Review from Single-Variable Calculus and Linear Algebra

This worksheet reviews some concepts and tools of calculation from linear algebra, differential calculus, and integral calculus that will be useful for this course.

1. Let $\mathbf{v} = \langle 1, 2, 1\rangle$ and $\mathbf{w} = \langle -1, 0, 2\rangle$. Compute $2\mathbf{v}$, $\mathbf{v} - \mathbf{w}$, and $\mathbf{v} \cdot \mathbf{w}$.

   Answer: $2\mathbf{v} = \langle 2, 4, 2\rangle$, $\mathbf{v} - \mathbf{w} = \langle 2, 2, -1\rangle$, $\mathbf{v} \cdot \mathbf{w} = 1$.

2. Let a, b, c be variables. Calculate

$$\det \begin{bmatrix} a & b & c \\ 1 & -1 & 2 \\ 4 & 3 & -7 \end{bmatrix}$$

   by expanding the determinant along the first row.

   Answer: $a + 15b + 7c$

3. Draw the vectors $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + 2\mathbf{j}$ starting at the origin in $\mathbb{R}^2$ to show that these vectors are orthogonal. Compute $\mathbf{u} \cdot \mathbf{v}$. What is the result?

   Answer: $\mathbf{u} \cdot \mathbf{v} = 0$

4. Let $\mathbf{v} = \langle 2, -4, \sqrt{5}\rangle$ and $\mathbf{u} = \langle -2, 4, -\sqrt{5}\rangle$. Compute the following:

   (a) $\mathbf{v} \cdot \mathbf{u}$

   (b) the cosine of the angle θ between $\mathbf{v}$ and $\mathbf{u}$. Recall that $\mathbf{v} \cdot \mathbf{u} = |\mathbf{v}||\mathbf{u}|\cos(\theta)$

   (c) $(3\mathbf{v}) \cdot (2\mathbf{u})$.

   Answer:

   (a) $\mathbf{v} \cdot \mathbf{u} = -25$

   (b) $\cos(\theta) = \dfrac{-25}{25} = -1$

   (c) $(3\mathbf{v}) \cdot (2\mathbf{u}) = 3(2)(-25) = -150$.

5. Compute the following integrals.

   (a) $\displaystyle\int_0^\pi (\sin(\theta))^5 \, d\theta$

   (b) $\displaystyle\int_0^\pi (\sin(\theta))^2 \, d\theta$

   (c) $\displaystyle\int_1^2 t \ln(t) \, dt$

   Answer:

(a) 16/15

(b) $\frac{\pi}{2}$

(c) $\ln(4) - \frac{3}{4}$

6. Find the maximum value of the function $f(t) = te^{t^2}$ on the interval from $t = 0$ to $t = 2$. Justify how you know this is the maximum.

   Answer: $f_{max} = 2e^4$, since the function has no critical points and is increasing on this interval.

7. Find the area enclosed by the curves $f(x) = x^2$ and $g(x) = 2 - x$.

   Answer: Area $= \frac{9}{2}$.

Section 12.1: Geometry in 3 Dimensions

Mechanics

1. Find the components of the vector in 3-space of length 3 lying in the yz-plane pointing upward at an angle of $\pi/6$ measured from the positive y-axis.

   Answer: $x = 0, y = 3\sqrt{3}/2, z = 3/2$.

2. Describe in words, and with a sketch, the region of $\mathbb{R}^3$ given by the following equations or inequalities.

   (a) $y = x^2$
   (b) $z = y$
   (c) $0 \leq z \leq 5$
   (d) $0 \leq z \leq 5$, $1 \leq y \leq 6$
   (e) $0 \leq z \leq 5$, $1 \leq y \leq 6$, $-2 \leq x \leq 3$

   Answer:

   (a) A parabolic cylinder.
   (b) A plane at a 45° angle to the xy-plane.
   (c) The region between the planes $z = 0$ and $z = 5$ (an infinite slab).
   (d) The region between the planes $z = 0$ and $z = 5$, and between the planes $y = 1$ and $y = 6$ (a rectangular prism extending infinitely in the x-direction).
   (e) The rectangular box bounded by the planes $z = 0$, $z = 5$, $y = 1$, $y = 6$, $x = -2$, and $x = 3$.

Applications

3. Which is traveling faster, a car whose velocity vector is $\langle 28, 33\rangle$ km/h or a car whose velocity vector is $\langle 40, 0\rangle$ km/h? At what speed is the faster car traveling?

   Answer: The first car is faster, at approx 43.28 km/h.

4. A pilot is steering a plane in the direction N45°W at an airspeed of 180 mph. A wind is blowing in the direction S30°E at a speed of 35 mph. Find the true course and the ground speed of the plane.

   Answer: The ground speed is approximately 146.94 mph and the true course is approximately N48.51°W (or a bearing of 311.49°)

5. Three forces act on an object. One of the force vectors is $\langle 25, 0, 0\rangle$ Newtons. The second lies on the xy-plane at an angle of 60° measured counterclockwise from the x-axis at a magnitude of 12N. The third is perpendicular to the plane of these two forces and has

magnitude 4N. Calculate the magnitude of a fourth force that would exactly counterbalance these three forces.

Answer: The magnitude of the counterbalancing force is 31.26 N

Extensions

6. Find the equation of a sphere of radius 4 in $\mathbb{R}^3$ centered at the point $(1, -2, 3)$.

   Answer: $(x-1)^2 + (y+2)^2 + (z-3)^2 = 16$.

7. Find the equation of an ellipsoid centered at the origin, passing through the points $(5, 0, 0)$, $(0, \frac{1}{4}, 0)$ and $(0, 0, 3)$

   Answer: $\frac{x^2}{25} + \frac{y^2}{\frac{1}{16}} + \frac{z^2}{9} = 1$.

8. Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

   Answer:

Section 12.4: Vector Products

Mechanics

1. Let $P = (1, -1, 2)$, $Q = (2, 0, -1)$, and $R = (0, 2, 1)$.

   (a) Find the area of the triangle determined by the points P, Q, and R.

   (b) Find a unit vector normal to the plane containing P, Q, and R.

   Answer:

   (a) $2\sqrt{6}$

   (b) $\frac{1}{\sqrt{6}}\langle 2, 1, 1\rangle$

2. Find the volume of the parallelepiped spanned by the vectors $\langle 1, 2, 3\rangle$, $\langle 3, 4, 5\rangle$, $\langle -2, -1, 0\rangle$.

   Answer: 0

3. Let $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ be any nonzero vector. What are all the vectors $\mathbf{w}$ for which $\mathbf{v} \times \mathbf{w} = \langle 0, 0, 0\rangle$?

   Answer: All scalar multiples of $\mathbf{v}$.

4. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$. If the statement is *always* true, answer true. If the statement is *ever* false, answer false. Justify your answer.

   (a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

   (b) $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^{\mathbf{2}}$

   (c) $(\mathbf{u} \times \mathbf{u}) \cdot \mathbf{u} = \mathbf{0}$

   Answer:

   (a) True

   (b) True

   (c) True

5. Suppose $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$. Which of the following make sense, and which do not? For those that make sense, is the result a vector or a scalar?

   (a) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

   (b) $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$

   (c) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

   (d) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

   Answer:

(a) Makes sense, scalar

(b) Does not make sense

(c) Makes sense, vector

(d) Does not make sense

Applications

6. In graphics rendering and CGI, vector products govern the way that light behaves. For example, the *law of reflection* demands that a light ray reflecting off of a flat surface must reflect outwards with the same angle that it came in with. Suppose then, that a light source located at $(1, 2, 3)$ points towards the origin. What is the angle of reflection?

   Answer: The angle of reflection is approximately $36.7°$ (relative to the surface normal)

Extensions

7. Let $\mathbf{u} = \langle 2, 3 \rangle$. Find the maximum possible value for $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{v}$ is a unit vector, and find a $\mathbf{v}$ which gives this maximum. Then repeat the problem with "maximum" replaced by "minimum."

   Answer: Max value is $\sqrt{13}$, given by $\mathbf{v} = \mathbf{u}/|\mathbf{u}|$ and min value is $-\sqrt{13}$, given by $\mathbf{v} = -\mathbf{u}/|\mathbf{u}|$.

8. Let $\mathbf{v} = (1, 1, 1)$. Describe the set of all vectors $\mathbf{u}, \mathbf{w}$ for which $\mathbf{u} \times \mathbf{w}$ is a scalar multiple of $\mathbf{v}$. Can you interpret your answer pictorially? How would your answer change if $\mathbf{v}$ was an arbitrary nonzero vector?

   Answer: The vectors $\mathbf{u}$ and $\mathbf{w}$ must both lie in the plane defined by the equation $x + y + z = 0$ (the plane perpendicular to $\mathbf{v}$). If $\mathbf{v}$ were an arbitrary nonzero vector, $\mathbf{u}$ and $\mathbf{w}$ would simply need to lie in the plane perpendicular to that specific vector $\mathbf{v}$.

Section 12.5: Lines and Planes in $\mathbb{R}^3$

G1: Lines and Planes. I can describe lines using the vector equation of a line. I can describe planes using the general equation of a plane. I can find the equations of planes using a point and a normal vector. I can find the intersections of lines and planes. I can describe the relationships of lines and planes to each other. I can solve problems with lines and planes.

Mechanics

1. Find an equation for the line passing through the points $P = (1, 2, -1)$ and $Q = (-1, 0, 1)$.

   Answer: $\mathbf{r}(t) = \langle 1 - 2t, 2 - 2t, -1 + 2t \rangle, \quad t \in \mathbb{R}$

2. Find an equation for the line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$.

   Answer: $\mathbf{r}(t) = \langle t, -7 + 2t, 2t \rangle, \quad t \in \mathbb{R}$

3. How can you tell when two planes $A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$ are parallel? Perpendicular? Justify your answer.

   Answer: Parallel: $\langle A_1, B_1, C_1 \rangle = \lambda \langle A_2, B_2, C_2 \rangle$ for some $\lambda \neq 0$

   Perpendicular: $\langle A_1, B_1, C_1 \rangle \cdot \langle A_2, B_2, C_2 \rangle = 0$

4. Find the point where the line $\mathbf{r}(t) = \langle 2, 3 + 2t, 1 + t \rangle$ intersects the plane $2x - y + 3z = 6$.

   Answer: $(2, 7, 3)$

5. Find 2 planes that are not parallel that both contain the points $P(1, -1, 1)$, $Q(3, 2, 0)$, and $R(5, 5, -1)$. When will 3 distinct points NOT determine a unique plane?

   Answer: Many possible answers, one is

   $$x + 2z = 3, \quad y + 3z = 2$$

   3 distinct points will not determine a unique plane when they are collinear.

Applications

6. The year is 1428, and France is backed into a corner in the Hundred Year's War. Her patron saint, Joan of Arc has recruited you, France's leading mathematician, to sharpen her archers. You are standing at the origin $(0, 0, 0) \in \mathbb{R}^3$ and aim your arrows at a square target with side length 10, centered at $(100, 0, 0)$. Assuming your arrows fly in perfect straight lines (i.e., paths look like $\mathbf{r}(t) = t\langle v_1, v_2, v_3 \rangle$), determine all values of $\langle v_1, v_2, v_3 \rangle$ for which your arrow will hit the target.

   Answer: $v_1 > 0$, $-0.05v_1 \leq v_2 \leq 0.05v_1$, $-0.05v_1 \leq v_3 \leq 0.05v_1$

Extensions

7. The planes $3x - 6y - 2z = 3$ and $2x + y - 2z = 2$ intersect in a line. Find an equation for this line.

   Answer: $\mathbf{r}(t) = \langle 1, 0, 0\rangle + t\langle 14, 2, 15\rangle, \quad t \in \mathbb{R}$

8. Consider the family of planes $Ax + 2y + 3z = 4$, where A is a real parameter. There is a unique line ℓ which lies in every plane in the family. Find an equation for ℓ. Using a 3D graphing software, explain qualitatively what is happening to the plane (relative to ℓ) as you change the parameter A.

   Answer: $\ell : \mathbf{r}(t) = \langle 0, 2, 1\rangle + t\langle 0, -3, 2\rangle, \quad t \in \mathbb{R}$

   As you change A, the plane pivots around the line ℓ.

9. Let ℓ_1 and ℓ_2 be skew lines. Show that you can find two parallel planes P_1 and P_2 so that P_1 contains ℓ_1 and P_2 contains ℓ_2.

   Answer: Let (a, b, c) be a point on ℓ_1 and (d, e, f) be a point on ℓ_2. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be direction vectors for the lines. Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. Then the planes

   $$\mathbf{n} \cdot (x - a, y - b, z - c) = 0$$

   and

   $$\mathbf{n} \cdot (x - d, y - e, z - f) = 0$$

   are parallel and contain ℓ_1 and ℓ_2 respectively.

Section 12.6: Quadric Surfaces

G4: Surfaces. I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.

Mechanics

1. Identify the type of surface described by each of the following. Then roughly sketch each surface, and label one or two key features (e.g., if it is a hyperboloid of one sheet, label the "tightest" circle).

   (a) $x^2 + 4z^2 = 16$.

   (b) $9x^2 + z^2 + y^2 = 9$.

   (c) $y^2 = 3x^2 + 3z^2$.

   (d) $z = x^2 + y^2 + 4$.

   (e) $x^2 + y^2 = 16 - z^2$

   Answer:

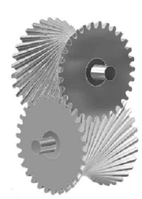
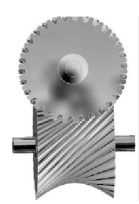
2. Consider the family of quadric surfaces $z^2 + t = x^2 + y^2$, where t is some parameter. What surface type do you get when $t > 0$? $t < 0$? And when $t = 0$?.

   Check your work with a 3D graphing software of your choice to plot this family, using a "slider" to smoothly vary the parameter t from -2 to 2. As you increase t, explain qualitatively what happens near the origin. Where might you witness such a phenomenon in real life?

   Answer:

Applications

3. Quadric surfaces are seen extensively in nature, design and engineering, owing in part to their symmetry. Spot the quadric surfaces in the pictures below, then provide realistic equations for each, including units.

