}_{} = F$

$\nabla F = \langle -2x, -1, -1 \rangle$

⇑ gradient parallel to

2) Compute gradient: $\nabla F = \langle 2x, 1, 1 \rangle$

3) Evaluate : $\nabla F(-2,1,-1) = \langle -4, 1, 1 \rangle$

4) Write plane: $\boxed{-4(x+2) + 1(y-1) + 1(z+1) = 0}$

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

This should look familiar: it's the linearization of $f(x,y)$

§ 14.7

Example 70. [Poll] Which of the following functions have a critical point at $(0,0)$?

$f(x,y) = 3x+y^3+2y^2$ | $g(x,y) = \cos(x)+\sin(y)$ | $h(x,y) = \dfrac{4}{x^2+y^2}$ | $k(x,y) = x^2+y^2$

(a,b) is a crit pt of $f(x,y) \iff Df(a,b) = \vec{0}$ or DNE & $(a,b) \in$ domain

$Df = [3 \quad 3y^2+4y]$, $Df(0,0) = [3 \quad 0]$ ✗

$Dg = [-\sin(x) \quad \cos(y)]$ $Dg(0,0) = [0 \quad 1]$ ✗

$Dh = \left[\dfrac{-8x}{(x^2+y^2)^2} \quad \dfrac{-8y}{(x^2+y^2)^2}\right]$ $Dh(0,0)$ DNE but $(0,0)$ is not in domain of h! ✗

$Dk = [2x \quad 2y]$ $Dk = [0 \quad 0]$ ✓

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix** of $f(x,y)$ at (a,b):

$$D^2 f(a,b) = Hf(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix} = D(\nabla f(a,b))$$

Theorem 71 (2nd Derivative Test). *Suppose (a,b) is a critical point of $f(x,y)$ and f has continuous second partial derivatives. Then we have:*

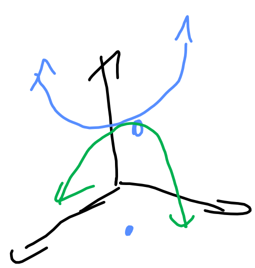
- *If $\det(Hf(a,b)) > 0$ and $\underline{f_{xx}(a,b) > 0}$, $f(a,b)$ is a local minimum*
- *If $\det(Hf(a,b)) > 0$ and $\underline{f_{xx}(a,b) < 0}$, $f(a,b)$ is a local maximum*
- *If $\det(Hf(a,b)) < 0$, f has a saddle point at (a,b)*
- *If $\det(Hf(a,b)) = 0$, the test is inconclusive.*

$$\det(Hf(a,b)) = \boxed{f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b) f_{yx}(a,b)}$$

"there are (at least) two directions where f has different concavity"

"f is changing the 'same' way in all directions"

$f_{xx} > 0$ & $f_{yy} < 0$



[Advanced] More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at $\mathbf{p}$ then

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, f is concave up in every direction from $\mathbf{p}$ and so has a local minimum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, f is concave down in every direction from $\mathbf{p}$ and so has a local maximum at $\mathbf{p}$.
- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, f is concave up in some directions from $\mathbf{p}$ and concave down in others, so has neither a local minimum or maximum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, f may have either a local minimum or neither at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, f may have either a local maximum or neither at $\mathbf{p}$.

A

$$\begin{bmatrix} \boxed{f_{xx}} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

Use $\det(Hf)$, $\det(A)$, $\det[f_{xx}]$

Example 72. Classify the critical points of $f(x,y) = x^3 + y^3 - 3xy$ from Example 68.

$$Df = \begin{bmatrix} 3x^2 - 3y & 3y^2 - 3x \end{bmatrix}$$

Crit pts: $(0,0)$ & $(1,1)$

Classify: $Hf(x,y) = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$

At $(0,0)$: $Hf(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$ $\quad \det Hf(0,0) = 0 - (-3)(-3) = -9 < 0$

so by 2^{nd} Deriv. Test, f has a saddle at $(0,0)$

At $(1,1)$: $Hf(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$ $\quad \det Hf(1,1) = 36 - 9 = 27 > 0$

& $f_{xx}(1,1) = 6 > 0$

so by 2^{nd} Deriv Test, f has a local min at $(1,1)$.

of $f(1,1) = -1$

Example 73. Find and classify the critical points of $f(x, y) = x^2y + y^2 + xy$.

1) Find crit pts.

$$Df = [2xy + y \quad x^2 + 2y + x] = [0 \quad 0]$$

$$\begin{cases} 2xy + y = 0 \\ x^2 + 2y + x = 0 \end{cases}$$

DON'T DIVIDE BY ANYTHING YOU AREN'T SURE IS NON-ZERO

$$(2x+1)y = 0$$

Case 1: $2x+1 = 0$ $x = -\frac{1}{2}$

$x^2 + 2y + x = 0 \Rightarrow$

$\frac{1}{4} + 2y - \frac{1}{2} = 0$

$2y = \frac{1}{4}$

$y = \frac{1}{8}$

$\left(-\frac{1}{2}, \frac{1}{8}\right)$ is a crit pt.

Case 2: $y = 0$

$x^2 + 2y + x = 0 \Rightarrow$

$x^2 + x = 0$

$x(x+1) = 0$

$x = 0$ or $x = -1$

$(0,0)$ & $(-1,0)$ are crit pt.

2) Classify:

$$Hf = \begin{bmatrix} 2y & 2x+1 \\ 2x+1 & 2 \end{bmatrix}$$

At $(0,0)$: $Hf = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

$\det Hf(0,0) = -1 < 0$

so f has a saddle point at $(0,0)$

At $(-1,0)$: $Hf = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$ $\det Hf(-1,0) = -1 < 0$

so f has a saddle point at $(-1,0)$

At $\left(-\frac{1}{2}, \frac{1}{8}\right)$: $Hf = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 2 \end{bmatrix}$ $\det Hf\left(-\frac{1}{2}, \frac{1}{8}\right) = \frac{1}{2} > 0$, $f_{xx}, f_{yy} > 0$

so f has a local min at $\left(-\frac{1}{2}, \frac{1}{8}\right)$

Two Local Maxima, No Local Minimum: The function $g(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2 + 2$ has two critical points, at $(-1, 0)$ and $(1, 2)$. Both are local maxima, and the function never has a local minimum!

A global maximum of $f(x, y)$ is like a local maximum, except we must have $f(a, b) \geq f(x, y)$ for **all** (x, y) in the domain of f. A global minimum is defined similarly.

Theorem 74 (Extreme Value Theorem)**.** *On a closed & bounded domain, any continuous function $f(x, y)$ attains a global minimum & maximum.*

Closed:

Bounded:

Day 11 Lecture

Daily Announcements & Reminders:

- HW D2/A1 due tomorrow
- Next week: - Q4 on D3 (W)
  - Tech Demo 2 (R - grace period through F)
    - TD1 back by W morning
- Do warmup
- Grades & Progress

Learning Targets:

- **D3: Optimization.** I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.

Goals for Today:

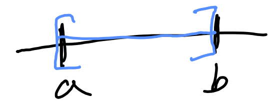
Sections 14.7, 14.8

- Learn how to find global extreme values on a closed & bounded domain
- Find global extreme values of continuous functions of two variables on closed & bounded domains
- Apply the method of Lagrange multipliers to find extreme values of functions of two or more variables subject to one or more constraints

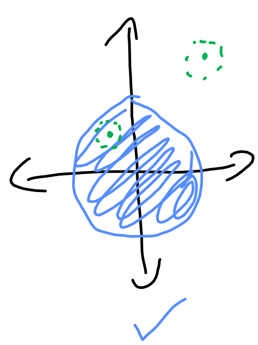
A global maximum of $f(x,y)$ is like a local maximum, except we must have $f(a,b) \geq f(x,y)$ for **all** (x,y) in the domain of f. A global minimum is defined similarly.