   Answer:

(a) **Hyperboloid of One Sheet**

Realistic Dimensions: Let's assume a tower height of 150 meters, a narrowest radius (the "throat") of 35 meters at $z = 0$, and a base radius of 60 meters.

$$\frac{x^2}{35^2} + \frac{y^2}{35^2} - \frac{z^2}{51^2} = 1 \quad \text{(units in meters)}$$

(b) **Paraboloid (Elliptic/Circular)**

Realistic Dimensions: A large radio telescope might have a diameter of 25 meters and a depth of 4 meters.

$$z = 0.0256(x^2 + y^2) \quad \text{(units in meters)}$$

(c) **Hyperboloid of One Sheet**

Realistic Dimensions: Let's assume the gears have a height of 10 centimeters, a narrowest radius of 2 centimeters at $z = 0$, and a base radius of 4 centimeters.

$$\frac{x^2}{2^2} + \frac{y^2}{2^2} - \frac{z^2}{5^2} = 1 \quad \text{(units in centimeters)}$$

(d) **Sphere**

Realistic Dimensions: A soap bubble might have a radius of 5 centimeters.

$$x^2 + y^2 + z^2 = 25 \quad \text{(units in centimeters)}$$

Extensions

4. What surface is defined by the equation $x^2 + x - y^2 - y = z^2 + z$?

   Answer: Hyperbolic paraboloid shifted to be centered at $(-1/2, -1/2, -1/2)$ with axis of symmetry along the line $(t, -1/2, -1/2), t \in \mathbb{R}$.

5. Many surfaces arising naturally are *not* quadric, yet the ideas we use to identify quadric surfaces can still be used to deduce their geometry. For example, a surface of extreme importance in statistics is given by $z = e^{-(x^2+y^2)}$. Explain why is this surface not quadric. Then, use the z-cross sections to sketch the surface.

   Answer: It is not quadratic because its equation is exponential in form, rather than polynomial of degree two.

Section 13.1: The Geometry of Curves in Space

G3: Geometry of Curves. I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.

G5: Parameterization. I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.

Mechanics

1. Describe the graph of the curve $\mathbf{r}(t) = \langle t\cos(t), t\sin(t), t\rangle$, $t \in \mathbb{R}$.

   Answer: This curve's graph is a spiral, narrowing to a point at the origin when $t = 0$ and widening outward around the z-axis for larger/smaller t.

2. One important family of curves that appear repeatedly in this course are the *graphs* of functions. For a function $y = f(x)$ in $\mathbb{R}^2$, find a parameterization $\mathbf{r}(t)$ which traverses the curve from left to right.

   Answer: $\mathbf{r}(t) = \langle t, f(t)\rangle, t \in$ domain of f

3. The motion of a particle in the xy-plane at time t is described by the vector function

$$\mathbf{r}(t) = e^t\mathbf{i} + \frac{2}{9}e^{2t}\mathbf{j}$$

   (a) Nodding to the previous problem, the graph of this curve is equal to the graph of some function $y = f(x)$ in the plane. What is $f(x)$? *[Hint: consider how $y(t)$ is related to $x(t)$ and what values $x(t)$ takes on]*

   (b) Find the particle's velocity and acceleration vectors at $t = \ln(3)$.

   (c) Sketch the path of the particle and include the particle's velocity and acceleration vectors at $t = \ln(3)$.

   Answer:

   (a) $y = \frac{2}{9}x^2$ for $x > 0$

   (b) $\mathbf{v}(\ln(3)) = 3\mathbf{i} + 4\mathbf{j}$
   $\mathbf{a}(\ln(3)) = 3\mathbf{i} + 8\mathbf{j}$.

4. Parameterize the tangent line to the space curve

$$\mathbf{r}(t) = \left\langle \ln t, \frac{t-1}{t+2}, t\ln t \right\rangle$$

when $t = 1$.

Answer: $x(s) = s, y(s) = \frac{s}{3}, z(s) = s$

Applications

5. Determine the point at which the two curves $\mathbf{f}(t) = \langle t, t^2, t^3\rangle$ and $\mathbf{g}(t) = \langle \cos(t), \cos(2t), t+1\rangle$ intersect, and find the angle between the curves at that point. *[Hint: You'll need to set this up like the line intersection problems you've seen before, writing one in s and one in t].*

   If these two functions were the trajectories of two bumblebees on the same scale of time, would the bees collide? Explain.

   Answer: $(1, 1, 1)$ (where the first parameter is 1 and the second is 0). The angle is $\arccos(3/\sqrt{14})$. The bees would not collide, since the first bee reaches the point at $t = 1$ and the second bee at $t = 0$.

6. You are an interior decorator for the secret permanent residence of James Bond. His loft consists of a cylindrical room with walls given by $x^2 + y^2 = 1$, with the first floor at $z = 0$ and second floor at $z = 5$. You are to build a spiraling staircase from the point $(1, 0, 0)$ on the first floor to the point $(1, 0, 5)$ on the second floor. Being the international superspy he is, Mr. Bond wants to discourage any intruders from going upstairs, and hence requests the staircase wrap 10 full (counterclockwise) times around the room until it reaches the desired point on the second floor. Parameterize the staircase you must build.

   Also, compute the total length of the staircase *[Hint: This will be easier to do next week, but you can still do this. Think about "unrolling" the walls of the cylinder.]*

   Answer: $\mathbf{r}(t) = \langle \cos(20\pi t), \sin(20\pi t), 5t\rangle, 0 \leq t \leq 1$.

   Length is $\sqrt{(20\pi)^2 + 5^2} = \sqrt{400\pi^2 + 25}$

Extensions

7. What is the difference between the parametric curves $\mathbf{f}(t) = \langle t, t, t^2\rangle, \mathbf{g}(t) = \langle t^2, t^2, t^4\rangle$, and $\mathbf{h}(t) = \langle \sin(t), \sin(t), \sin^2(t)\rangle$, for $t \in \mathbb{R}$? What if $t \in [0, \pi]$?

   Answer: All three functions describe part of the same set of points in $\mathbb{R}^3$, which lie above the line $y = x$ in the xy-plane and form a parabola in the plane $x = y$. $\mathbf{f}$ traces out all the points on this parabola, $\mathbf{g}$ only those in the first octant, and $\mathbf{h}$ only those which lie above the square $[-1, 1] \times [-1, 1]$.

8. Find a vector-valued function for the curve of intersection of the cylinder $x^2 + y^2 = 9$ and the plane $y + z = 2$. [*Hint: How could you parameterize the circle $x^2 + y^2 = 9$ in the plane?*]

   Answer: $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 2 - 3\sin(t)\rangle, 0 \le t \le 2\pi$

9. Let $\mathbf{r}(t)$ from $a \le t \le b$ be a parameterization of a space curve. Observe that implicit in the equation is an associated direction; the curve goes *from* $\mathbf{r}(a)$ *to* $\mathbf{r}(b)$. One may instead wish for their curve to go the other way: *from* $\mathbf{r}(b)$ *to* $\mathbf{r}(a)$. So given a parametric curve $\mathbf{r}(t)$, find a related parameterization (along with time bounds) that traverses the same path, but backwards. Use this to find a parameterization of the unit circle in $\mathbb{R}^2$, oriented clockwise

   Answer: Let $s = b + (a - b)t$ and take $\mathbf{r}(s)$ for $0 \le s \le 1$.

   Unit circle: $\mathbf{r}(t) = \langle \cos(2\pi - 2\pi t), \sin(2\pi - 2\pi t)\rangle, 0 \le t \le 1$ (or drop the 2π factors and let $0 \le t \le 2\pi$).

10. Find the equation of the plane perpendicular to the curve $\langle \cos(t), \sin(t), \cos(6t)\rangle$ when $t = \pi/4$.

    Answer: $-\frac{1}{\sqrt{2}}(x - \frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{2}}(y - \frac{1}{\sqrt{2}}) + 6(z - 0) = 0$

    OR $x - y - 6\sqrt{2}z = 0$

Section 13.2: Integrals of Vector Valued Functions

G2: Calculus of Curves. I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.

Mechanics

1. Solve the following initial value problems:

   (a) $\mathbf{r}''(t) = -\mathbf{i} - \mathbf{j} - \mathbf{k}, \quad t \geq 0, \qquad \mathbf{r}'(0) = 5\mathbf{i}, \qquad \mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$

   (b) $\mathbf{r}''(t) = \langle t, e^t, e^{-1} \rangle \quad t \in \mathbb{R}, \qquad \mathbf{r}'(0) = \langle 0, 0, 1 \rangle, \qquad \mathbf{r}(0) = \langle 0, 1, 1 \rangle$

   Answer:

   (a) $\mathbf{r}(t) = (-\frac{1}{2}t^2 + 5t + 10)\mathbf{i} + (-\frac{1}{2}t^2 + 10)\mathbf{j} + (-\frac{1}{2}t^2 + 10)\mathbf{k}, \quad t \geq 0.$

   (b) $\mathbf{r}(t) = \langle \frac{1}{6}t^3, e^t - t, \frac{e^{-1}}{2}t^2 + t + 1 \rangle, \quad t \in \mathbb{R}$

Applications

2. A baseball is hit when it is 2.5 ft above the ground. It leaves the bat with an initial velocity of 140 ft/sec at a launch angle of 30°. At the instant the ball is hit, an instantaneous gust of wind blows against the ball, adding a component of $-14\mathbf{i}$ (ft/sec) to the ball's initial velocity. A 15 ft high fence lies 400 ft from the home plate in the direction of the flight. (Note that the acceleration due to gravity is $g = -32\mathbf{j}$ ft/sec^2)

   (a) Find a vector equation for the path of the baseball.

   (b) How high does the baseball go, and when does it reach maximum height?

   (c) Find the range and flight time of the baseball, assuming that the ball is not caught.

   (d) When is the baseball 20 ft high? How far (ground distance) is the baseball from home plate at that height?

   (e) Has the batter hit a home run? Explain.

   Answer:

   (a) $\vec{r}(t) = (140\cos 30^\circ - 14)t\hat{i} + (2.5 + (140\sin 30^\circ)t - 16t^2)\hat{j} = (70\sqrt{3} - 14)t\hat{i} + (2.5 + 70t - 16t^2)\hat{j}$.

   (b) $y_{\text{max}} = \frac{(140\sin 30^\circ)^2}{64} + 2.5 = \frac{70^2}{64} + 2.5 = 79.0625$ ft., which is reached at $t = \frac{140\sin 30^\circ}{32} = \frac{70}{32} = 2.1875$ s.

   (c) For the time, solve $y = 2.5 + 70t - 16t^2 = 0$ for t. Using quadratic formula, we have $t = 4.41$s. Then, the range at $t = 4.41$ is $x(4.41) = (140\cos 30^\circ - 14)(4.41) = 472.94$ ft.

(d) For the time, solve $y = 2.5 + 70t - 16t^2 = 20$ for t. Using quadratic formula, we have $t = 0.27,\ 4.11$ seconds. Then, the range at those times are $x(0.27) = 29$ ft and $x(4.11) = 441$ ft.

(e) Yes, according to part (d), the ball is still 20 feet above the ground when it is 441 feet from home plate.

3. A fortified medieval city is surrounded walls with length 500m and height 15m. You are the commander of an attacking army and the closest you can get to the wall is 100m. Your plan is to set fire to the city by catapulting heated rocks over the wall at an initial speed of 80 ms^{-1}. At what range of angles should you tell your army to set the catapult? You may assume the path of the rocks is perpendicular to the wall, and that you are as close as possible. *[Note: in contrast to the previous problem, here we need to use* $g = -9.8\mathbf{j}\ ms^{-2}$*].*

   Answer: $78.31^\circ \le \theta \le 85.54^\circ$

4. There are 5 seconds left in the annual Tech-UGA football game, and as the targeted receiver in the final play, down 3 points, you've got one chance to send the Dawgs packing.

   The defense drops back on the snap, allowing your quarterback 2 seconds to throw after you start running. They sling the ball at an initial angle of 40° at a speed of 20 ms^{-1}. Your initial speed is 6 ms^{-1}, but as soon as you see the ball come out, you realize that you need to speed up to catch the ball. Assuming your acceleration is constant, how fast do you need to accelerate to catch the ball "in stride" (i.e., intercept the ball as it's about to hit the ground) and go on to score the game-winning touchdown? Also, how fast will you be running when you make the catch? *[Note: be careful about the scale of time.]*

   Answer: acceleration: 3.62 m/s^2

   final speed: 15.48 m/s

Extensions

5. Show that a projectile reaches three-quarters of its maximum height in half the time needed to reach its maximum height.

   Answer:

Section 13.3: The Arc Length Formula

G3: Geometry of Curves. I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.

Mechanics

1. Find the length of the curve

$$\mathbf{r}(t) = \langle \sqrt{2}t, \sqrt{3}t, (1-t) \rangle$$

   from $(0,0,1)$ to $(\sqrt{2}, \sqrt{3}, 0)$

   Answer: $\sqrt{6}$

2. Let $\mathbf{r}(t) = \langle 6\sin 2t, 6\cos 2t, 5t \rangle$. Find the unit tangent vector of $\mathbf{r}(t)$ and find the length of the portion of the graph of $\mathbf{r}(t)$ where $0 \leq t \leq \pi$.

   Answer: $\mathbf{T}(t) = \frac{1}{13}\langle 12\cos(2t), -12\sin(2t), 5 \rangle$ and length: 13π

3. Suppose an object's position is given by $\mathbf{r}(t) = (2\ln(t+1))\mathbf{i} + (e^{2t} + t)\mathbf{j} + (\sin^2(t))\mathbf{k}$. Set up but do not evaluate the appropriate integral with limits to find the distance the object traveled from the point $A(0, 1, 0)$ to the point $B(\ln 4, e^2 + 1, \sin^2(1))$.

   Answer: $\displaystyle\int_0^1 \sqrt{(\frac{2}{t+1})^2 + (2e^{2t} + 1)^2 + (2\sin(t)\cos(t))^2}\ dt$

4. Suppose you are standing at the point $(0, 0, 5)$ and walk 13 units along the path $\langle 5\sin t, 12t, 5\cos t \rangle$ in the direction of increasing t. Where do you end up?

   Answer: $\mathbf{r}(1) = \langle 5\sin(1), 12, 5\cos(1) \rangle$

Applications

5. A DNA molecule has the structure of a double helix (i.e., two intertwined helices). Its radius is about 1 nanometer, and each loop of the helix rises about 3.4 nanometers. Supposing that a full molecule consists of 2.9×10^8 loops, compute the approximate length of a molecule of DNA.

   Answer: 2.07 meters

Extensions

6. Find the point on the curve

$$\mathbf{r}(t) = (5\sin t)\mathbf{i} + (5\cos t)\mathbf{j} + 12t\mathbf{k}$$

at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length.

Answer: $(0, 5, 24\pi)$

7. This problem is meant to generalize the previous one. Let $\mathbf{r}(t)$ be a smooth curve defined for $t \geq 0$. Suppose further that $\|\mathbf{r}'(t)\| \geq 0.01$ for all t. Show that if you are given a positive number D, you can find a (unique!) corresponding point P on the curve, so that the distance from $\mathbf{r}(0)$ to P along the curve equals D. Also, show that if we just assumed that $\|\mathbf{r}'(t)\| > 0$ for all t, this statement is no longer true.

Answer:

.

Section 13.4: Curvature and Normals

G3: Geometry of Curves. I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.

Mechanics

1. Find $\mathbf{T}, \mathbf{N}$ and κ for the curve $\mathbf{r}(t) = \cos(2t)\mathbf{i} - \sin(2t)\mathbf{j} + 6t\mathbf{k}$, for $t \geq 0$. What do you notice about κ? Explain (perhaps with a picture) why this happens.

   Answer: $\mathbf{T}(t) = \dfrac{1}{\sqrt{10}}(-\sin(2t)\mathbf{i} - \cos(2t)\mathbf{j} + 3\mathbf{k})$,

   $\mathbf{N}(t) = -\cos(2t)\mathbf{i} + \sin(2t)\mathbf{j}$

   $\kappa(t) = \dfrac{1}{10}$. The curvature is constant.

2. Compute the unit tangent vector, unit normal vector, and curvature of the curve $\mathbf{r}(t) = \langle \sqrt{2}t, 1+t, e^t \rangle$ for all $t \in \mathbb{R}$.

   Answer: $\mathbf{T}(t) = \dfrac{1}{\sqrt{3+e^{2t}}}\langle \sqrt{2}, 1, e^t \rangle$,

   $\mathbf{N}(t) = \dfrac{1}{\sqrt{9+3e^{2t}}}\langle -\sqrt{2}e^t, -e^t, 3 \rangle$,

   $\kappa(t) = \dfrac{\sqrt{3}e^t}{(3+e^{2t})^{3/2}}$.

3. Compute $\mathbf{N}$ for the curve $\mathbf{r}(t) = \langle t, (1/3)t^3 \rangle, t \in \mathbb{R}$ for $t \neq 0$.

   Does $\mathbf{N}$ exist at $t = 0$? Graph the curve, along with its normal vectors at the times $t = -1, -0.5, 0.5, 1$ and explain what is happening to $\mathbf{N}$ as t passes through $(0, 0)$

   Answer: $\mathbf{N} = \langle \dfrac{-t^2}{\sqrt{1+t^4}}, \dfrac{1}{\sqrt{1+t^4}} \rangle$ if $t > 0$ and $\langle \dfrac{t^2}{\sqrt{1+t^4}}, \dfrac{-1}{\sqrt{1+t^4}} \rangle$ if $t < 0$.

   The normal vector does not exist when $t = 0$; as t passes from negative to positive values the normal vector changes which side of the curve it is on.

Applications

4. You are an engineer overseeing the construction of a certain bridge on campus. The blueprint shows that the bridge has a side view profile which looks like the parabola $y = x^2$. Unfortunately, the material that the bridge is supposed to be built with is extremely rigid, and can only support curves with $\kappa \leq 1.5$ units. Can this bridge be safely built with this material? [*Hint: Where is the curvature the greatest?*]

   Answer: It cannot; the greatest curvature is $\kappa = 2$ units.

5. Imagine that you are an ant travelling along the space curve

$$\mathbf{r}_1(t) = \left(\frac{3}{2}t^2 + 2t, 4t - 1, -3t^2 + 10t\right)$$

while your ant-friend is travelling along a different space curve

$$\mathbf{r}_2(t) = \left(2t^2 - 3t + 10, -\frac{1}{2}t^2 + 9t, -2t^2\right)$$

Assuming you are both looking "forwards" and are on the same scale of time, is there a time t when you are both looking in the same direction? If so, at what time?

Answer: Yes, at $t = 5$.

Extensions

6. For a smooth curve $\mathbf{r}(t)$, define its *binormal vector* $\mathbf{B}(t)$ at a time t to be $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, where the $\times$ is the vector cross product. Compute $\mathbf{B}$ for $\mathbf{r}(t) = (t, 3\cos t, 3\sin t)$.

Answer: $\mathbf{B}(t) = \dfrac{1}{\sqrt{10}}\langle 3, \sin(t), -\cos(t)\rangle$

7. Give an example of a parametric curve in $\mathbb{R}^2$ which has $\mathbf{N}(t) = \left(\frac{-3}{\sqrt{e^{2t}+9}}, \frac{e^t}{\sqrt{e^{2t}+9}}\right)$. You may want to use the fact that $\|(e^t, 3)\| = \sqrt{e^{2t} + 9}$. [*Hint: First deduce a possible* $\mathbf{T}$, *then use the given fact, and integrate.*]

Answer: $\mathbf{r}(t) = \langle -e^t, -3t\rangle$

Section 14.1: Multivariate Functions

G4: Surfaces. I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.

Mechanics

1. Algebraically describe the domains of each of the following functions. Then sketch them on (separate) xy-planes.

   (a) $f(x, y) = \sqrt{x - y - 1}$.

   (b) $f(x, y) = \sqrt{(x - 4)(y^2 - 1)}$.

   (c) $f(x, y) = \cos^{-1}(y - 4x^2)$.

   (d) $f(x, y) = \dfrac{1}{4 - x^2 - y^2}$.

   (e) $f(x, y) = \dfrac{1}{\ln(4 - x^2 - y^2)}$

   Answer:

   (a) $\{(x, y) \mid x - y \geq 1\}$

   (b) $\{(x, y) \mid x \geq 4, |y| \geq 1\} \cup \{(x, y) \mid x < 4, |y| < 1\}$

   (c) $\{(x, y) \mid 4x^2 - 1 \leq y \leq 4x^2 + 1\}$

   (d) All of $\mathbb{R}^2$ except the circle $x^2 + y^2 = 4$

   (e) All of the disk $x^2 + y^2 < 4$ except the circle $x^2 + y^2 = 3$.

2. For each of the surfaces (a)-(g), determine if the proposed descriptions of the level curves are correct. If not, give a correct descriptor. *[Note: consider a point as a circle/ellipse of radius 0]*

   (a) $z = 2x^2 - 3y^2$; Level curves are concentric ellipses.

   (b) $z = x^2 + y^2$; Level curves are concentric circles

   (c) $z = \dfrac{1}{x + y}$; Level curves are lines, whenever $x \neq -y$.

   (d) $z = 2x + 3y$; Level curves are parallel planes.

   (e) $z = \sqrt{25 - x^2 - y^2}$; Level curves are concentric circles, but only if $z > 5$ or $z < -5$

   (f) $z = \sqrt{x^2 + y^2}$; Level curves are concentric circles, but only if $z \geq 0$.

   (g) $z = xy$; Level curves are hyperbolas.

   Answer:

   (a) Not correct; level curves are hyperbolas

(b) Correct

(c) Correct

(d) Not correct; level curves are parallel lines

(e) Not correct; level curves are concentric circles, but only if $0 \leq z \leq 5$.

(f) Correct

(g) Correct

Applications

3. Multivariable functions are often used in economic models to describe how one should price an asset, or how to determine the utility of a product. For example, consider a *utility function* $u(x, y, z)$, where x, y, z represent three independent properties of an object (e.g., price, quantity, quality), and u tells you how much you value that item. In this context, what economic significance do the level surfaces $u(x, y, z) = C$ have (assume C is a constant)? Give an example of how this phenomenon might manifest in your day-to-day life.

   Answer: The level surfaces tell you all combinations of price, quantity, and quality that you value the amount C.

Extensions

4. Find an equation for the level surface of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ passing through $(1, 1, 1)$. Sketch a plot of this level surface in $\mathbb{R}^3$.

   Answer: The sphere $3 = x^2 + y^2 + z^2$

5. Let $f(x, y) = (x - y)^2$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, and $x = y$, and describe the level curves. Use this information to produce a sketch of the graph of the surface. Confirm your sketch using a 3d graphing utility.

   Answer: When $x = 0$, the cross-section is the parabola $z = y^2$.

   When $y = 0$, the cross-section is the parabola $z = x^2$.

   When $x = y$, the cross-section is the line $z = 0$.

   The level curves are pairs of parallel lines $y = x \pm \sqrt{k}$.

Section 14.2: Limits and Continuity

D1: Limits of Functions. I can calculate the limits of some functions of two variables or and apply the Two-Path Test to determine if they do not exist. I can state the definition of continuity for functions of multiple variables.

Mechanics

1. Let $f(x,y) = \left(\frac{1}{x} + \frac{1}{y}\right)^2$. Find $\lim_{(x,y)\to(2,-3)} f(x,y)$ or show it does not exist.

   Answer: $\frac{1}{36}$

2. Let $f(x,y) = \frac{x-2y}{x^3-8y^3}$. Find $\lim_{(x,y)\to(2,1)} f(x,y)$ or show it does not exist.

   Answer: $\frac{1}{12}$

3. Let $f(x,y) = \frac{\sqrt{2x-y}-2}{2x-y-4}$. Find $\lim_{(x,y)\to(2,0)} f(x,y)$ or show it does not exist.

   Answer: $\frac{1}{4}$

4. Let $f(x,y) = \frac{y^2}{x^2+y^2}$. Find $\lim_{(x,y)\to(0,0)} f(x,y)$ or show it does not exist.

   Answer: Does not exist.

Extensions

5. At what points (x,y) in the plane is $f(x,y) = \cos\left(\frac{1}{xy}\right)$ continuous?

   Answer: f is continuous on its entire domain: all (x,y) such that neither $x=0$ nor $y=0$.

6. At what points (x,y,z) is $h(x,y,z) = \frac{1}{1-\ln(x^2+y^2+z^2)}$ continuous?

   Answer: f is continuous on its entire domain: all (x,y,z) except the sphere $x^2+y^2+z^2 = e$.

7. Let

$$f(x,y) = \begin{cases} |x|\sin\left(\frac{1}{y}\right) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Use the Squeeze Theorem for functions of two variables to show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. Show also that

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) \neq \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right).$$

Explain what is happening. Why is this not a contradiction?

Answer:

Review for Exam 1

G1: Lines and Planes. I can describe lines using the vector equation of a line. I can describe planes using the general equation of a plane. I can find the equations of planes using a point and a normal vector. I can find the intersections of lines and planes. I can describe the relationships of lines and planes to each other. I can solve problems with lines and planes.

G2: Calculus of Curves. I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.

G3: Geometry of Curves. I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.

G4: Surfaces. I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.

G5: Parameterization. I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.

D1: Limits of Functions. I can calculate the limits of some functions of two variables or and apply the Two-Path Test to determine if they do not exist. I can state the definition of continuity for functions of multiple variables.

1. Set up the integral to find the arc length of the curve $y = e^x$ from the point $(0, 1)$ to the point $(1, e)$. Focus on finding a parameterization, and on what values of t give these two points. Is this an integral you would want to compute? Why or why not?

   Answer: $\int_0^1 \sqrt{1 + e^{2t}}\, dt$

2. Parameterize the line tangent to the curve

   $$\mathbf{r}(t) = \langle \cos^2(t), \sin(t)\cos(t), \cos(t)\rangle$$

   at the point where $t = \pi/2$.

   Answer: $\ell(s) = \langle 0, -s, -s\rangle$

3. Compute the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ to the circle

   $$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle.$$

   Before checking, should the normal vector be pointing into or out of the circle? Why?

   Answer: $\mathbf{T}(t) = \langle -\sin(t), \cos(t)\rangle$

   $\mathbf{N}(t) = \langle -\cos(t), -\sin(t)\rangle$

   Into

4. We have seen that the curvature of a circle with radius a is $1/a$. Thinking about the geometry of a helix with radius a, do you think its curvature will be greater than or less than $1/a$? Why? Compute the curvature using the parameterization

$$\mathbf{r}(t) = \langle a\cos(t), t, a\sin(t)\rangle$$

to confirm or challenge your intuition.

Answer: $\kappa = \dfrac{a}{1+a^2}$

5. The function $\ell(t)$ below describes a line. There is a particular plane that $\ell(t)$ is normal to at the point $t = 0$. Find an equation of this plane.

$$\ell(t) = \langle 3 - 3t, 2 + t, -2t\rangle.$$

Where does this line intersect the different plane $3x - y + 2z = -7$?

Answer: $-3(x-3) + (y-2) - 2z = 0$
Intersection point is $(0, 3, -2)$, when $t = 1$.

6. Find and sketch the domain of each of the following functions of two variables:

   (a) $\sqrt{9 - x^2} + \sqrt{y^2 - 4}$
   (b) $\arcsin(x^2 + y^2 - 2)$
   (c) $\sqrt{16 - x^2 - 4y^2}$

   Answer:

   (a) $\{(x, y) \mid |x| \leq 3, |y| \geq 2\}$
   (b) $1 \leq x^2 + y^2 \leq 3$
   (c) $\dfrac{x^2}{16} + \dfrac{y^2}{4} \leq 1$

7. Solve the differential equation below, together with its given initial conditions. Remember that this means finding all functions $\mathbf{r}(t)$ which satisfy the given equations.

$$\mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}, \quad \mathbf{r}'(1) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(1) = \mathbf{i} + \mathbf{j}$$

Answer: $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + \frac{2}{3}(t^{3/2} - 1)\mathbf{k}$

8. Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$. Is it possible to define $f(0, 0)$ in a way that makes f continuous at the origin? Why?

Answer: No, because the limit of f as $(x, y) \to (0, 0)$ does not exist.

Section 14.3: Partial Derivatives

D2: Computing Derivatives. I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.

Mechanics

1. Find all first and second partial derivatives for $f(x, y) = e^x + x\ln(y)$.

   Answer:
$$f_x = e^x + \ln(y), \quad f_y = \frac{x}{y}$$
$$f_{xx} = e^x, \quad f_{xy} = \frac{1}{y} = f_{yx}, \quad f_{yy} = -\frac{x}{y^2}$$

2. Find f_x, f_y, f_z, and f_{xzz} for the function $f(x, y, z) = x\sin(yz)$.

   Answer:
$$f_x = \sin(yz), \quad f_y = xz\cos(yz), \quad f_z = xy\cos(yz), \quad f_{xzz} = -y^2\sin(yz)$$

3. Find the total derivative Df at the given point for each function below. Remember that Df is the matrix of (partial) derivatives of the function and if f is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ then Df is an $m \times n$ matrix.

   (a) $f(x) = 2x^3 + 7$ at $x = 2$.