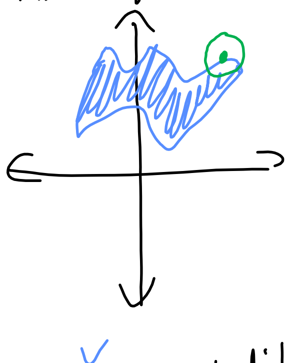
14.7 - Applying Extreme Value Theorem

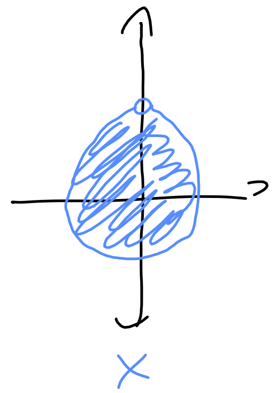
Theorem 75 (Extreme Value Theorem). *On a closed & bounded domain, any continuous function $f(x, y)$ attains a global minimum & maximum.*

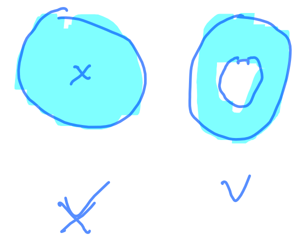


Closed: Contains its edge/boundary





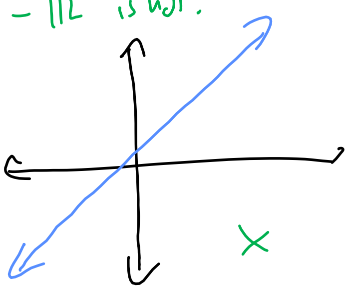




$\mathbb{R}^2$ ✓

Bounded: fits in a large enough disk

- first 5 examples above are bounded
- $\mathbb{R}^2$ is not.

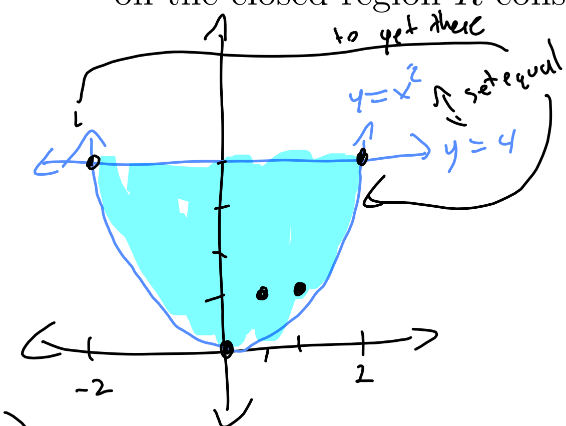


Example 76. [Poll] Now you try! Which of the following domains are closed? Which of the following are bounded?

Strategy for finding global min/max of continuous $f(x,y)$ on a closed & bounded domain R

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R ← reduce # variables & do single-var process
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 77. Find the global minimum and maximum of $f(x,y) = 4x^2 - 4xy + 2y$ on the closed region R consisting of those points with $x^2 \le y \le 4$.



1) $Df = [8x-4y \quad -4x+2] = [0 \quad 0]$

$x = \frac{1}{2}$ $(-4x+2=0)$

so $8x-4y=0 \Rightarrow 4-4y=0 \Rightarrow y=1$

$(\frac{1}{2}, 1)$ is in region ✓

2) Boundary:

1) Replace variable / Substitute

e.g. on $y=4$: $-2 \le x \le 2$

$f(x,y) = f(x,4) = \underbrace{4x^2 - 16x + 8}_{g(x)}, \quad -2 \le x \le 2$

$g'(x) = 8x - 16 = 0 \Rightarrow x = 2$

Also check endpoints: $(-2,4)$ & $(2,4)$

2) Parameterize [Useful esp. if boundary is a circle/ellipse

e.g. on $y=x^2$; $-2 \le x \le 2$

$\vec{r}(t) = \langle t, t^2 \rangle, \quad -2 \le t \le 2$ (endpoints same)

$f(\vec{r}(t)) = 4(t)^2 - 4(t)(t^2) + 2(t^2)$

$h(t) = 6t^2 - 4t^3, \quad -2 \le t \le 2$

$h'(t) = 12t - 12t^2 = 0 \Rightarrow t=0, t=1$

$(0,0)$ $(1,1)$

3)

(x,y)	$f(x,y)$
$(\frac{1}{2}, 1)$	1
$(2,4)$	-8
$(-2,4)$	56
$(0,0)$	0
$(1,1)$	2

4) f_{max} is 56

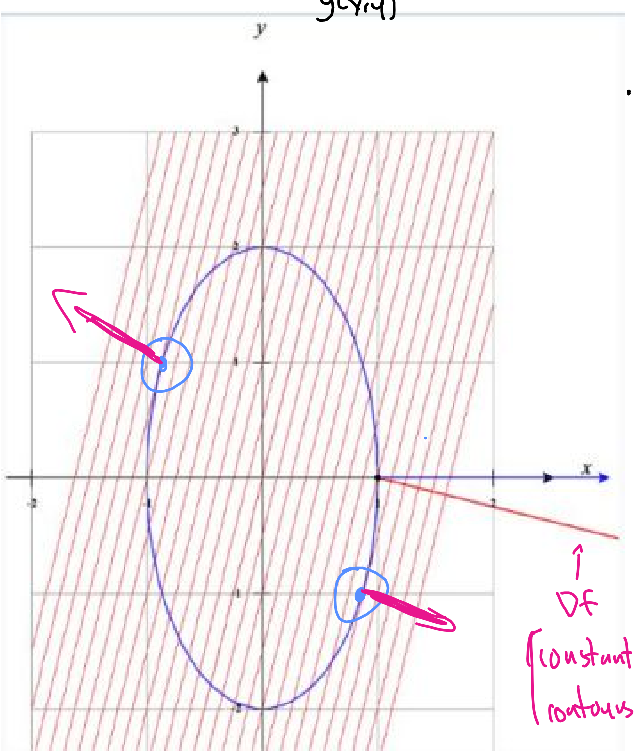
f_{min} is -8

14.8 - Lagrange Multipliers

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\underbrace{\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)}_{\nabla f \in \text{Span}(\nabla g, \nabla h)}$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

Example 78 (Poll)**.** Find the points where the value of the function of f with contours given in red (the lines) may take a minimum or maximum value subject to the constraint $\underbrace{4x^2 + y^2}_{g(x,y)} = 1$ (the ellipse).



$g(x,y) = 4x^2 + y^2$

We need $\nabla f = \lambda \nabla g$

∇f

(constant since contours are parallel lines)

Example 79. Find the constrained maxima and minima of $f(x,y) = 2x + y$ given that $x^2 + y^2 = 4$

Objective: $f(x,y) = 2x+y$

λ = rate of change of min/max as c changes

Constraint: $g(x,y) = x^2+y^2 = 4$

Lagrange Multipliers: $\begin{cases} \nabla f = \lambda \nabla g \\ g = 4 \end{cases} \Rightarrow \begin{cases} \langle 2, 1\rangle = \lambda \langle 2x, 2y\rangle \\ x^2+y^2=4 \end{cases}$

b/c LHS $\neq 0$, $x, y, \lambda \neq 0$

so try eliminating λ: ①: $\lambda = \frac{1}{x} \overset{\text{into ②}}{\Rightarrow} 1 = 2\left(\frac{1}{x}\right)y$

$\begin{cases} ① \; 2 = 2\lambda x \\ ② \; 1 = 2\lambda y \\ ③ \; x^2+y^2=4 \end{cases}$

Plug in to ②: $\begin{cases} x = 2y & ④ \\ x^2+y^2=4 & ⑤ \end{cases}$

Plug ④ into ⑤: $4y^2+y^2=4$

$y^2 = 4/5$

$y = \pm\sqrt{4/5} = \pm \frac{2}{\sqrt5}$

Test pts: $f\left(\frac{4}{\sqrt5}, \frac{2}{\sqrt5}\right) = \frac{10}{\sqrt5} = f_{max}$

$f\left(-\frac{4}{\sqrt5}, -\frac{2}{\sqrt5}\right) = \frac{-10}{\sqrt5} = f_{min}$.

See supplement videos

Example 80. Set up a system of equations to find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

Use Lagrange multipliers:

1) objective: $\cancel{z^2 = xy + 4}$?

$\boxed{\text{distance to origin? } f = \sqrt{x^2+y^2+z^2}}$

$\cancel{x+y+z=0}$?

constraint: $z^2 = xy + 4$

$\underbrace{z^2 - xy - 4}_{g(x,y,z)} = 0$

• $f(u) = \sqrt{u}$ is monotonic (also e^u, $\ln(u)$)

so min/max of $\sqrt{u}$ happens at a min/max of u

Use: objective $f = x^2+y^2+z^2 = \text{dist}^2$ to origin

constraint $z^2 - xy - 4 = 0$.

do this to make algebra nicer

allowed b/c

System:
$$\begin{cases} 2x = \lambda(-y) \\ 2y = \lambda(-x) \\ 2z = \lambda(2z) \\ z^2 - xy - 4 = 0 \end{cases}$$

Example 81. Set up the system of equations from the method of Lagrange multipliers to find the extreme values of the function $f(x, y, z) = x^2 + y^2 + z^2$ on the curve of intersection of the surfaces $x^2 + y^2 - z = 3$ and $x + 2y - 2z = 2$.