   (b) $\mathbf{f}(t) = \langle 2\cos(t), 2\sin(t), t\rangle$ at $t = \pi/2$.

   (c) $f(x, y) = \sqrt{y - x}$ at $(x, y) = (1, 2)$.

   (d) $f(x, y, z) = e^{2y-x} + z^2 + 4$ at $(x, y, z) = (1, 2, 3)$.

   (e) $\mathbf{f}(s, t) = \langle 2s + 3t, t - s\rangle$ at $(s, t) = (1, 1)$.
   Note: The graph of this function is a surface (in this case all of $\mathbb{R}^2$) parameterized by two variables just like the graph of the function in (b) is a curve parameterized by one variable - we'll see these more later! Another way of thinking about this is that this is a *change of variables* for $\mathbb{R}^2$ between the system of coordinates (s, t) and (x, y).

   Answer:

   (a) $Df(2) = f'(2) = [24]$

   (b) $D\mathbf{f}(\pi/2) = \mathbf{f}'(\pi/2) = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

   (c) $Df(1, 2) = \begin{bmatrix} -1/2 & 1/2 \end{bmatrix}$

   (d) $Df(1, 2, 3) = \begin{bmatrix} -e^3 & 2e^3 & 6 \end{bmatrix}$

   (e) $D\mathbf{f}(1, 1) = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$

Applications

4. The speed of sound C traveling through ocean water is a function of temperature, salinity, and depth. It may be modeled by the function

$$C(T, S, D) = 1450 + 4.5T - 0.05T^2 + 0.0003T^3 + (1.5 - 0.01T)(S - 35) + 0.015D,$$

where C is the speed of sound in meters/second, T is the temperature in degrees Celsius, S is the salinity in grams/liter of water, and D is the depth below the ocean surface in meters.

(a) State the units in which each of the partial derivatives C_T, C_S, and C_D are expressed and explain the physical meaning of each.

(b) Find the partial derivatives C_T, C_S, and C_D.

(c) Evaluate each of the three partial derivatives at the point where $T = 10, S = 35$, and $D = 100$. What does the sign of each partial derivative tell us about the behavior of the function C at the point $(10, 35, 100)$?

Answer:

(a) C_T: (meters/second)/ degree Celsius - this gives the change in speed for each one degree C of temperature increase. C_S (meters/second)/(grams/liter) - this gives the change in speed for each one gram/liter increase in salinity C_D: (meters/second)/meter - this gives the change in speed for each one meter increase in depth below the surface

(b) $C_T = 4.5 - 0.1T + 0.0009T^2 - 0.01(S - 35)$ $C_S = 1.5 - 0.01T$ $C_D = 0.015$

(c) At $(T, S, D) = (10, 35, 100)$, we have $C_T = 3.59, C_S = 1.4, C_D = 0.015$. This tells us that if we increase the temperature, salinity, or depth from these conditions the speed of sound will increase as well.

5. Recall from last week's worksheet that a utility function is a multivariable function $u(x, y, z)$, where x, y, z represent three independent properties of an object (e.g., price, quantity, quality), and u tells you how much you value that item. The *marginal utility functions* are the partial derivatives u_x, u_y and u_z. What is the economic interpretation of the marginal utilities?

Answer: The marginal utility functions tell you how much your utility (value) changes when you change one of the properties of the object, while keeping the other two properties fixed.

Extensions

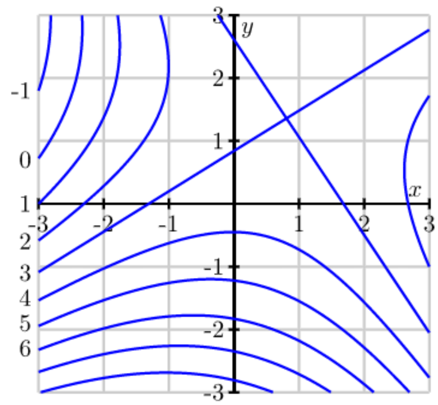
6. Below is a contour plot for a function $f(x, y)$, with values for some of the contours (level curves) indicated on the *left* of the figure.

(a) Find the sign of the partial derivatives $f_x(-2,-1)$ and $f_y(-2,-1)$.

(b) At the point $(0,-1/2)$, which is larger? f_x or f_y?

(c) Find all (x,y) where $f_x(x,y)=0$.

(d) Locate, if possible, one point (x,y) where $f_x(x,y)<0$.



Answer:

(a) $f_x(-2,-1) > 0$ and $f_y(-2,-1) < 0$

(b) $f_y(0,-1/2) > f_x(0,-1/2)$

(c) $f_x(x,y) = 0$ along the tops of the arcs labeled $4, 5, 6, \ldots$

(d) One such point is $(1,-3/2)$

7. The fifth-order partial derivative $\partial^5 f/\partial x^2 \partial y^3$ is zero for each of the following functions. To show this as quickly as possible, which variable would you differentiate with respect to first: x or y?

Try to answer without writing anything down. Why did you make the choice you did?

(a) $f(x,y) = y^2x^4e^x + 2$

(b) $f(x,y) = y^2 + y(\sin(x) - x^4)$

(c) $f(x,y) = x^2 + 5xy + \sin(x) + 7e^x$

(d) $f(x,y) = xe^{y/2}$

Answer: Note this does not have a definitive right answer - some differences may arise and that's good! Discuss!

(a) First y since $\partial^3 f/\partial y^3 = 0$ and the y-partial derivatives are easier

(b) First y, since $\partial^3 f/\partial y^3 = 0$

(c) First y, since $\partial^2 f/\partial y^2 = 0$

(d) First x, since $\partial^2 f/\partial x^2 = 0$ and the x-partial derivatives are easier.

A common theme is to work with the variable with lower powers/simpler expressions first when taking mixed partials.

8. Let A be any 2×2 matrix, and let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$. Compute the total derivative $D\mathbf{f}$. What do you notice? What familiar family of functions from single-variable calculus does this remind you of? Can you generalize this result?

Section 14.4: The Chain Rule

D2: Computing Derivatives. I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.

A1: Interpreting Derivatives. I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.

Mechanics

1. Use the chain rule to compute the total derivatives of the following at the prescribed points. [*Recall that Df for $f : \mathbb{R}^n \to \mathbb{R}^m$ is an $m \times n$ matrix.*]

   (a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = h(g(t))$, where $g(t) = (t+1, t^2)$, $h(x, y) = xy$ at $t = 2$.

   (b) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = (b \circ a)(x, y)$, where $a(x, y) = x \sin y$, $b(t) = 8t - t^2$ at $(x, y) = (4, \pi/3)$.

   (c) $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(t) = u(v(t))$, where $v(t) = (1, t, t^2)$, $u(x, y, z) = (xy, yz)$ at $t = 1$.

   (d) $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = h(g(x, y))$, where $g(x, y) = (3x + 4y, 5x + 7y)$, $h(u, v) = (7u - 4v, -5u + 3v)$ at $(x, y) = (0, 0)$.
   What is interesting about (d)? How are g and h related?

   Answer:

   (a) $Df(2) = \begin{bmatrix}16\end{bmatrix}$

   (b) $Df(4, \pi/3) = \begin{bmatrix}4\sqrt{3} - 6 & 16 - 8\sqrt{3}\end{bmatrix}$

   (c) $Df(1) = \begin{bmatrix}1 \\ 3\end{bmatrix}$

   (d) $Df(0, 0) = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$

2. Find the values of t where $\dfrac{dz}{dt} = 0$ if $z = 3x + 4y$, $x = t^2$, and $y = 2t$.

   Answer: $t = -4/3$

3. Let $w(x, y, z) = xy + yz + zx$, where $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$. Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial \theta}$ when $r = 2$ and $\theta = \dfrac{\pi}{2}$.

   Answer: $\dfrac{\partial w}{\partial r}(2, \pi/2) = 2\pi$ $\qquad \dfrac{\partial w}{\partial \theta}(2, \pi/2) = -2\pi$

Applications

4. You are a myrmecologist (ant scientist) studying the behavior of ants on anthills. You equipped an ant with a tracker to track its motion. Unfortunately, the exact geometry of the hill, as well as the ant's precise motion are too delicate to measure. Fortunately, at time $t = 5$ seconds it is known that

$$\frac{\partial z}{\partial x} = 5, \quad \frac{\partial z}{\partial y} = -2, \quad \frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 7$$

   where z denotes the height of the ant relative to the ground. Use this information to determine if the ant is going uphill or downhill (or neither) at $t = 5$ seconds.

   Answer: The ant is going uphill.

5. The multivariable chain rule is a key tool in modern machine learning. In the big picture, neural networks "learn" parameters through an algorithm called *gradient descent*. This algorithm involves computing total derivatives of long chains of functions in high dimensions, which is in general extremely hard to do. The chain rule tells us that instead of contending with such a long chain of functions all at once, one can instead study each "layer" by itself, then combine everything with matrix multiplication, which is relatively easier. This is called *backpropagation*.

   Answer: Right now this is just an interesting piece of information.

Extensions

6. Suppose we have a differentiable function $w = g(x, y)$, x and y are differentiable functions of t, and we know the following information.

$$g(1,0) = 1,\ g_x(1,0) = -2,\ g_y(1,0) = 2,\ g(-1,2) = 3,\ g_x(-1,2) = 1,\ g_y(-1,2) = -2,$$

$$x(2) = 1,\ y(2) = 0,\ x(1) = 1,\ y(1) = 3,\ x'(2) = 4,\ y'(2) = -1,\ x'(1) = 0,\ y'(1) = 2$$

   If possible, find $\frac{dw}{dt}(1)$ and $\frac{dw}{dt}(2)$ or explain why the given information is not enough to do so. Which of these pieces of information would you not use at all to compute either value?

   Answer: $\frac{dw}{dt}(2) = -10$. $\frac{dw}{dt}(1)$ cannot be computed from the given information because we do not know the values of g_x or g_y at $(x(1), y(1)) = (1, 3)$. We do not use the values of $g(1,0), g(-1,2), g_x(-1,2), g_y(-1,2)$.

7. Give an example of a nonconstant, differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ for which Df is **not** invertible at $(0, 0)$, but invertible at $(1, 0)$.

   Answer: Many possible answers. One is

$$f(x, y) = \begin{bmatrix} x^2 \\ y \end{bmatrix}$$

Section 14.5: Gradients and Directional Derivatives

D2: Computing Derivatives. I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.

A1: Interpreting Derivatives. I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.

Mechanics

1. Compute the gradients of each of the following:

   (a) $f(x, y) = x^2 y$

   (b) $f(x, y, z) = x\sin(y) + z\cos(x)$

   (c) $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$

   Answer:

   (a) $\langle 2xy, x^2\rangle$

   (b) $\langle \sin(y) - z\sin(x), x\cos(y), \cos(x)\rangle$

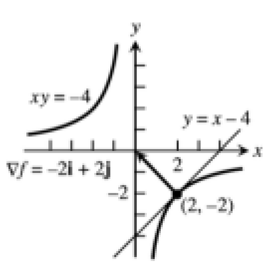
   (c) $\langle 2x, 2y, 2z, 2w\rangle$

2. Find the derivative of $g(x, y) = \dfrac{x-y}{xy+2}$ at $(1, -1)$ in the direction of $\langle 12, 5\rangle$

   Answer: $D_{\mathbf{u}}g(1, -1) = \dfrac{21}{13}$

3. Let $f(x, y) = xy$. Sketch the curve $f(x, y) = -4$ together with $\nabla f(2, -2)$ and the tangent line at $(2, -2)$. Then, find an equation for the tangent line. What do you notice?

   Answer: Tangent line: $-2(x-2) + 2(y+2) = 0$



Applications

4. Gradients form the basis for "learning" in machine learning through a process called *gradient descent*. Here is a setup and overview of the thematic ideas: we are interested in minimizing a (differentiable) error function $\mathcal{L}(x, y, z)$ (*e.g., $\mathcal{L}$ might represent the difference between a predicted quantity vs. the true value*). Though we are able to plug in points, the function $\mathcal{L}$ may be difficult to write down, thus we cannot do "regular calculus" (e.g., first derivative test) with it.

   Gradients give us a workaround to approximate a local minimum as follows: start by randomly choosing point (x_0, y_0, z_0). Calculate the gradient at this point, and then take a small step in the direction of $-\nabla\mathcal{L}(x_0, y_0, z_0)$ [why this direction?] to arrive at a new point (x_1, y_1, z_1). Now, repeat this process with successively smaller step sizes. As it turns out, if we choose our initial point and step sizes cleverly, we are (sometimes) able to get closer and closer to a local minimum, without having much knowledge of the function $\mathcal{L}$! Can you anticipate some shortcomings of this algorithm?

   Answer: Answers will vary. May include issues with stability or overshooting.

5. Suppose you are climbing a hill whose shape is given by the equation

$$z = 1000 - 0.005x^2 - 0.01y^2,$$

   where x, y, and z are measured in meters, and you are standing at a point with coordinates $(60, 40, 966)$. The positive x-axis points east and the positive y-axis points north.

   (a) If you walk due south, will you start to ascend or descend? At what rate?

   (b) If you walk northwest, will you start to ascend or descend? At what rate?

   (c) In which direction is the slope largest? What is the rate of ascent in that direction?

   Answer:

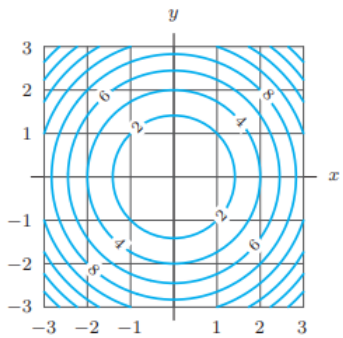
   (a) Ascend at a rate of 0.8 vertical meters per horizontal meter

   (b) Descend at a rate of $\sqrt{2}/10$ vertical meters per horizontal meter

   (c) $\langle -0.6, -0.8 \rangle$ is the direction of largest slope with rate of ascent 1 vertical meter per horizontal meter.

Extensions

6. Use the contour diagram of the differentiable function f given below to decide if the specified directional derivative is positive, negative, or approximately zero.



(a) At the point $(-2, 2)$ in the direction $\mathbf{i}$

(b) At the point $(0, -2)$ in the direction $\mathbf{j}$

(c) At the point $(-1, 1)$ in the direction $\mathbf{i} + \mathbf{j}$

(d) At the point $(-1, 1)$ in the direction $-\mathbf{i} + \mathbf{j}$

(e) At the point $(0, -2)$ in the direction $\mathbf{i} - 2\mathbf{j}$

Answer:

(a) Negative

(b) Negative

(c) Approximately zero

(d) Positive

(e) Positive

7. Let $f(x, y) = -x^2y + xy^2 + xy$ and $P = (2, 1)$.

(a) Find the direction of maximal increase of f at P.