Method 1: Parametrize curve, plug into f to get a function of t, optimize

Method 2: L.M w/ 2 constraints

Objective: $f(x,y,z) = x^2+y^2+z^2$

Constraints: $g(x,y,z) = x^2+y^2-z = 3$

$h(x,y,z) = x+2y-2z = 2$

$\nabla g = \langle 2x, 2y, -1\rangle$

$\nabla h = \langle 1, 2, -1\rangle$

System: $\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = 3 \\ h = 2 \end{cases}$

$$= \begin{cases} 2x = \lambda(2x) + \mu \\ 2y = \lambda(2y) + 2\mu \\ 2z = -\lambda - \mu \\ x^2+y^2-z=3 \\ x+2y-2z=2 \end{cases}$$

Day 12 Lecture

Daily Announcements & Reminders:

- HW D3 due F
- Quiz 4 on D3 tomorrow
- Tech Demo 2 due Th
  - be careful w/ default settings
- Midterm Progress Reports: ≥ 4 LTs at C or M $\Rightarrow$ S
  ≤ 3 LTs at C or M $\Rightarrow$ U
- Do warmup on Poll Ev

Learning Targets:

- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **I2: Iterated Integrals.** I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.

Goals for Today:

Sections 15.1, 15.2

- Introduce double and iterated integrals for functions of two variables on rectangles
- Use Fubini's Theorem to change the order of integration of a iterated integral
- Be able to set up & evaluate a double integral over a general region
- Change the order of integration for general regions

$\iint_R f(x,y)\,dA$

R rectangle $[a,b]\times[c,d]$

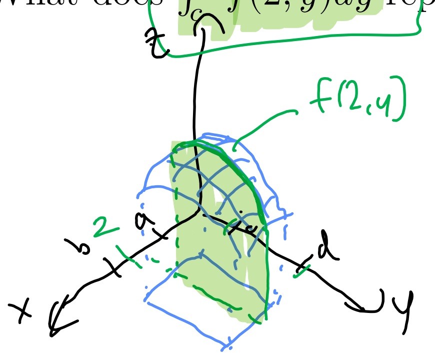
Question: How can we compute a double integral?

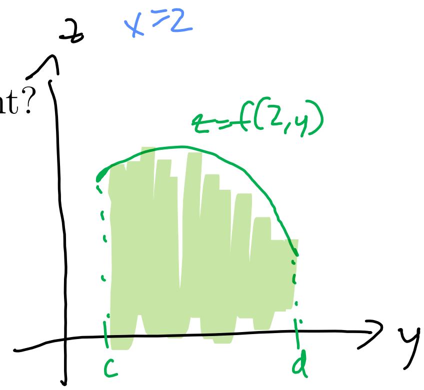
Answer: Iterated integrals

Integration is accumulation of values of a function over a domain.

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a,b] \times [c,d]$.

What does $\int_c^d f(2,y)dy$ represent?





$\int_c^d f(2,y)\,dy$ is cross-sectional area of slice $x=2$

What about $\boxed{\int_c^d f(x,y)dy?}$ Is cross-section area obtained for any fixed x between $x=a$ and $x=b$

$$\int_c^d x+2y\,dy$$

Let $\boxed{A(x)} = \int_c^d f(x,y)dy$. Then,

$$\text{volume of region} = \int_a^b A(x)dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

This is called an iterated integral.

• inside to outside • MUST write differentials

Example 83. Evaluate $\displaystyle\int_1^2 \int_3^4 6x^2y\,dy\,dx$. $\overset{?}{=} \displaystyle\int_3^4 \int_1^2 6x^2y\,dx\,dy$

$a=1$ $b=2$
$c=3$ $d=4$

$$= \int_1^2 \left(\int_3^4 6x^2 y\,dy \right) dx$$

$$= \int_1^2 \left(\frac{6x^2y^2}{2} \Big|_{y=3}^{y=4} \right) dx$$

$$= \int_1^2 21x^2\,dx = 7x^3 \Big|_1^2 = \boxed{49}$$

yes, sometimes

$$= \int_3^4 2x^3y \Big|_{x=1}^{x=2} dy$$

$$= \int_3^4 14y\,dy$$

$$= 7y^2 \Big|_3^4$$

$$= \boxed{49}$$

Theorem 84 (Fubini's Theorem)**.** *If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then*

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

continuous $[a,b]\times[c,d]$

Example 85. Compute $\iint_R \boxed{xe^{e^{e^y}}}\,dA$, where R is the rectangle $[-1,1]\times[0,4]$.

$$\int_0^4\int_{-1}^1 xe^{e^{e^y}}\,dx\,dy = \int_{-1}^1\int_0^4 xe^{e^{e^y}}\,dy\,dx$$

$\int e^{e^y}\,dy = ???$

Really hard

$$= \int_0^4 e^{e^{e^y}}\cdot \frac{1}{2}x^2\Big|_{x=-1}^{x=1}\,dy$$

$$= \int_0^4 0\,dy$$

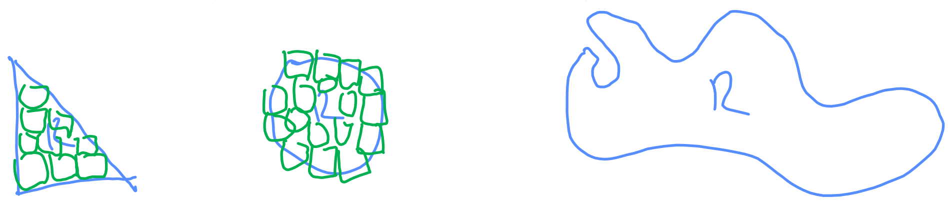
$= 0$ ← half volume is above xy-plane ("positive")
half is below xy-plane ("negative")

$\iint_R f(x,y)\,dA$

$f(x,y)$ – integrand : thing we are integrating

R – domain of integration : where we are integrating

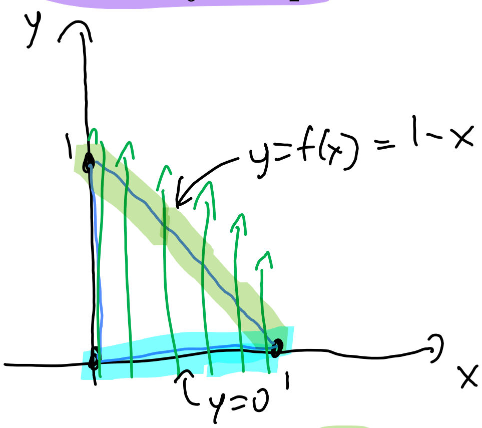
Question: What if the region R we wish to integrate over is not a rectangle?

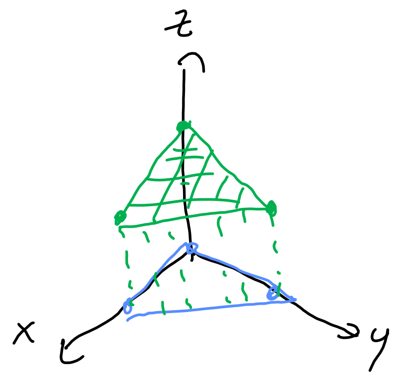


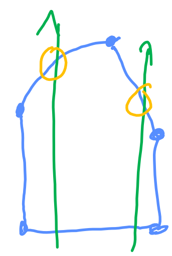
Answer: Repeat same procedure - it will work if the boundary of R is smooth and f is continuous.

Example 86. Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the xy-plane and whose top is $z = 2 - x - y$.

Vertically simple:







$$\text{volume} = \int_0^1 \int_{y=0}^{y=1-x} 2-x-y \, dy \, dx = \int_0^1 2y - xy - \frac{1}{2}y^2 \Big|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 2-2x-x+x^2-\frac{1}{2}(1-x)^2 \, dx$$

$$= 2x - \frac{3}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{6}(1-x)^3 \Big|_0^1$$

$$= \left(2 - \frac{3}{2} + \frac{1}{3} + 0\right) - \left(0 - 0 + 0 + \frac{1}{6}\right)$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \boxed{\frac{2}{3}}$$

Horizontally simple:

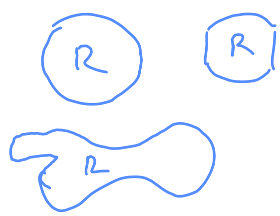
Day 14 - Area, Average Value, Polar Coordinates

Pre-Lecture

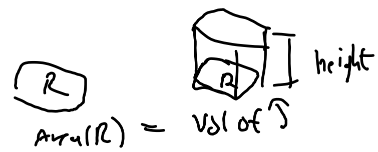
Section 15.3: Area and Average Value

Two other applications of double integrals are computing the area of a region in the plane and finding the average value of a function over some domain.

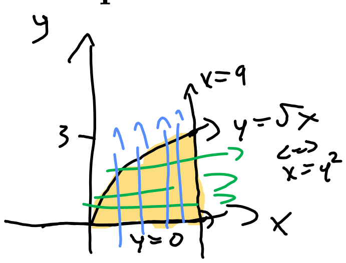
Area: If R is a region bounded by smooth curves, then



$$\text{Area}(R) = \iint_R 1 \, dA$$



Example 79. Find the area of the region R bounded by $y = \sqrt{x}, y = 0,$ and $x = 9$.



$$\text{Area}(R) = \iint_R 1 \, dA = \int_0^9 \int_0^{\sqrt{x}} dy\, dx = \int_0^3 \int_{y^2}^9 1 \, dx\, dy$$
$$= \int_0^3 9 - y^2 \, dy$$
$$= 9y - \tfrac{1}{3}y^3 \Big|_0^3$$
$$= \boxed{18}$$

Average Value: The average value of $f(x, y)$ on a region R contained in $\mathbb{R}^2$ is

$f: \mathbb{R} \to \mathbb{R}$

f_{avg} on $[a,b]$: $\underbrace{\frac{1}{b-a}}_{\text{size of interval}} \underbrace{\int_a^b f(x)\,dx}_{\text{accumulation of } f}$

$$f_{avg} = \frac{\iint_R f(x,y)\, dA}{\text{Area}(R)}$$

Example 80. Find the average temperature on the region R in the previous example if the temperature at each point is given by $T(x, y) = 4xy^2$. °C

$$T_{avg} = \frac{\int_0^3 \int_{y^2}^9 4xy^2 \, dx\, dy}{18} = \frac{1}{9} \int_0^3 x^2 y^2 \Big|_{y^2}^9 dy$$
$$= \frac{1}{9} \int_0^3 81y^2 - y^6 \, dy$$
$$= \frac{1}{9} \left(27y^3 - \frac{1}{7}y^7\right)\Big|_0^3$$
$$= \frac{1}{9}\left(3^6 - \frac{3^7}{7}\right)$$
$$= \frac{324}{7} \approx \boxed{46.3^\circ C}$$

Day 13 Lecture

Daily Announcements & Reminders:

- HW D3 due F
- Tech Demo 2 due tonight
- No class M/T - Fall Break!
- Checkpoint 2 on W: D2, D3, $\boxed{\text{A1}}$
- Do warmup question on PollEv

practice up soon

Learning Targets:

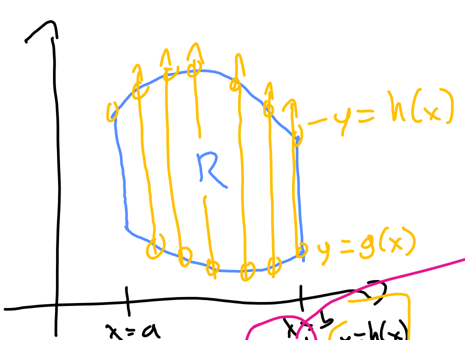
- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **I2: Iterated Integrals.** I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.
- **A2: Integral Applications.** I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.

Goals for Today:

Sections 15.2, 15.3

- Be able to set up & evaluate a double integral over a general region
- Compute areas of general regions in the plane
- Compute the average value of a function of two variables

Vertically simple:

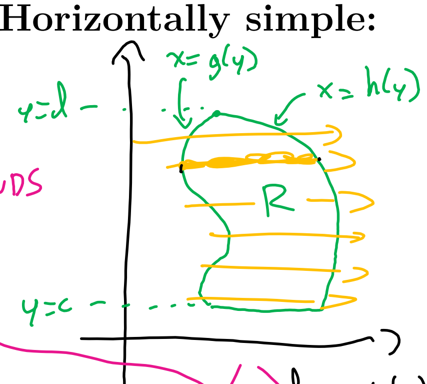


OUTER BOUNDS MUST BE CONSTANTS

$$\iint_R f(x,y)\,dA = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x,y)\,dy\,dx$$

only x's, no y's

Horizontally simple:



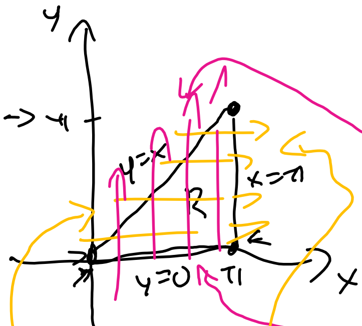
$$\iint_R f(x,y)\,dA = \int_{y=c}^{y=d} \int_{x=g(y)}^{x=h(y)} f(x,y)\,dx\,dy$$

Example 89. Set up an iterated integral to evaluate the double integral

$$\iint_R \cos(x)\sin(y)\ dA,$$

where R is the triangle with vertices $(0,0)$, $(\pi,0)$, and (π,π).

1) Sketch R



2) Choose direction

– both horizontally & vertically simple

Vert. simple: dy dx

$$\int_0^{\pi} \int_0^{x} \cos(x)\sin(y)\ dy\ dx$$

← big

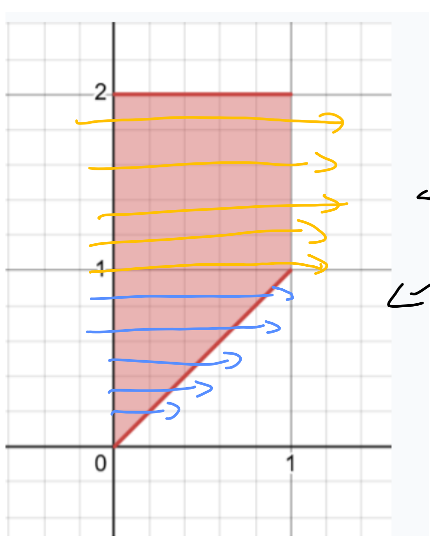
← small

Horiz. simple: dx dy

$$\int_0^{\pi} \int_y^{\pi} \cos(x)\sin(y)\ dx\ dy$$

← big

← small



Two behaviors ⇒ not horizontally simple

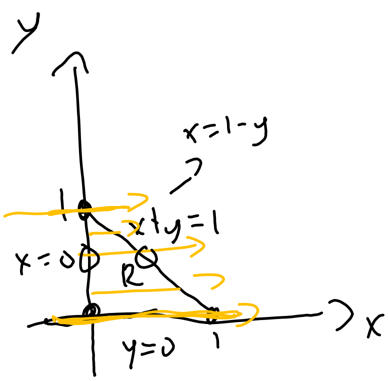
• cannot write $\iint_R f(x,y)\,dA$ as a single iterated integral in the dx dy

Example 90. Set up an iterated integral to compute the average value of

$$f(x,y) = e^{1-x^2-y^2}$$

over the triangle with vertices $(0,0), (1,0)$, and $(0,1)$, treating the region as horizontally simple.

- $$f_{avg} = \frac{\boxed{\iint_R f(x,y)\,dA}}{\text{Area}(R)}$$



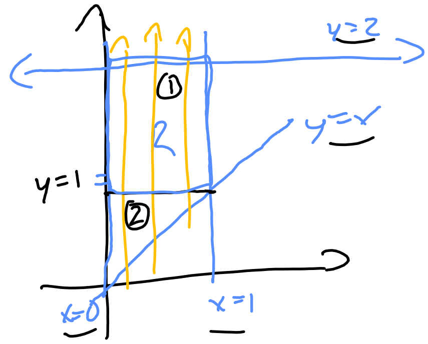
$$\text{Area}(R) = \frac{1}{2}\cdot 1 \cdot 1 = \frac{1}{2}$$

$$f_{avg} = \frac{\int_0^1 \int_0^{1-y} e^{1-x^2-y^2}\,dx\,dy}{1/2}$$

$$= \boxed{2\int_0^1 \int_0^{1-y} e^{1-x^2-y^2}\,dx\,dy}$$

Example 91. [Poll] Set up an iterated integral to evaluate the double integral $\iint_R 6x^2y\,dA$, where R is the region bounded by $x = 0, x = 1, y = 2$, and $y = x$.