(b) What is the maximum rate of change of f at P?

(c) Find the direction of maximal decrease of f at P.

(d) Find a direction $\mathbf{u}$ such that $D_{\mathbf{u}}f(P) = 0$ (note this forces $\mathbf{u}$ to be a unit vector!).

Answer:

(a) $\langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$

(b) $2\sqrt{2}$

(c) $\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$

(d) $\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$

Section 14.6: Linearization and Tangent Planes

D3: Tangent Planes and Linear Approximations. I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.

Mechanics

1. Find the linearization of $f(x, y) = e^{2y-x}$ at $(1, 2)$. Without doing any more calculations, find an equation of the tangent plane of the surface $f(x, y)$ at $(1, 2)$.

   Answer: $L(x, y) = e^3 - e^3(x - 1) + 2e^3(y - 2)$

   tangent plane: $z = e^3 - e^3(x - 1) + 2e^3(y - 2)$

2. Find the linearization of $f(x, y, z) = \arctan(xyz)$ at $(1, 1, 0)$.

   Answer: $L(x, y, z) = z$

3. Use the linearization to approximate $f(2.95, 7.1)$ for the function $f(x, y) = \sqrt{x^2 + y}$, knowing that $f(3, 7) = 4$.

   Answer: $f(2.95, 7.1) \approx 4 - 1/40$

4. Find an equation of the tangent plane to the unit sphere $x^2 + y^2 + z^2 = 1$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right)$.

   Answer: $\sqrt{2}\left(x - \frac{1}{\sqrt{2}}\right) + \left(y - \frac{1}{2}\right) + \left(z - \frac{1}{2}\right) = 0$

Applications

5. Suppose you are shining a flashlight on a smooth surface. The *angle of incidence*, θ_i is the angle at which a light beam hits the surface, measured with respect to the surface normal (i.e., the normal vector to the tangent plane at point of contact). The *angle of reflection* θ_r is the angle of the reflected light beam measured with respect to the surface normal. The *law of reflection* states that in a vacuum, we must have $\theta_i = \theta_r$. Draw a labeled picture to convince yourself that this is reasonable.

   Now, consider the paraboloid $z = x^2 + y^2$, and a light ray traveling along the path $\mathbf{r}(t) = (-2, -3, 2)t + (3, 4, 0)$. Compute the angle of reflection at the point $(1, 1, 2)$ [*Hint: How can one find the angle between two vectors?*].

   Answer: $\theta_r = \arccos(\frac{4}{\sqrt{17}}) \approx 0.245$ rad

Extensions

6. Use software to graph the function $z = x^{1/3}y^{1/3}$. Examine the graph at $(0, 0)$ - does it look like the function has a tangent plane there? Use this to deduce a necessary condition for a function $f(x, y)$ to have a tangent plane.

   Answer: No, there are two different tangent planes. The function cannot have a cusp at the point with the tangent plane.

Section 14.7: Optimization

D4: Optimization. I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.

Mechanics

1. Find and classify all critical points for the function $x^3 + 3xy + y^3$.

   Answer: Saddle point at $(0,0)$ and local minimum at $(0,-2)$.

2. Find all the local maxima, local minima, and saddle points of $f(x,y) = e^y(x^2 - y^2)$.

   Answer: Saddle point at $(0,0)$ and local maximum at $(-1,-1)$.

3. Find the absolute maxima and minima of the function $f(x,y) = x^2 - xy + y^2 + 1$ on the closed triangular plate bounded by lines $x = 0$, $y = 4$, $y = x$ in the first quadrant.

   Answer: The absolute maximum is 17, achieved at $(0,4)$ and $(4,4)$, and the absolute minimum is 1, achieved at $(0,0)$.

4. Give an example of a differentiable function $f(x,y)$ with no critical points.

   Answer: Many possible examples, e.g. any linear function $f(x,y) = ax + by + c$.

Applications

5. A terrible recipe for lemonade calls for you to just mix lemon juice, denoted by ℓ, and water, denoted by w (units in tonnes). Suppose that given a pair (ℓ, w), you are able to make $f(\ell, w) = \ell^2 - \ell w + w$ liters of lemonade. Given that have only 2 tonnes of lemon juice and 3 tonnes of water, what is the maximum amount of (a very acidic) lemonade can you make? How much of each ingredient is used?

   Answer: $f_{max} = 4$ is the unique maximum attained at $(\ell, w) = (2,0)$. This means that the lemonade is pure lemon juice :)

6. In an alternate universe, Atlanta is famous for her extravagant beaches and pristine waters. In this universe, it is known that Atlanta's waters are extremely wavy, and the height of the water (relative to ground level) may be modeled by the function $h(x,y) = \sin(x)\cos(y)$, for $x \in (0, 2\pi)$ and $y \in (0, 2\pi)$. Visualize this using a 3D graphing software, and compute the height of the highest tides as well as the depth of the lowest troughs. At which coordinates (x,y) do these tides and troughs occur?

   Answer: The highest tides are at 1 occurring at $(3\pi/2, \pi)$ and the lowest troughs at -1 occurring at $(\pi/2, \pi)$.

7. Let us interpret local maxima as peaks of mountains, local minima as valleys and saddle points as passes between mountain peaks. Consider the statement: "It is impossible to have two mountain peaks without some sort of valley or pass connecting them. Therefore, if a function has two local maxima, there must also be a saddle point or a local minimum." Do you agree with this? Verify your answer by using software to graph the function $f(x, y) = 4x^2e^y - 2x^4 - e^4y$.

   Answer: The statement is false! The function given has precisely two local maximums and no other critical points.

Extensions

8. Can you conclude anything about $f(a, b)$, if f and its first and second partial derivatives are continuous around the critical point (a, b) and $f_{xx}(a, b)$ and $f_{yy}(a, b)$ have opposite signs? Justify your answer.

   Answer: Yes, this must be a saddle point because $f_{xx}(a, b)f_{yy}(a, b) < 0$ so $\det(Hf) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b) < 0$.

9. In each case, the origin is a critical point of f and $f_{xx}f_{yy} - (f_{xy})^2 = 0$ at the origin, so the Second Derivative Test fails at the origin. Use some other method to determine whether the function f has a maximum, a minimum, or neither at the origin.

   (a) $f(x, y) = x^2y^2$
   (b) $f(x, y) = 1 - x^2y^2$
   (c) $f(x, y) = xy^2$
   (d) $f(x, y) = x^3y^2$
   (e) $f(x, y) = x^3y^3$
   (f) $f(x, y) = x^4y^4$

   Answer:

   (a) Minimum is 0 at $(0, 0)$ since $f(x, y) > 0$ for all other (x, y).
   (b) Maximum is 1 at $(0, 0)$ since $f(x, y) < 1$ for all other (x, y).
   (c) Neither since $f(x, y) < 0$ for $x < 0$ and $f(x, y) > 0$ for $x > 0$.
   (d) Neither since $f(x, y) < 0$ for $x < 0$ and $f(x, y) > 0$ for $x > 0$.
   (e) Neither since $f(x, y) < 0$ for $x < 0$ and $y > 0$, but $f(x, y) > 0$ for $x > 0$ and $y > 0$.
   (f) Minimum is 0 at $(0, 0)$ since $f(x, y) > 0$ for all other (x, y).

10. Among all rectangular boxes of volume 27 cm^3, what are the dimensions of the box with the smallest surface area? What is the smallest possible surface area? (you may assume this occurs at a local min of the surface area function)

    Answer: The dimensions are $3 \times 3 \times 3$ and the surface area is 54.

Section 14.8: Lagrange Multipliers

D4: Optimization. I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.

Mechanics

1. Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

   Answer: The extreme values are 1 and 2.

2. Find the extreme values of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x^4 + y^4 + z^4 = 1$.

   Answer: The extreme values are 1 and $\sqrt{3}$.

Applications

3. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

   Answer: $V(2, 2, 1) = 4$ cubic units

4. A niche restaurant in midtown Atlanta serves only garlic bread (denoted by g) and bunches of kale (denoted by k). The cost of producing these goods is given by the function $C(g, k) = 5g^2 + 2gk + 3k^2 + 10$. Assuming that the total amount of items to be produced is 40, compute the minimal production cost.

   Answer: $C_{min} = \frac{11230}{3} \approx \$3743.33$.

5. The height of a mountain is given by $h(x, y) = 300 - (3x^2 + 4xy + 3y^2)$. Compute the height of the lowest point on the mountain within $\sqrt{200}$ units of the origin. Where might this point occur? Are there multiple points where this occurs? *[Hint: First, justify why none of x, y, λ can be zero. Then solve for λ.]*

   Answer: The two minimums occur at $(x, y) = (\pm 10, \pm 10)$, where the height is -700 units. Warning: The two other solutions to the Lagrange system are $(x, y) = (\pm 10, \mp 10)$, and they are maximums at height 100.

Extensions

6. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin. *[Hint: It may be helpful algebraically to work with the square of the distance to the origin.]*

   Answer: The closest point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the farthest point is $(-1, -1, 2)$.

7. Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r.

   Answer: $\dfrac{8r^3}{3\sqrt{3}}$

Section 15.1: Double Integrals on Rectangles

I1: Double & Triple Integrals. I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.

I2: Iterated Integrals. I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.

Mechanics

1. Compute $\iint_R (xy - 3xy^2)\, dA$, where R is the square $0 \leq x \leq 2, 1 \leq y \leq 2$.

   Answer: -11

2. Use Fubini's Theorem to evaluate the integral

$$\int_0^1 \int_0^3 xe^{xy}\, dx\, dy$$

   Why was it a good idea to exchange the order of integration?

   Answer: $e^3 - 4$. Exchanging the order of integration allows us to do a substitution for the inner integral rather than integration by parts.

3. Find the volume of the region bounded above by the paraboloid $z = 16 - x^2 - y^2$ and below by the square $R : 0 \leq x \leq 2, 0 \leq y \leq 2$.

   Answer: 160/3 cubic units.

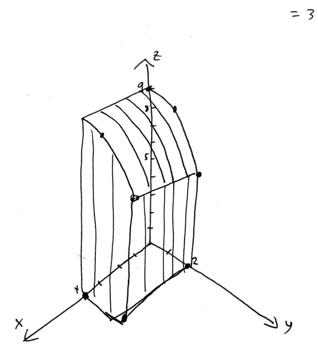
Extensions

4. Evaluate the double integral $\iint_R (4-2y)\, dA$, where $R = [0,1]\times[0,1]$ **without integrating** by identifying it as the volume of a solid *[Hint: It is a prism cut by some plane.]*

   Answer: 3

5. The integral $\iint_R \sqrt{9-y^2}$, where $R = [0,4] \times [0,2]$, represents the volume of a solid. Sketch the solid.

   Answer:



6. This problem explores a failure of Fubini's theorem. Consider the two iterated integrals, which differ by swapping the order of integration:

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dy\, dx \quad \text{and} \quad \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dx\, dy$$

Use the fact that

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2}$$

and that $\frac{d}{dt}\arctan(t) = 1/(1+t^2)$ to show that the iterated integrals are different. Why does Fubini's theorem fail?

Answer: The integrals evaluate to $\pi/4$ and $-\pi/4$ respectively. This does not violate Fubini's theorem because this function is not continuous on $[0,1]\times[0,1]$ (it has an asymptote at $(0,0)$)

Section 15.2: General Double Integrals

I1: Double & Triple Integrals. I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.

I2: Iterated Integrals. I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.

Mechanics

1. Write iterated integrals for $\iint_R 1\, dA$ over the regions R in both orders $dxdy$ and $dydx$.
   (a) R bounded by $y = e^{-x}$, $y = 1$, and $x = \ln 3$.
   (b) R bounded by $y = x^2$ and $y = x + 2$

   Answer:
   (a) $\int_0^{\ln(3)} \int_{e^{-x}}^{1} \, dy\, dx$ and $\int_{1/3}^{1} \int_{-\ln(y)}^{\ln(3)} \, dx\, dy$
   (b) $\int_{-1}^{2} \int_{x^2}^{x+2} \, dy\, dx$ and $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \, dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} \, dx\, dy$

2. Evaluate each of the following by sketching the region of integration and deciding on the best order of integration. Why did you choose the order that you did?
   (a)
   $$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\, dx\, dy$$
   (b)
   $$\int_0^{\pi} \int_0^{x} x\sin(y)\, dy\, dx$$
   (c)
   $$\iint_R xy^2\, dA$$
   where R is the region bounded by $x = 0$ and $x = \sqrt{1 - y^2}$

   Answer:
   (a) 0; switch the order
   (b) $\frac{\pi^2}{2} + 2$; keep the order
   (c) $\frac{2}{15}$; $dx\, dy$ is easier

Applications

3. Find the mass of a triangular plate with vertices $(0, 0), (4, 0)$ and $(0, 1)$, given that its density at any point is $\rho(x, y) = x^2 + y^2$

   Answer: 17/3

Extensions

4. Find the volume of the solid bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant.

   Answer: 16/3

Section 15.3: More Double Integrals

I1: Double & Triple Integrals. I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.

I2: Iterated Integrals. I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.

Mechanics

1. Consider the function $f(x, y) = xy$. Without performing any computations, do you think the average value of f is larger over the square $0 \leq x \leq 1, 0 \leq y \leq 1$, or over the quarter circle $x^2 + y^2 \leq 1$ *in the first quadrant*? Verify your guess by integrating

   Answer: On square: $f_{avg} = \frac{1}{1} \cdot \frac{1}{4} = \frac{1}{4}$

   On quarter circle: $f_{avg} = \frac{1}{\pi/4} \cdot \frac{1}{8} = \frac{1}{2\pi}$

2. A metal triangular plate with vertices $(0, 0)$, $(2, 0)$ and $(2, 4)$ has temperature equal to $C(x, y) = x^2 e^{xy}$ degrees Celsius. Compute the average temperature of the plate. *[Hint: Choose a favorable order of integration.]*

   Answer: $\dfrac{e^8 - 9}{16} \approx 185.7$ degrees Celsius

Applications

3. If $f(x, y) = 100(y + 1)$ represents the population density in people per square mile of a planar region on Earth, where x and y are measured in miles, find the number of people in the region bounded by the curves $x = y^2$ and $x = 2y - y^2$.