1) Sketch R



$\int_0^1 \int_2^x \_\_\_\, dy\, dx$

$\Rightarrow$ error

$0 \le x \le 1$ & $2 \le y \le x$ } no points

2) Choose order & set up: $dy\,dx$

$$\iint_R 6x^2y\,dA = \int_0^1 \int_x^2 6x^2y\,dy\,dx$$

In the $dx\,dy$ order:

$$\iint_R 6x^2y\,dA = \iint_{①} 6x^2y\,dA + \iint_{②} 6x^2y\,dA$$

$$= \int_1^2 \int_{x=0}^{x=1} 6x^2y\,dx\,dy + \int_0^1 \int_0^y 6x^2y\,dx\,dy$$

Example 92. Sketch the region of integration for the integral expression

$$\underbrace{\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\ dy\ dx}_{1} + \underbrace{\int_1^4 \int_{x-2}^{\sqrt{x}} f(x,y)\ dy\ dx}_{2}.$$

Then write an equivalent iterated integral expression in the order $dx\ dy$.

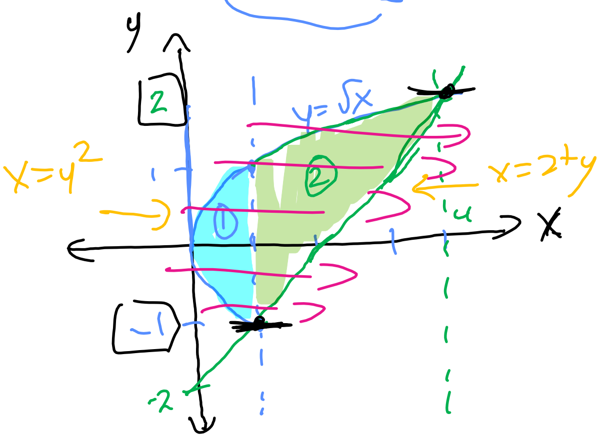
1) Sketch region.

① : $-\sqrt{x} \le y \le \sqrt{x}$

$0 \le x \le 1$

② $x-2 \le y \le \sqrt{x}$

$1 \le x \le 4$



2) Reverse order.

$$\boxed{\int_{-1}^{2} \int_{y^2}^{2+y} f(x,y)\, dx\, dy}$$

$2+y = y^2 \iff y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

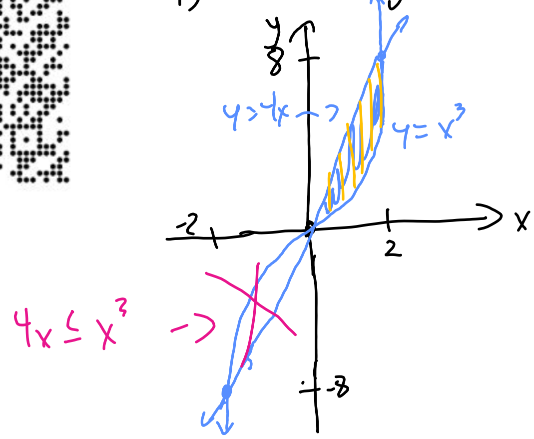
finite/bounded

Example 93. [Poll] Sketch the region with $x^3 \le y \le 4x$ and compute its area.

intersection: $x^3 = 4x \Rightarrow x=0$ or $x=2$ or $x=-2$

$x^3 \le 4x$

1) Sketch region



2) $\text{Area} = \iint_R 1 \, dA = \int_0^2 \int_{x^3}^{4x} 1 \, dy \, dx = 4$

Day 14 Lecture

Daily Announcements & Reminders:

- Webwork due F: I1 & I2 part 1
- Exam 2 in 1.5 weeks (T Oct 21)
  - opportunity for all G, D, I, A LTs
- Do warmup on PollEv →

Learning Targets:

- **I3: Change of Variables.** I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.

Goals for Today:

Sections 15.4

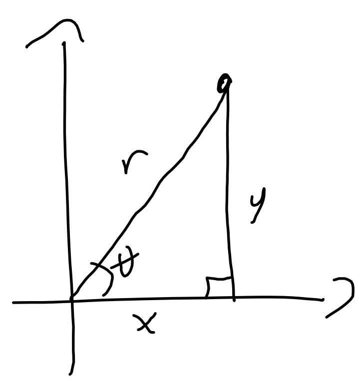
- Convert double integrals to iterated polar integrals
- Compute iterated polar integrals

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\tan\theta = \frac{y}{x}$



Example 95.

(a) Write the function $f(x, y) = \sqrt{x^2 + y^2}$ in polar coordinates.

$f(r,\theta) = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)}$

$= \sqrt{r^2}$

$= r$ (since we take $r > 0$)

$\parallel$

$\sqrt{r^2}$

(b/c $x^2+y^2=r^2$)

(b) [Poll] Write a Cartesian equation describing the points that satisfy $\boxed{r = 2\sin(\theta).}$

① $\sqrt{x^2+y^2} = 2\tan\theta \cos\theta = 2\,\frac{y}{x}\cdot\frac{x}{r}$

$= \frac{2y}{r}$

①' $r^2 = 2r\sin\theta \Rightarrow \boxed{x^2+y^2 = 2y}$

Check that $r=0$ was a solution originally

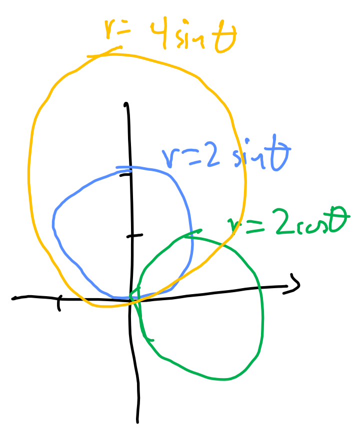
$x = r\cos(\theta)$
$y = r\sin(\theta)$

$x^2+y^2=r^2$

$\tan(\theta) = \frac{y}{x}$

$x^2+y^2-2y+(-1)^2 = 0+1$

$x^2+(y-1)^2 = 1$



Guess: $r = 2\cos\theta$ (flip x & y)

$r = 4\sin\theta$:

$r = a\cos\theta \rightarrow$
$(x-\frac{a}{2})^2 + y^2 = (\frac{a}{2})^2$

15.4: Double Integrals in Polar Coordinates

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r,\theta)$ on R, compute $\iint_R f(r,\theta)\ dA$.

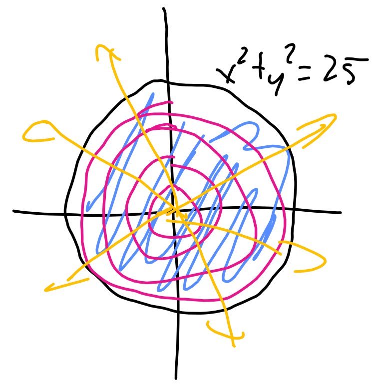
Example 96. Compute the area of the disk of radius 5 centered at $(0,0)$.

25π

$$\text{Area} = \iint_R 1\, dA$$

$$= \int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} dy\, dx$$

$$= \int_{-5}^{5} 2\sqrt{25-x^2}\, dx \quad \leftarrow \text{trig sub, hard}$$



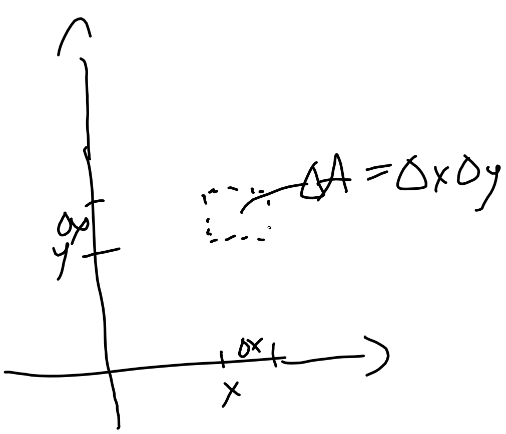
Polar: $0 \le r \le 5$

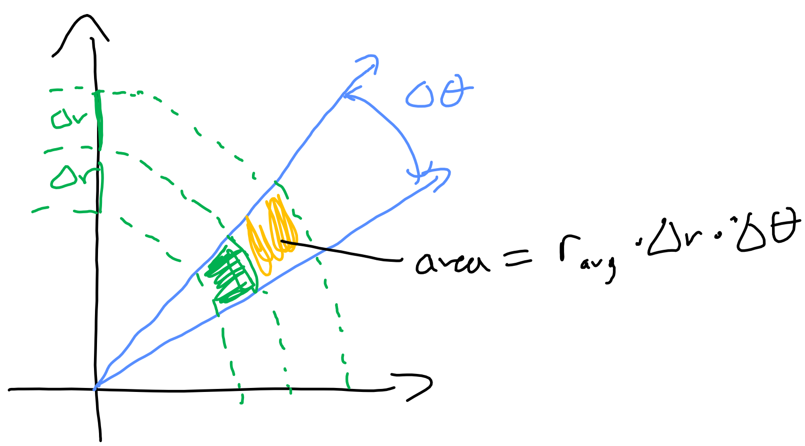
$0 \le \theta \le 2\pi$

in polar:

$$= \int_0^{2\pi} \int_0^5 1\, r\, dr\, d\theta$$

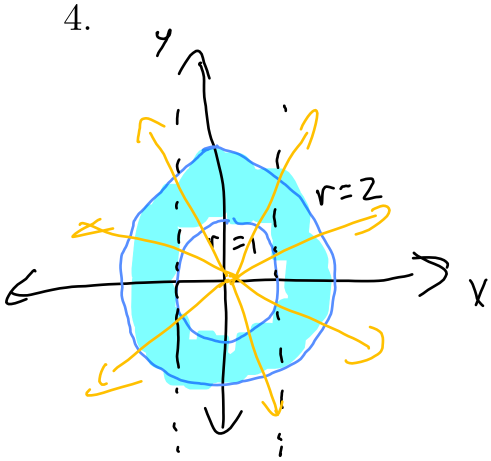
$$= \int_0^{2\pi} \cancel{r}\, \frac{r^2}{2}\Big|_0^5\, d\theta = \int_0^{2\pi} \cancel{5}\, \frac{25}{2}\, d\theta = \cancel{10\pi}\ 25\pi$$





Remember: In polar coordinates, the area form $dA =$ $r\, dr\, d\theta$

Example 97. Compute $\iint_D e^{-(x^2+y^2)}\,dA$ on the washer-shaped region $1 \le x^2+y^2 \le 4$.



Polar: $1 \le r \le 2$; $0 \le \theta \le 2\pi$

$(u = -r^2)$

$$\iint_D e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_1^2 e^{-r^2}\, r\,dr\,d\theta$$

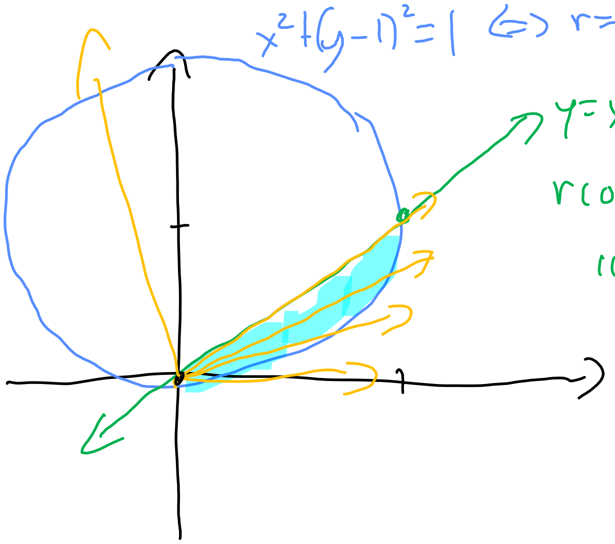
$$= \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_1^2\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left(e^{-1}-e^{-4}\right) d\theta$$

$$= \boxed{\pi\left(e^{-1}-e^{-4}\right)}$$

Example 98. Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$.

$x^2+(y-1)^2=1 \iff r=2\sin\theta$



$y=x \iff \theta=\pi/4$

$r\cos\theta = r\sin\theta$

$\cos\theta = \sin\theta$

$\Rightarrow$ Polar coords:

$0 \le r \le 2\sin\theta$

$0 \le \theta \le \pi/4$

$$\text{Area} = \int_0^{\pi/4}\int_0^{2\sin\theta} 1 \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}r^2\Big|_0^{2\sin\theta}\, d\theta$$

$$= \int_0^{\pi/4} 2\sin^2\theta\, d\theta$$

$$= \int_0^{\pi/4} 1-\cos 2\theta\, d\theta$$

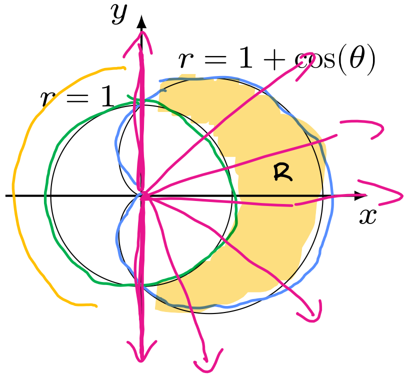
$$= \theta - \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/4}$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}(1)\right) - (0-0) = \boxed{\frac{\pi}{4}-\frac{1}{2}}$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

Example 99 (Poll)**.** Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



Volume under $z=x$ on R $= \iint_R x \, dA$

X $3\pi/2 \to$

$$= \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} (r\cos(\theta)) \cdot r \, dr \, d\theta$$

X $\pi/2 \to$

Day 15 Lecture

Daily Announcements & Reminders:

- I3 Webwork part 1 (pdav) due F
- Quiz 5 tomorrow: 1 problem
  - -> set up integral for I1 credit
  - -> integrate correctly for I2 credit
  - independent
- Exam 2 next T: 13 problems, all G, D, I, A LTs.
  - 1 sheet of handwritten notes allowed again
- Do warmpull poll

Learning Targets:

- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **A~~3~~2: Integral Applications.** I can use multiple integrals to solve physical problems, such as finding area, average value, volume, mass of a lamina, or solid.

Goals for Today:

Sections 15.5, 15.6

- Learn how to write triple integrals as iterated integrals.
- Compute triple iterated integrals
- Change the order of integration in a triple iterated integral.
- Apply our work to find the mass and center of mass of objects in $\mathbb{R}^2$ and $\mathbb{R}^3$

Example 101. Let's practice with triple integrals.

CONSTANTS ← AT MOST 1 VARIABLE

1. **Mechanics:** Compute $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} 1 \, dz\, dy\, dx.$

$$= \int_0^1 \int_0^{2-x} z \Big|_0^{2-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{2-x} 2-x-y \, dy\, dx$$

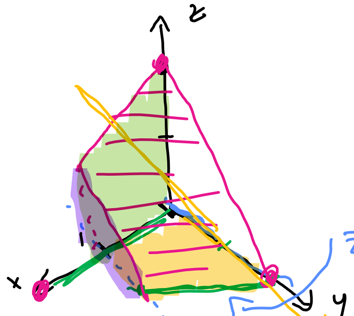
$$= \int_0^1 (2-x)y - \tfrac{1}{2}y^2 \Big|_0^{2-x} dx$$

$$= \int_0^1 \tfrac{1}{2}(2-x)^2\, dx = -\tfrac{1}{6}(2-x)^3 \Big|_0^1 = -\tfrac{1}{6} + \tfrac{8}{6} = \boxed{\tfrac{7}{6}}$$

2. **Interpretation:** What shape is this the volume of?

bottom $\boxed{0 \le z \le 2-x-y}$ top

shadow $\begin{cases} 0 \le y \le 2-x \\ 0 \le x \le 1 \end{cases}$ give as Restriction



$x+y+z=2$

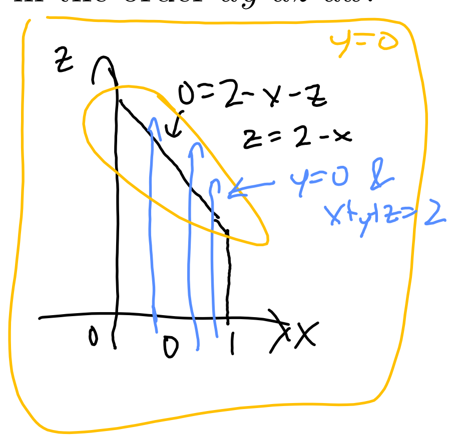
$\downarrow$

$y = 2-x-z$

$z=0$ & $x+y+z=2 \iff y = 2-x$

3. **Rearrange:** Write an equivalent iterated integral in the order $dy\, dz\, dx$.