   Answer: 50 people

4. A rectangular can of Pringles chips may be modeled by the prism $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $0 \leq z \leq 5$. Assuming that the Pringles container is filled up with chips until the surface $z = x^2 - y^2 + 3$, are there more chips or air in the can? *[Note: The Pringles enthusiast may complain that their containers are supposed to be cylinders, not prisms. This nuance will be addressed when we work with polar coordinates.]*

   Answer: More chips (3 units vs 2 units of air)

Extensions

5. An organism can be initially described as the solid with base $[0, 1] \times [0, 1]$ and height $z = e^{x+y}$. Suppose that the base of this organism grows at a rate of t units per second in both the positive x and positive y directions. Compute the rate of change of the volume

of the organism at $t = 4$ seconds. *[Hint: Set up an integral expression for the volume in terms of t, evaluate the integral, then differentiate with respect to t.]*

Answer: $8e^{18} - 8e^{9}$ cubic units per second

Section 15.4: Polar Coordinates

I3: Change of Variables. I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.

Mechanics

1. Use polar coordinates to compute the integral:

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-x^2-y^2}\ dx\ dy$$

   Answer: $\frac{\pi}{2}(1-e^{-4})$

2. Evaluate $\iint_D y^2 + 3x\ dA$ where D is the region in the 3rd quadrant between $x^2+y^2=1$ and $x^2+y^2=9$.

   Answer: $5\pi - 26$

3. Use a double integral to determine the volume of the solid that is inside the cylinder $x^2+y^2=16$, below $z=2x^2+2y^2$, and above the xy-plane.

   Answer: 256π

4. Give an example of region and function you would *not* want to use polar coordinates to integrate. Justify your answer.

   Answer: Many answers are possible; generally regions that are rectangular in nature (e.g. a square or rectangle) and/or functions that do not have circular symmetry (e.g. $f(x,y)=x+y$) are poor choices for polar coordinates.

Applications

5. The previous worksheet featured a problem involving a peculiar Pringles can. This problem rectifies that inconsistency.

   A true can of Pringles chips may be modeled by the cylinder $x^2+y^2=1$ bounded above and below like $0 \le z \le 5$. Assuming that the Pringles container is filled up with chips until the surface $z=x^2-y^2+3$, are there more chips or air in the can? *[Hint: Use the identity* $\cos^2\theta - \sin^2\theta = \cos 2\theta$*].*

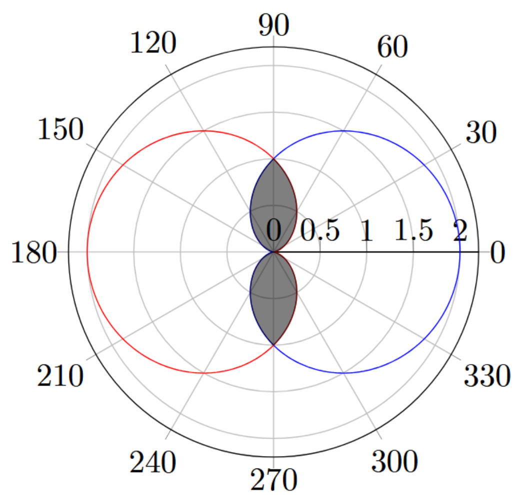
   Answer: More chips: volume of chips is 3π while volume of air is 2π.

6. In the town of Churchill in northern Canada, the density of polar bears around the town dump is given by $p(x,y) = e^{x^2+y^2}$ bears per square mile. Use polar coordinates to compute the average number of polar bears in the region $1 \le x^2 + y^2 \le 2$.

   Answer: $e^2 - e$ bears per square mile

Extensions

7. Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$. *[Hint: Use symmetry to restrict your calculation to only the first quadrant.]*



   Answer: $\dfrac{3\pi}{2} - 4$

8. An integral of great importance in statistics is the Gaussian integral $I = \displaystyle\int_0^\infty e^{-x^2}\,dx$.

   The function $f(x) = e^{-x^2}$ has no elementary antiderivative, so this integral is hard to compute in the usual way. Fortunately, polar coordinates provide a solution.

   Notice that $I^2 = \left(\displaystyle\int_0^\infty e^{-x^2}\,dx\right)\left(\displaystyle\int_0^\infty e^{-y^2}\,dy\right) = \displaystyle\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dx dy.$

   (a) The domain of the above double integral is the first quadrant $[0,\infty) \times [0,\infty)$. Describe this region using polar coordinates, and transform I^2 into an (improper) polar integral.

   (b) Evaluate your double integral to compute the value of I^2. Use this to find the value of the original Gaussian integral I.

   You can find some history of this integral here.

   Answer:

(a) $I^2 = \lim_{R \to \infty} \int_0^2 \pi \int_0^R e^{-r^2} r \, dr \, d\theta$

(b) $I^2 = \frac{\pi}{4}$, so $I = \frac{\sqrt{\pi}}{2}$

Exam 2 Review

D2: Computing Derivatives. I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.

D3: Tangent Planes and Linear Approximations. I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.

D4: Optimization. I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.

I1: Double & Triple Integrals. I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.

I2: Iterated Integrals. I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.

I3: Change of Variables. I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.

A1: Interpreting Derivatives. I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.

A2: Integral Applications. I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.

1. Which of the following statements are true if $f(x, y)$ is differentiable at (x_0, y_0)? Give reasons for your answers.

   (a) If $\mathbf{u}$ is a unit vector, the derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is $(f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \mathbf{u}$.

   (b) The derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is a vector.

   (c) The directional derivative of f at (x_0, y_0) has its greatest value in the direction of ∇f.

   (d) At (x_0, y_0), the vector ∇f is normal to the curve $f(x, y) = f(x_0, y_0)$.

   Answer: All are true except (b).

2. Find dw/dt at $t = 0$ if $w = \sin(xy + \pi), x = e^t$, and $y = \ln(t + 1)$.

   Answer: -1

3. Find the extreme values of $f(x, y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$.

   Answer: ± 1

4. Test the function $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$ for local maxima and minima and saddle points and find the function's value at these points.

   Answer: Saddle at $(0, 0)$ with $f(0, 0) = 0$, local min at $(0, 2)$ of -4, local max at $(-2, 0)$ of 4, saddle at $(-2, 2)$ with $f(-2, 2) = 0$

5. Find the points on the surface $xy + yz + zx - x - z^2 = 0$ where the tangent plane is parallel to the xy-plane.

   Answer: $(-1/2, 1/2, 1/2)$ and $(0, 1, 0)$

6. Evaluate the integral $\int_0^1 \int_{2y}^2 4\cos(x^2)\ dx\ dy$. Describe why you made any choices you did in the course of evaluating this integral.

   Answer: $\sin(4)$

7. If $f(x, y) \geq 2$ for all (x, y), is it possible that the average value of $f(x, y)$ on a unit disk centered at the origin is $\frac{2}{\pi}$?

   Answer: No, this is less than $f(x, y)$ at all points, so it cannot possibly be the average value.

8. A swimming pool is circular with a 40 foot diameter. The depth is constant along east-west lines and increases linearly from 2 feet at the south end to 7 feet at the north end. Find the volume of water in the pool.

   Answer: 1800π cubic feet

Section 15.5: Triple Integrals

I1: Double & Triple Integrals. I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.

I2: Iterated Integrals. I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.

Mechanics

1. Triple integrals can compute volumes, just like double integrals can, so when might you prefer to use one over the other? Give an example of a solid whose volume is easier to compute with a double integral, and vice versa.

   Answer: Many possible answers.

2. Evaluate the triple iterated integral

   $$\int_{-1}^{1}\int_{0}^{4}\int_{0}^{1}(z^3 - 4x^2y)\,dz\,dy\,dx$$

   Recall that to evaluate the innermost integral, treat both x and y as constants and take the antiderivative with respect to z.

   Describe the region of integration.

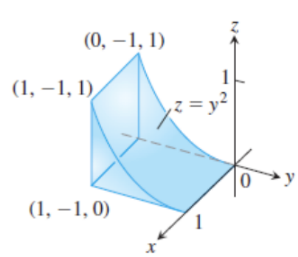
   Answer: $\dfrac{-58}{3}$, region of integration the rectangular prism $[-1,1]\times[0,4]\times[0,1]$ or $-1 \le x \le 1, 0 \le y \le 4, 0 \le z \le 1$.

3. Set up a triple iterated integral for $\iiint_E z\,dV$, where E is the solid tetrahedron in the first octant bounded above by $x+y+z=1$. It may be helpful to make a sketch of the solid.

   Answer: Various possibilities depending on order of integration. E.g.

   $$\int_{0}^{1}\int_{0}^{1-x}\int_{0}^{1-x-y} z\,dz\,dy\,dx$$

4. Set up integrals that would calculate the volume of the region below, using the specified orders of integration.



(a) $dy\ dz\ dx$ (b) $dy\ dx\ dz$ (c) $dx\ dy\ dz$ (d) $dx\ dz\ dy$ (e) $dz\ dx\ dy$

Answer:

(a) $\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\ dz\ dx$

(b) $\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\ dx\ dz$

(c) $\int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx\ dy\ dz$

(d) $\int_{-1}^0 \int_0^{y^2} \int_0^1 dx\ dz\ dy$

(e) $\int_{-1}^0 \int_0^1 \int_0^{y^2} dz\ dx\ dy$

Applications

5. A solid with density $\delta(x, y, z) = 3x^2yz$ is bounded below by the plane $z = 0$, on the sides by the elliptical cylinder $x^2 + 4y^2 = 4$, and above by the plane $z = 2 - x$. Set up all the necessary triple integrals to compute its center of mass. You do not need to compute any integrals (unless you want to). It may be helpful to sketch the solid first.

   Answer:

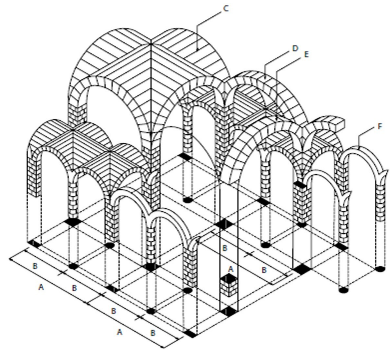
$$M = \int_{-2}^{2} \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} \int_0^{2-x} 3x^2yz\,dz\,dy\,dx \qquad M_{yz} = \int_{-2}^{2} \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} \int_0^{2-x} 3x^3yz\,dz\,dy\,dx$$

$$M_{xz} = \int_{-2}^{2} \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} \int_0^{2-x} 3x^2y^2z\,dz\,dy\,dx \qquad M_{xy} = \int_{-2}^{2} \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} \int_0^{2-x} 3x^2yz^2\,dz\,dy\,dx$$

6. The triple integral below represents the volume of a particular solid, called the *Steinmetz* solid:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\ dz\ dx$$

   Perform this integral, and then unravel the bounds to realize it as the intersection of two simpler solids. Though mathematician Charles Steinmetz is cited with the study of this in the 17/18th century, evidence of its study traces back to ancient Greece and China, as well as the early Renaissance. Can you spot the Steinmetz solids in the following pictures?



Answer: The volume is $\dfrac{16}{3}$. The Steinmetz solid is the intersection of two cylinders of radius 1, centered along the x and y-axes, respectively.

Extensions

7. Use a clever swap of the order of integration to evaluate

$$\int_0^9 \int_0^y \int_{\sqrt{z}}^3 z\cos(y^6)\, dy\, dx\, dz$$

Answer: $\frac{1}{12}\sin(729)$

8. Let D be the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$, i.e.

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \le z \le 2y\}.$$

Write triple iterated integrals in the orders $dz\; dy\; dx$ and $dx\; dz\; dy$ that give the volume of D. Can you write a single triple iterated integral for this volume using any other orders of integration?

Answer: $\displaystyle\int_{-1}^{1} \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \int_{x^2+y^2}^{2y} dz\; dy\; dx.$

$$\int_0^2 \int_{y^2}^{2y} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\; dz\; dy.$$

Yes, all orders of integration result in a single iterated integral.

Section 15.7: Cylindrical and Spherical Coordinates

I3: Change of Variables. I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.

During studio, focus on *setting up* the integrals. However, don't forget to carry out the actual integration in your own time. Both the correct triple integral and correct final answer are given in the answers for this worksheet.

Mechanics

1. Use spherical coordinates to verify that the volume of a sphere with radius R is $\frac{4}{3}\pi R^3$.

   Answer: Integral: $\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$

2. Use cylindrical coordinates to compute $\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_0^{x} (x^2+y^2)\,dz\,dx\,dy$

   Answer: Integral: $\int_{-\pi/2}^{\pi/2}\int_0^{1}\int_0^{r\cos(\theta)} r^3\,dz\,dr\,d\theta$

   Answer: $\frac{2}{5}$.

3. Find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy-plane, and below the cone $z=\sqrt{x^2+y^2}$.

   Answer: Integral: $\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2} \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$

   Answer: $\frac{8\sqrt{2}\pi}{3}$

4. Find the volume of the region bounded above by the paraboloid $z=9-x^2-y^2$, below by the xy-plane, and lying *outside* the cylinder $x^2+y^2=1$.

   Answer: Integral: $\int_0^{2\pi}\int_1^{3}\int_0^{9-r^2} r\,dz\,dr\,d\theta$

   Answer: 32π

5. Suppose $a \geq 0$. Find the volume of the region cut from the solid sphere $\rho \leq a$ by the half-planes $\theta = 0$ and $\theta = \pi/6$ in the first octant.

Answer: Integral: $\displaystyle\int_0^{\pi/6}\int_0^{\pi/2}\int_0^{a} \rho^2 \sin(\varphi)\; d\rho\; d\varphi\; d\theta$

Answer: $\dfrac{a^3\pi}{18}$.

6. When might you prefer to use cylindrical coordinates over spherical ones? In other words, is there a particular type of symmetry whose presence/absence suggests that cylindrical coordinates may be more useful?

   Answer: Many possible answers. One example: Cylindrical coordinates are often preferred when the solid has symmetry around an axis (usually the z-axis) but not around a point. For example, solids bounded by cylinders, planes, and paraboloids are often easier to describe in cylindrical coordinates.

Applications

7. Let D be the right circular cylinder whose base is the circle $r = 2\sin\theta$ in the xy-plane and whose top is the plane $z = 4 - y$. Recall that $r = 2\sin\theta$ describes a circle centered at $(0, 1)$ with radius 1 in the xy-plane. Using cylindrical coordinates,

   (a) find the volume of the region D, and

   (b) find the $\bar{x}$ component of the centroid of the region. *[Hint: Use symmetry.]*

   Answer:

   (a) Integral: $\displaystyle\int_0^{\pi}\int_0^{2\sin(\theta)}\int_0^{4-r\sin(\theta)} r\; dz\; dr\; d\theta$

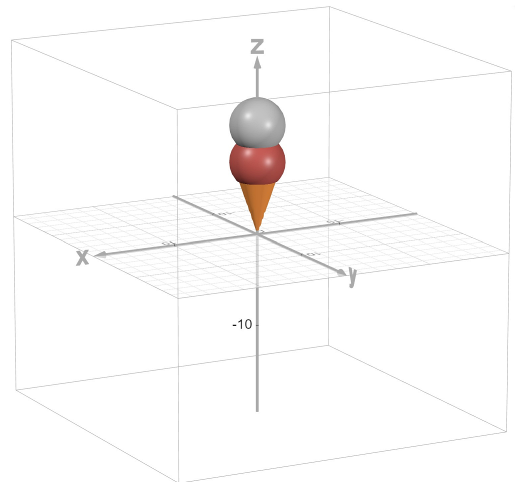
   Answer: 3π.

   (b) $\bar{x} = 0$.

8. A double-scoop ice cream cone can be modeled by the region bound by the three component surfaces

Cone:	$9x^2 + 9y^2 - z^2 = 0$	for $0 \le z \le 6$
Scoop One:	$x^2 + y^2 + (z-8)^2 = 9$	for $6 \le z \le 10$
Scoop Two:	$x^2 + y^2 + (z-12)^2 = 9$	for $10 \le z$

   Compute the volume of the entire dessert (including what is inside the cone, presumably more ice cream). *[Hint: Do not compute the volume of the scoops as they are presented. Translate each scoop down so they are centered at the origin.]*



Answer: 72π

Extensions

9. Let B be the unit ball given by $x^2 + y^2 + z^2 \leq 1$. Compute the average distance of a point in B to the origin. Before you do any calculations, do you expect the average to be less than or greater than 0.5? Why?

   Answer: $3/4$

10. Find the volume of the solid that is between the spheres $\rho = \sqrt{2}$ and $\rho = 2$, but outside the circular cylinder $x^2 + y^2 = 1$. It will be helpful to draw a cross-section in a plane $\theta = c$ for this problem and to use symmetry.

    Answer: Integral: $2\left(\int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_{\csc(\varphi)}^{2} \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta + \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_{\sqrt{2}}^{2} \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta\right)$

    Answer: $\dfrac{12\sqrt{3}-4}{3}\pi$.

Section 15.8: The Change of Variables Formula

Mechanics

1. Find the Jacobian determinant of the transformation $x = e^{-r}\sin(\theta), y = e^{r}\cos(\theta)$.

   Answer: $\sin^2(\theta) - \cos^2(\theta) = -\cos(2\theta)$

2. An ellipse in the plane has the equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ for positive numbers a and b. Use a change of variables to compute the area of this ellipse. *[Note: Without a change of variables, you would likely need a trig substitution to do this!]* πab

3. Compute $\iint_R e^{x+y}\, dA$, where R is the region given by the inequality $|x| + |y| \leq 1$. *[Hint: Sketching the region should reveal a natural choice of transformation to use.]*

   Answer: $e + \frac{1}{e}$

4. Use the change-of-variables formula to justify the cylindrical and spherical volume elements $dx\ dy\ dz = rdr\ d\theta\ dz = \rho^2 \sin\varphi\ d\rho\ d\varphi\ d\theta$ we learned last week.

   Answer: Compute $\det(D\mathbf{T}_c(r,\theta,z))$ and $\det(D\mathbf{T}_s(\rho,\varphi,\theta))$ where

   $$\mathbf{T}_c(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$$

   and

   $$\mathbf{T}_s(\rho,\varphi,\theta) = (\rho\sin(\varphi)\cos(\theta), \rho\sin(\varphi)\sin(\theta), \rho\cos(\varphi)).$$

5. Solve the system

   $$u = 2x - 3y, v = -x + y$$

   for x and y in terms of u and v. Then find the value of the Jacobian and find the image of the parallelogram R in the xy-plane with boundaries $x = -3, x = 0, y = x$, and $y = x + 1$ under this transformation. Sketch the transformed region in the uv-plane. Use your results to rewrite the integral

   $$\iint_R 2(x-y)\ dx\ dy$$

   as an integral in uv-coordinates.

   Answer: $\displaystyle\int_0^1 \int_{-3v}^{-3v+3} -2v\ du\ dv$

Extensions

6. Find equations for a transformation $\mathbf{T}$ that maps a rectangular region S in the uv-plane whose sides are parallel to the u- and v-axes onto the region R bounded by the hyperbolas $y = 1/x, y = 3/x$ and the lines $y = x, y = 3x$ in the first quadrant.

   Answer: $x = \sqrt{\frac{u}{v}}, y = \sqrt{uv}$ maps $[1,3] \times [1,3]$ onto R

7. Use a change of variables to compute

$$\iint_R xy \, dA,$$

where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1, xy = 3$. *[Hint: Consider the above problem.]*

Answer: $\ln(9)$

Section 16.1: Scalar Line Integrals

V1: Line Integrals. I can set up and evaluate scalar and vector field line integrals in two and three dimensions.

A2: Integral Applications. I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.

Mechanics

1. For each curve, find a parameterization of the curve with the specified orientation.

   (a) The line segment in $\mathbb{R}^3$ from $(0, 1, -2)$ to $(3, -1, 2)$.

   (b) The line segment in $\mathbb{R}^3$ from $(3, -1, 2)$ to $(0, 1, -2)$.

   (c) The circle of radius 3 in $\mathbb{R}^2$ centered at the origin, beginning at the point $(0, -3)$ and proceeding clockwise around the circle.

   (d) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in $\mathbb{R}^2$ beginning at the point $(a, 0)$, $a > 0$ and proceeding counterclockwise around the ellipse.

   (e) In $\mathbb{R}^2$, the graph of *any* smooth function $y = f(x)$ from $(x_0, f(x_0))$ to $(x_1, f(x_1))$, where $x_0 \leq x_1$.

   (f) In $\mathbb{R}^2$, the graph of *any* smooth function $x = g(y)$ from $(g(y_0), y_0)$ to $(g(y_1), y_1)$, where $y_0 \leq y_1$.

   Answer: There are many possible correct answers! Here are some.

   (a) $\mathbf{r}(t) = \langle 3t, -2t + 1, 4t - 2\rangle, \quad 0 \leq t \leq 1$

   (b) $\mathbf{r}(t) = \langle -3t + 3, 2t - 1, -4t + 2\rangle, \quad 0 \leq t \leq 1$

   (c) $\mathbf{r}(t) = \langle 3\sin(t), 3\cos(t)\rangle, \quad \pi \leq t \leq 3\pi$

   (d) $\mathbf{r}(t) = \langle a\cos(t), b\sin(t)\rangle, \quad 0 \leq t \leq 2\pi$

   (e) $\mathbf{r}(t) = \langle t, f(t)\rangle, \quad x_0 \leq t \leq x_1$

   (f) $\mathbf{r}(t) = \langle g(t), t\rangle, \quad y_0 \leq t \leq y_1$

2. Find the line integral of $f(x, y) = \sqrt{4x + 1}$ over C where C is the part of the curve $x = y^2$ from the point $(4, -2)$ to $(1, 1)$.

   Answer: The integral to compute is $\int_{-2}^{1} 4t^2 + 1\, dt$. Its value is 15.

3. Find the line integral of $f(x, y, z) = \sqrt{x^2 + y^2}$ over the curve $\mathbf{r}(t) = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} + 3t\mathbf{k}$, where $t \in [0, 2\pi]$.

Answer: The integral to compute is $\int_0^{2\pi} 20\ dt$ since $f(\mathbf{r}(t)) = 4$ and $\|\mathbf{r}'(t)\| = 5$. Its value is 40π.

4. Evaluate the line integral

$$\int_C (x^2 - yz)ds$$

where C is parameterized by $\mathbf{r}_1(t) = (2t, 5t+1, -t)$ for $t \in [0, 10]$. Recalculate the integral using the parameterization $\mathbf{r}_2(t) = (4t, 10t+2, -2t)$ for $t \in [0, 5]$. What do you notice? Is this a coincidence? Summarize your findings by filling in the blank:

Scalar line integrals are independent of ________________ (so long as the path is traversed exactly once).

Answer: $3050\sqrt{30}$

Blank should be filled with: "parameterization"

Applications

5. An awkwardly shaped animal enclosure involves a fence with base parameterized by the closed curve $\langle t^2 - 1, t(t^2 - 1)\rangle$ for $-1 \le t \le 1$ and height equal to $\sin(xy) + 2$. Set up a line integral to compute the surface area of the fence.

   Answer: $\int_{-1}^{1} (\sin(t^5 - 2t^3 + t) + 2)\sqrt{9t^4 - 2t^2 + 1}\ dt$

6. Suppose you are designing a custom tablecloth for a circular dinner table modelled by $x^2+y^2 \le 1$. Your customer would like the cloth to drape down $f(x, y) = -x^2-y^2+xy-4$ units off of the boundary of the table. Compute the total area of fabric needed for the tablecloth.

   Answer: 11π

7. Let C a thin wire parameterized by

$$\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k}, \quad t \in [0, \pi].$$

Find the mass of C if the density along C is $\delta(x, y, z) = z^{-1}$.

Answer: The integral to compute is $\int_0^{\pi} e^{-t}e^t\sqrt{3}\ dt = \sqrt{3}\pi$

Extensions

8. Consider the line integral

$$\int_{C_k} (x^2 + y^2 - z)ds$$

where C_k is the part of the helix $\langle \cos t, \sin t, t \rangle$, for $0 \le t \le k$. Find the value of k which maximizes the integral.

Answer: $k = 1$

Section 16.2: Vector Line Integrals

V1: Line Integrals. I can set up and evaluate scalar and vector field line integrals in two and three dimensions.

A2: Integral Applications. I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.

Mechanics

1. Evaluate $\int_C (2x - y)\mathbf{i} \cdot \, d\mathbf{r}$ where C is parameterized by $\mathbf{r}(t) = (t^2)\mathbf{i} + (3t-2)\mathbf{j}$, $t \in [0, 1]$.
   Answer: 1

2. Consider the closed curve C consisting of a semicircle and a straight line segment as follows:

$$\mathbf{r}_1(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j},\ t \in [0, \pi], \qquad \mathbf{r}_2(t) = t\mathbf{i},\ t \in [-2, 2]$$

   Let the vector field $\mathbf{F}$ be given by

$$\mathbf{F}(x, y) = -y^2\mathbf{i} + x^2\mathbf{j}.$$

   Find the circulation of $\mathbf{F}$ around C and the flux of $\mathbf{F}$ across C.
   Answer: Circulation: $\frac{32}{3}$
   Flux: 0

3. Find the work done by the force $\mathbf{F} = xy\mathbf{i} + (y - x)\mathbf{j}$ over the straight line from $(1, 1)$ to $(2, 3)$.
   Answer: $\frac{25}{6}$

Applications

4. *Ampere's Law* states that for a magnetic field $\mathbf{B}$, and closed curve C, we have that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I_{\text{enc}}$$

   where μ_0 is a physical constant, and I_{enc} is the total current in the region enclosed by C. Compute I_{enc} in the unit square $[0, 1] \times [0, 1]$ induced by the magnetic field $\mathbf{B}(x, y) = \langle x^2y, 5 - xy^2\rangle$. Also compute the magnetic flux across the same square.
   Answer: $I_{\text{enc}} = -\frac{2}{3\mu_0}$
   Magnetic Flux: 0

5. Consider a radial force field $\mathbf{F}(x,y) = \langle Kx, Ky \rangle$ for $K > 0$, and a particle moving counterclockwise on the unit circle. How much work does the force field exert on the particle over one loop around the circle? Give an example of a real life force whose field look like $\mathbf{F}$ (up to choosing a constant K).

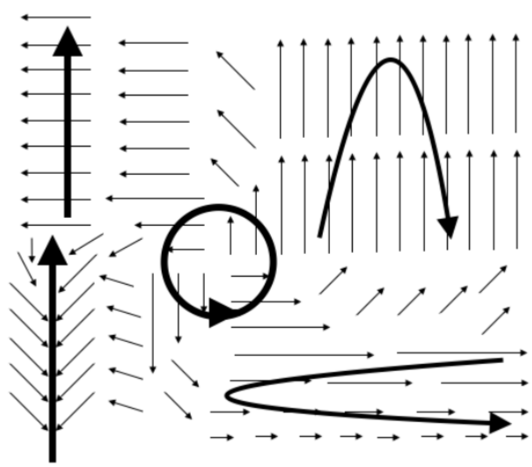
   Answer: Work is 0.

Extensions

6. Consider the vector field $\mathbf{F}$ (thin arrows) and let $\mathbf{T}$ denote the unit tangent vector to the oriented curves shown below (thick arrows). Determine whether

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

   is positive, negative, or zero for each oriented curve C. In other words, determine whether the work done by the vector field on each curve is positive, negative, or zero.



   Answer: Top left: 0
   Bottom left: Negative
   Center: Positive
   Top right: 0
   Bottom right: Negative

7. *The Work-Energy Theorem* tells us that the total work done by a *conservative* force field $\mathbf{F}$ equals the difference in potential energy:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = PE_f - PE_i$$

   On the other hand, the principle of conservation of energy tells us that when we traverse any closed path, no (potential) energy can be gained or lost; it must always be preserved. Using only this fact, verify that the following vector fields are not conservative force fields by integrating over an appropriate loop:

(a) $\mathbf{F}(x, y) = (x^2y, 5 - xy^2)$

(b) $\mathbf{F}(x, y) = (xy, x^2)$

(c) $\mathbf{F}(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ *[Hint: Those fractions would look a lot nicer if their denominators were equal to 1.]*

Answer:

(a) The work done by $\mathbf{F}$ around the square $[0, 1] \times [0, 1]$ is $-\dfrac{2}{3} \neq 0$, so $\mathbf{F}$ is not conservative.

(b) The work done by $\mathbf{F}$ around the square $[0, 1] \times [0, 1]$ is $\dfrac{1}{2} \neq 0$, so $\mathbf{F}$ is not conservative.

(c) The work done by $\mathbf{F}$ around the unit circle is $2\pi \neq 0$, so $\mathbf{F}$ is not conservative.

Section 16.3: Conservative Vector Fields

V2: Conservative Vector Fields. I can test for conservative vector fields and find potential functions. I can state and apply the Fundamental Theorem of Line Integrals.