$$\int_0^1 \int_0^{2-x} \int_0^{2-x-z} 1 \, dy\, dz\, dx$$



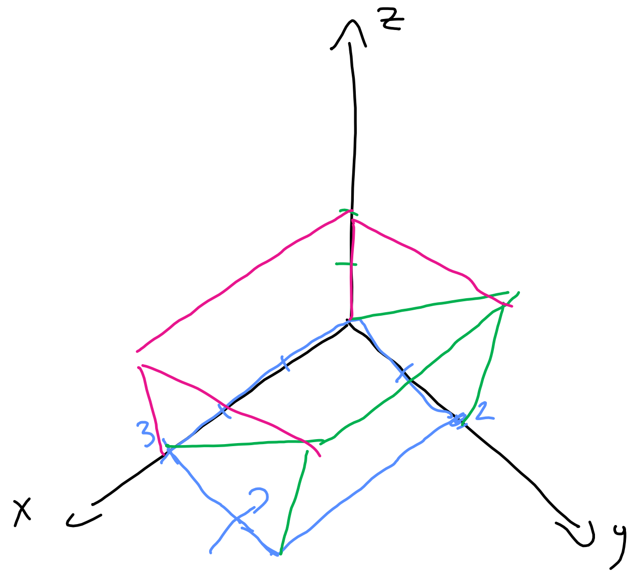
Example 102. [Poll] Which of the following integrals is equal to

$$\int_0^3 \int_0^2 \int_0^y f(x,y,z)\ dz\ dy\ dx?$$

$0 \le x \le 3$

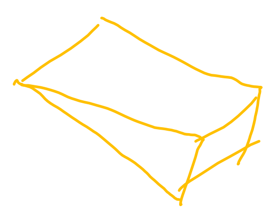
$0 \le y \le 2$

$0 \le z \le y$



1. $\int_0^2 \int_0^3 \int_0^y \boxed{f(x,y,z)}\ dz\ dx\ dy$ ✓

2. $\int_0^2 \int_0^3 \int_0^y f(x,y,z)\ dz\ dy\ dx$ ✗



$x \le y$

3. $\int_0^3 \int_0^2 \int_0^y f(x,y,z)\ dx\ dy\ dz$ ✗

4. $\int_0^3 \int_0^2 \int_0^z f(x,y,z)\ dy\ dz\ dx$

$0 \le x \le 3$

$0 \le z \le 2$

$0 \le y \le z$ ✗ $z \le y \le 2$

Example 103. Write an integral for the mass of the solid D in the first octant bounded by the planes

$$x = 0, z = 0, x = y, \text{ and } x + y + z = 2$$

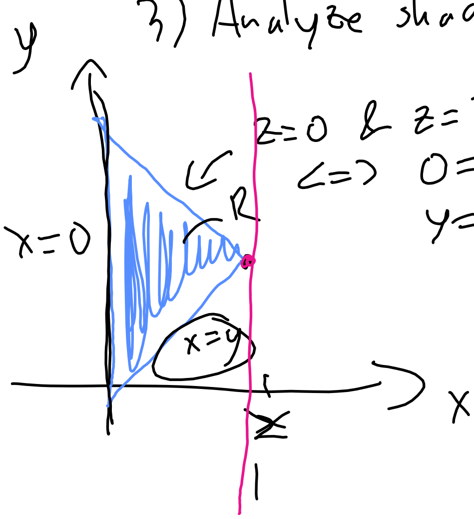
kg/m^3

with density $\delta(x, y, z) = x^2 y$ using the shadow method and using the cross-section method. Which orders of integration work well for each method?

Shadows: 1) Pick inner variable: z

2) Identify top/bottom surfaces: $0 \le z \le 2-x-y$

3) Analyze shadow in $z=0$:



$z=0$ & $z=2-x-y$.
$\Leftrightarrow$ $0=2-x-y$
$y=2-x$

so we have

$$\text{mass} = \iiint_D x^2 y \, dV$$

$$= \iint_R \left(\int_0^{2-x-y} x^2 y \, dz \right) dA$$

$$= \int_0^{\boxed{1}} \int_x^{2-x} \int_0^{2-x-y} x^2 y \, dz \, dy \, dx$$

$y=x$ meets $y=2-x$

$2-x=x$

$2=2x$

$x=1$

Example 103 (cont.)

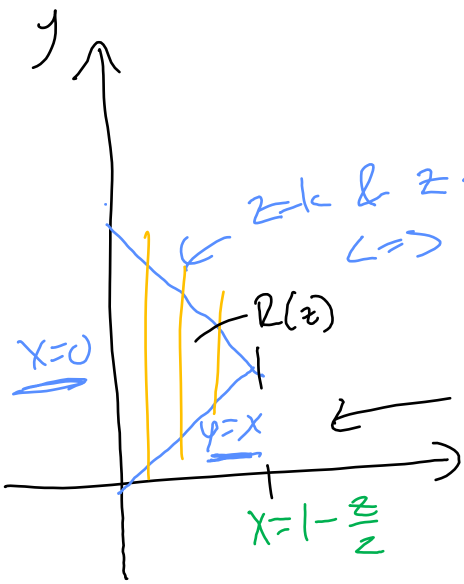
Cross-sections:

1) Pick outer variable (z)

2) Find extreme z-values $\quad 0 \le z \le 2$

3) Analyze cross-section in $z=k$ for ↑

$z=k$



$z=k$ & $z=2-x-y$

$\Leftrightarrow \quad k = 2-x-y$

$y = (2-k) - x$

$\rightarrow y = (2-z) - x$

z is a constant

$x = 2-z-x$

$2x = 2-z$

$x = 1 - \frac{z}{2}$

$$\text{mass} = \iiint_D x^2 y \, dV$$

$$= \int_0^2 \left(\iint_{\text{cross-section}} x^2 y \, dA \right) dz$$

$$= \int_0^2 \left(\int_0^{1-\frac{z}{2}} \int_x^{2-z-x} x^2 y \, dy \, dx \right) dz$$

$R(z)$

Day 16 - Triple Integrals Algebraically & Applications

Pre-Lecture

Rules for Triple Integrals for the Sketching Impaired

Rules for Triple Integrals for the Sketching Impaired (credit to Wm. Douglas Withers)

Basic

Rule 1: Choose a variable appearing exactly twice for the next integral.

Rule 2: After setting up an integral, cross out any constraints involving the variable just used.

Rule 3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

Intermediate

Rule 4: A square variable counts twice.

Rule 5: The argument of a square root must be non-negative.

Rule 6: If you do not know which is the upper limit and which is the lower, take a guess - but be prepared to backtrack.

Advanced

Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Example 105. Set up an integral for the volume of the region D defined by

$$\cancel{x + y^2 \le 8}, \quad \cancel{y^2 + 2z^2 \le x}, \quad y \ge 0$$

Rule 4

Rule 1: Pick a variable appearing exactly twice (count squares as 2)

$x: 1+1+0 = 2$

$y: 2+2+1 = 5$

$z: 0+2+0 = 2$

$x + y^2 \le 8 \Rightarrow x \le 8 - y^2$

$$\text{Volume} = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{y^2+2z^2}^{8-y^2} 1 \, dx \, dz \, dy$$

Rule 2: Cross out used constraints

Rule 3: Make a new constraint from limits:

$y^2 + 2z^2 \le 8 - y^2$ (new), $\quad y \ge 0$ (old)

$2y^2 + 2z^2 \le 8$

$\cancel{y^2 + z^2 \le 4}$

Rule 1/4: $y: 2+1 = 3$ $\quad z: 2+0 = 2$

$z^2 \le 4 - y^2$

$-\sqrt{4-y^2} \le z \le \sqrt{4-y^2}$

Rule 2: Cross out

Rule 3: Make new constraint: $-\sqrt{4-y^2} \le \sqrt{4-y^2}$ Not useful

Rule 5: Argument of square root is nonnegative:

$4 - y^2 \ge 0$ $\quad y \ge 0$

$\hookrightarrow y^2 \le 4$

$-2 \le y \le 2$ $\quad \Rightarrow 0 \le y \le 2$

Day 16 Lecture

Daily Announcements & Reminders:

- Webwork I3 (polar) due F
- Quiz 5 grades back M
- Checkpoint 2 messed up in Canvas, will fix
- Exam 2 on T, 1 problem for each G, D, F, A LT
  - bring 1 sheet handwritten notes
- Do warmup on PollEv

Learning Targets:

- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.

- **A2: Integral Applications.** I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.

Goals for Today: Section 15.5,15.6

- Learn how to write triple integrals as iterated integrals.
- Apply triple integrals to solve problems in $\mathbb{R}^3$

$x \geq 0,\ y \geq 0,\ z \geq 0$

Example 105. Write an integral for the mass of the solid D in the first octant with $2y \leq z \leq 3 - x^2 - y^2$ with density $\delta(x, y, z) = 3x + y^{1.5} + 0.2$ using any method and order of integration. Which orders of integration work well?

$$\text{mass} = \iiint_D \delta(x,y,z)\, dV$$

$\leftarrow$ or $\sqrt{3-y^2-2y}$

$$= \int_0^1 \int_0^{\sqrt{4-(y+1)^2}} \int_{2y}^{3-x^2-y^2} 3x + y^{1.5} + 0.2 \, dz\, dx\, dy$$

Bounds

$\cancel{2y \leq z}$

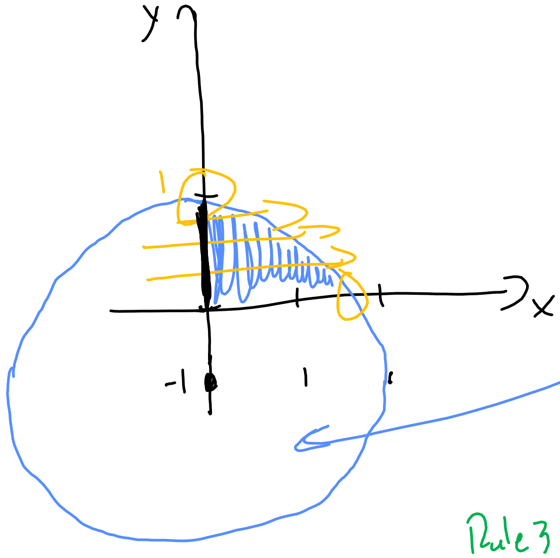
$\cancel{z \leq 3 - x^2 - y^2}$

$\cancel{x \geq 0}$

$y \geq 0$

$2y \leq 3 - x^2 - y^2$

Draw xy-shadow



$2y \leq 3 - x^2 - y^2$

$x^2 + y^2 + 2y + 1 \leq 3 + 1$

$x^2 + (y+1)^2 \leq 4$

$x^2 \leq 4 - (y+1)^2$

$\cancel{-\sqrt{4-(y+1)^2}} \leq x \leq \sqrt{4-(y+1)^2}$

$x \geq 0$

Rule 3: $0 \leq \sqrt{4-(y+1)^2}$ $\quad y \geq 0$

Rule 5: $4 - (y+1)^2 \geq 0$

$(y+1)^2 \leq 4$

$-2 \leq y+1 \leq 2$

$-3 \leq y \leq 1$

$0 \leq y \leq 1$

$2y \leq 3 - x^2 - y^2$

$x^2 \leq 3 - y^2 - 2y$

$-\sqrt{3-y^2-2y} \leq x \leq \sqrt{3-y^2-2y}$

$0 \leq x$

Example 105 (cont.)

Let's do x first:

$$\int_0^1 \int_{2y}^{3-y} \int_0^{\sqrt{3-y^2-z}} 3x + y^{1.5} + 0.2 \, dx \, dz \, dy$$

$2y \le z$

$\cancel{z \le 3 - x^2 - y^2}$

$\cancel{x \ge 0}, \; y \ge 0, \; z \ge 0$

$x^2 \le 3 - y^2 - z$

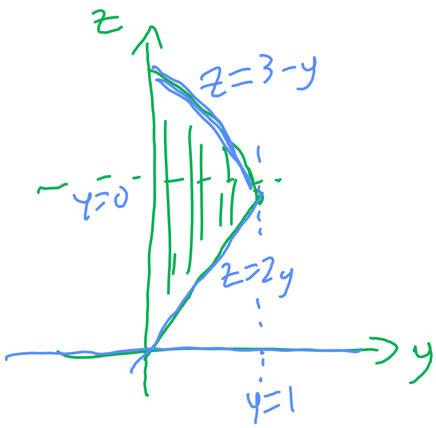
$\cancel{-\sqrt{3-y^2-z}}\ 0 \le x \le \sqrt{3-y^2-z}$

$\Downarrow$

Bounds: $2y \le z, \; y \ge 0, \; \boxed{\cancel{z \ge 0}}$

$0 \le \sqrt{3-y^2-z} \Rightarrow 3 - y^2 - z \ge 0$

not useful



y-first:

$2y \le z \qquad y \ge 0 \qquad x \ge 0 \qquad z \le 3 - y^2 - x^2$

$\Downarrow$

$y \le \frac{z}{2} \qquad y \ge 0 \qquad y^2 \le 3 - x^2 - z$

$\cancel{-\sqrt{3-x^2-z} \le}\ y \le \sqrt{3-x^2-z}$

always bigger

① $y \le \frac{z}{2}$ ② $y \le \sqrt{3-x^2-z}$

for some (x,y,z) in D ① is a stronger bound, For others ② is a stronger bound

Example 106. Set up a triple iterated integral for the triple integral of $f(x, y, z) = x^3y$ over the region D bounded by

$$x^2 + y^2 = 1, \quad z = 0, \quad x + y + z = 2.$$

Rule 1: $x : 3$ $y : 3$ $\boxed{z = 2}$

$z = 0$ $\quad z = 2 - x - y$

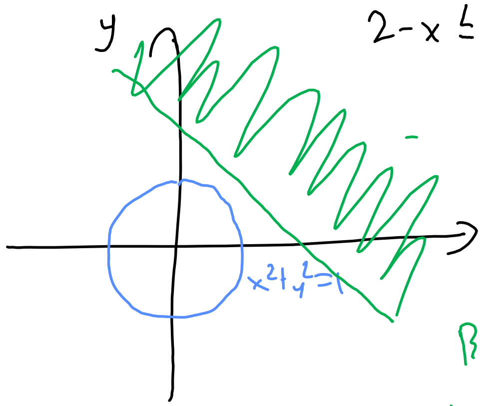
$2 - x - y \leq z \leq 0$

Rule 3: $2 - x - y \geq 0$

$$\boxed{\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{2-x-y} x^3 y \, dz \, dy \, dx}$$

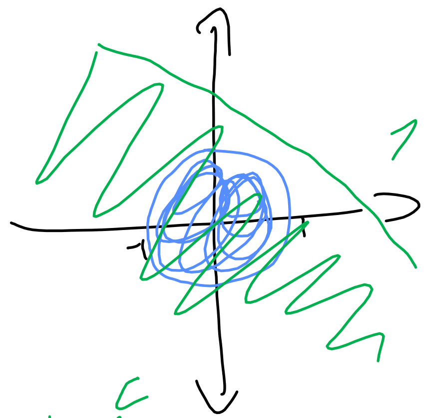
Again: $2 - x - y \leq 0$ $\quad x^2 + y^2 = 1$

$2 - x \leq y$



Bad!

Want finite region! $\Rightarrow$ z choice is backwards



Example 107. Set up triple iterated integrals to compute the center of mass of an object that occupies the volume of the upper hemisphere of $x^2 + y^2 + z^2 \leq 4$ with density z at (x, y, z).

$$\text{mass} = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z \, dz \, dy \, dx$$

$$\text{center of mass} = (\bar{x}, \bar{y}, \bar{z})$$

$$= \left(\frac{\iiint x\delta \, dV}{\iiint \delta \, dV}, \frac{\iiint y\delta \, dV}{\iiint \delta \, dV}, \frac{\iiint z\delta \, dV}{\iiint \delta \, dV} \right)$$

$$= \left(\underbrace{0, 0}_{\text{xy symmetry of shape \& density}}, \frac{\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \, dz \, dy \, dx}{\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z \, dz \, dy \, dx} \right)$$

Another application of integration is to compute probabilities. A *joint density function* of two random variables X and Y (such as the height and weight of a randomly chosen person) is a function $f(x, y)$ such that the probability that (X, Y) lies in a region R of possible values is

$$P((X, Y) \in R) = \iint_R f(x,y)\, dA$$

The function f must satisfy two properties:

- $f(x, y) \geq 0$
- $\iint_{\mathbb{R}^2} f(x, y)\, dA = 1$

Example 108. Suppose the joint density function for random variables X and Y is

$$f(x, y) = \begin{cases} C(x + 2y) & 0 \leq x \leq 10, 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}.$$

Find the value of C and the probability that $X + Y \leq 10$.