Mechanics

1. Show that the vector field $\mathbf{F} = 12xy\mathbf{i} + 6(x^2+y^2)\mathbf{j}$ is conservative using the mixed partials test, then find a potential function f such that $\mathbf{F} = \nabla f$.

   Answer: $f(x, y) = 12x^2y + 2y^3$

2. Find a potential function for $\mathbf{F}(x, y, z) = \langle e^x \cos(y) + yz, xz - e^x \sin(y), xy + z\rangle$.

   Answer: $f(x, y, z) = e^x \cos(y) + xyz + \dfrac{z^2}{2}$

3. Find a potential function f for $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 - z^2)\mathbf{j} - 2yz\mathbf{k}$ and compute

   $$\int_C \mathbf{F} \cdot d\mathbf{r}$$

   where C is any path from $(0, 0, 0)$ to $(1, 2, 3)$.

   Answer: $f(x, y, z) = x^2y - z^2y$ and $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = -16$.

4. Find a potential function f for

   $$\mathbf{F}(x, y, z) = \left\langle \frac{1}{y}, -\frac{x}{y^2}, 2z - 1 \right\rangle$$

   and use it to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve $C : \mathbf{r}(t) = \langle \sqrt{t}, t + 1, t^2\rangle, 0 \le t \le 1$.

   Answer: $f = \frac{x}{y} + z^2 - z$ and the value of the line integral is 1/2.

Applications

5. The namesake of "conservative" vector fields has roots in the fact that many observable force fields are conservative, as they obey the law of conservation of energy. Let's verify this fact for the gravitational field in a 2-body system:

   $$\mathbf{F}(x, y, z) = -\frac{GMm}{(x^2 + y^2 + z^2)^{3/2}}\langle x, y, z\rangle$$

   where M, m represent the masses of the two bodies and G is a physical constant. Use the curl test to show that $\mathbf{F}$ is conservative. Verify that the gravitational potential may be given by $V(x, y, z) = \dfrac{GMm}{\sqrt{x^2 + y^2 + z^2}}$.

   Answer: Do what it says.

6. Your classmate in ECON 101 has recently gotten into the honey-selling business and needs to come up with a model describing how much it would cost to ship 10000L of honey from a factory to a destination. They suggest the following honey-transporting scheme: for some conservative vector field $\mathbf{F}$, the price of moving 10000L of honey from point A to B is about $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where C is any curve linking A and B. Using your MATH 2551 expertise, what would you tell your classmate regarding the accuracy of their model?

   Answer: It seems unlikely that the cost of transporting honey would be independent of the path taken, as is implied by the use of a conservative vector field.

Extensions

7. Let a, b, c, d, e be real numbers and

$$\begin{aligned} P(x, y, z) &= 3x + 7y + 2z; \\ Q(x, y, z) &= ax + by + 4z; \\ R(x, y, z) &= cx + dy + ez. \end{aligned}$$

   For which values of the constants a, b, c, d, e is $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ a conservative vector field?

   Answer: $7 = a, 2 = c, 4 = d$, no restriction on b or e.

8. This problem highlights an intricacy in the curl test. Consider the "vortex" vector field $\mathbf{F} = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$.

   (a) What is the domain of $\mathbf{F}$?

   (b) Show that $\mathbf{F}$ passes the curl test everywhere on its domain.

   (c) Show that

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds \neq 0$$

   where C is the unit circle.

   (d) But $\mathbf{F}$ satisfies the curl test! By the fundamental theorem of line integrals, shouldn't the above integral be zero? What goes wrong?

   (e) Recompute the integral in (c) but replace C with the boundary of the square $[1, 2] \times [1, 2]$. What do you notice? Summarize your findings by filling in the blank below:

   The integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ equals zero whenever the region bounded inside C does *not* contain ______________.

   Answer:

   (a) The domain is $\mathbb{R}^2 \setminus \{(0, 0)\}$.

   (b) The curl is zero everywhere on the domain.

(c) The integral is 2π.

(d) The domain is not simply connected; there is a hole at the origin.

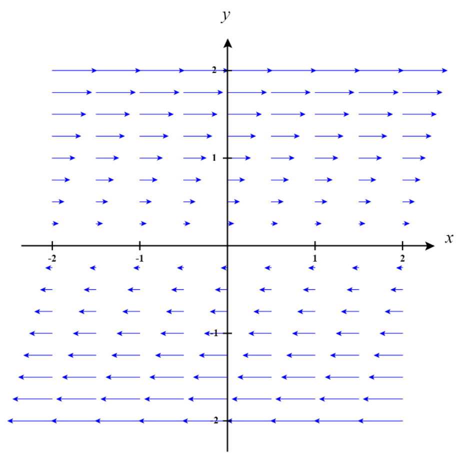
(e) The integral is zero. Thus, the blank is "the origin".

Section 16.4: Curl, Divergence and Green's Theorems

V3: Generalizations of the FTC. I can state and apply Green's Theorem, Stokes' Theorem and the Divergence Theorem to solve problems in two and three dimensions. I can choose which theorem is appropriate for different integrals. I can compute curl and divergence of vector fields.

Mechanics

1. Below is a plot of the vector field $\mathbf{F}(x, y) = \langle \frac{y}{4}, 0 \rangle$. Based on the plot, decide the sign of $\operatorname{curl} \mathbf{F} \cdot \mathbf{k}$ and $\operatorname{div} \mathbf{F}$. Then verify your predictions by computing both quantities.



Answer: Signs: $\nabla \cdot \mathbf{F}$ is 0 in all quadrants.
$(\nabla \times \mathbf{F}) \cdot \mathbf{k}$ is negative in all quadrants.

Computations: $\nabla \cdot \mathbf{F} = 0$
$(\nabla \times \mathbf{F}) \cdot \mathbf{k} = -\frac{1}{4}$.

2. Let C be the ellipse
$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1.$$

   (a) Parameterize this ellipse to give it a positive orientation.

   (b) Let $\mathbf{F}(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$. Use Green's theorem to find the circulation of $\mathbf{F}$ around C and its flux across C.

Answer:

(a) $\mathbf{r}(t) = \langle 3\cos(t), 4\sin(t) \rangle$, $0 \le t \le 2\pi$.

(b) Circulation: 0
Flux: 48π

3. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = \langle x(x + y), xy^2\rangle$ in moving a particle from the origin along the x-axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y-axis.

   Answer: $-1/12$

4. Let R be the region in the xy-plane bounded above by the curve $y = 3 - x^2$ and below by the curve $y = x^4 + 1$. Orient this boundary positively. Let

$$\mathbf{F}(x, y) = (y + e^x \ln y)\mathbf{i} + (e^x/y)\mathbf{j}.$$

   Use Green's theorem to find the circulation of $\mathbf{F}$ around C. What happens when you try to use Green's theorem to evaluate the flux of $\mathbf{F}$ across C? Should you use Green's theorem to evaluate the flux integral?

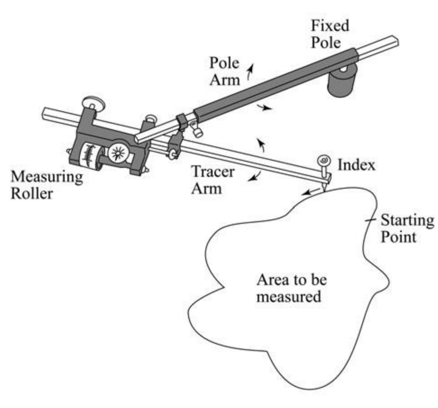
   Answer: Circulation: $-44/15$
   Flux: The integrand is very difficult to work with, so we should not use Green's theorem here.

Applications

5. One way to interpret Green's theorem is that it relates information on the boundary (the line integral) to information in the interior (the double integral). For example, consider the vector field $\mathbf{A}(x, y) = \frac{1}{2}\langle -y, x\rangle$. Use Green's theorem to verify that for a closed curve C,

$$\int_C \mathbf{A} \cdot \mathbf{T}\, ds = \text{ the area of the region enclosed by } C$$

   In fact, this principle underpins the *planimeter*, an area-measuring device dating back to the 17th century. The planimeter consists of a fixed pole, roller, tracer arm and index needle which traverses a boundary curve C.



   Using Green's theorem, one can show that the area of the region enclosed by C (which is probably very complicated) is simply the distance that the roller rolls multiplied by a constant depending on the specifics of the device. One advantage to this approach is that it produces the *exact* area, in contrast to approximation techniques, such as Riemann sums. As such, it has seen use in vastly diverse fields, such as medical imagery (finding the cross-sectional area of a tumor), botany (finding the area of irregularly shaped leaves) and cartography (creating area-accurate maps).

Answer: $\int_C \mathbf{A} \cdot \mathbf{T}\, ds = \iint_R \operatorname{curl} \mathbf{A} \cdot \mathbf{k}\, dA = \iint_R 1\, dA =$ area of R

Extensions

6. Compute the flow of the vector field $\mathbf{F}(x, y) = \langle y^2 \sin x, -2y \cos x + 3x \rangle$ over the positively oriented semicircle $x^2 + y^2 = 1$, with $y \geq 0$. *[Hint: This is a flow problem on a curve which is not closed... what's it doing in the Green's theorem worksheet?]*

   Answer: $3\pi/2$

7. Consider the vector field $\mathbf{F}(x, y) = \langle \frac{1}{16}x^2 y + \frac{1}{27}y^3, x \rangle$. Among all positively oriented simple closed curves in the plane, which one maximizes the work by $\mathbf{F}$? That is, find the closed curve C which maximizes the quantity

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

   Answer: The ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ maximizes the work.

Section 16.5: Parametric Surfaces

G5: Parameterization. I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.

V4: Surface Integrals. I can set up and compute surface integrals for scalar and vector valued functions.

Mechanics

1. Consider the surface cut from the parabolic cylinder $y = 4 - x^2$ by the planes $z = 0$, $z = 2$, and $y = 0$. Sketch S and find a parameterization of S.

   Answer: One answer: $\mathbf{r}(u, v) = \langle u, 4 - u^2, v\rangle$, $-2 \le u \le 2, 0 \le v \le 2$.

2. Consider the surface given by $x^2 + y^2 + z^2 = 4$, where $z \ge \sqrt{3}$. Sketch S, parameterize S and find the surface area of S.

   Answer: One answer: $\mathbf{r}(\phi, \theta) = \langle 2\sin(\phi)\cos(\theta), 2\sin(\phi)\sin(\theta), 2\cos(\phi)\rangle$ with $0 \le \phi \le \pi/6$ and $0 \le \theta \le 2\pi$.
   SA: $8\pi\left(1 - \frac{\sqrt{3}}{2}\right)$

3. Find the area of the surface cut from the "nose" of the paraboloid $x = 1 - y^2 - z^2$ by the yz-plane.

   Answer: $\dfrac{\pi}{6}(5^{3/2} - 1)$

4. Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

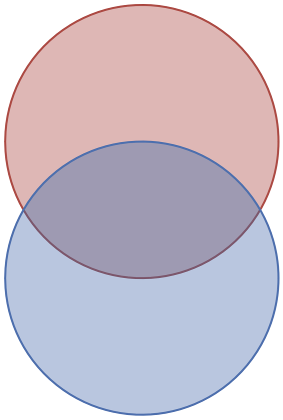
   Answer: $\dfrac{2}{3}\pi(2^{3/2} - 1)$

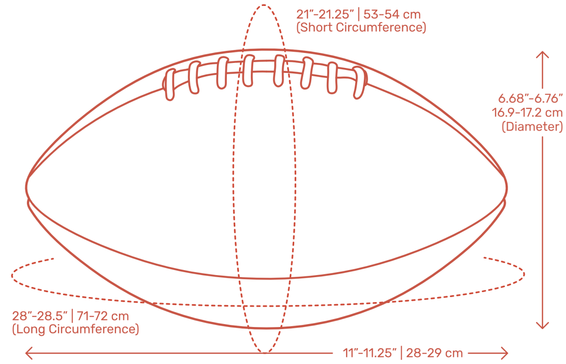
Applications

5. Recall from single-variable calculus that one can form a surface by revolving the graph of a *non-negative* function $y = f(x)$ about the x-axis. Translating this into the language of parameterized surfaces, convince yourself that a parameterization can be given by

   $$r(x, \theta) = (x, f(x)\cos\theta, f(x)\sin\theta)$$

   [Hint: Each x-cross section is a circle with what radius?] A typical (American) football can be roughly modeled as the revolution of the area of intersection of the two circles $x^2 + (y - 1)^2 = 4$ and $x^2 + (y + 1)^2 = 4$ about the x-axis. Set up a surface integral to compute the surface area of a football. [Warning: this is not to scale!!]





Answer:

$$S = \int_0^{2\pi} \int_{-\sqrt{3}}^{\sqrt{3}} \left(2 - \frac{2}{\sqrt{4 - x^2}}\right) \, dx \, d\theta$$

Extensions

6. The tangent plane at a point $P_0 = (f(u_0, v_0), g(u_0, v_0), h(u_0, v_0))$ on a parameterized surface $\mathbf{r}(u, v) = \langle f(u, v), g(u, v), h(u, v)\rangle$ is the plane through P_0 with normal vector equal to $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$.

   Use this to find an equation to the tangent plane of the surface parameterized by $\mathbf{r}(r, \theta) = (r\cos(\theta))\mathbf{i} + (r\sin(\theta))\mathbf{j} + r\mathbf{k}$, with $r \geq 0, 0 \leq \theta \leq 2\pi$ at the point where $(r, \theta) = (2, \pi/4)$.

   What is a Cartesian equation for this surface? Sketch it and the tangent plane.

   Answer: The tangent plane is $-\sqrt{2}(x - \sqrt{2}) - \sqrt{2}(y - \sqrt{2}) + 2(z - 2) = 0$. This is the cone $z = \sqrt{x^2 + y^2}$.

7. The notion of curvature for curves that we explored in Section 13.4 has an extension for parametric surfaces, and is fundamental in a field of math called *differential geometry*. Recall from the above problem that the *unit* normal vector to a parameterized surface $\mathbf{r}(u, v)$ is given by the vector $\mathbf{n} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$. Consider the following scalar quantities for the surface $\mathbf{r}(u, v)$:

$$E = \mathbf{r}_u \cdot \mathbf{r}_u$$
$$F = \mathbf{r}_u \cdot \mathbf{r}_v$$
$$G = \mathbf{r}_v \cdot \mathbf{r}_v$$
$$L = \mathbf{r}_{uu} \cdot \mathbf{n}$$
$$M = \mathbf{r}_{uv} \cdot \mathbf{n}$$
$$N = \mathbf{r}_{vv} \cdot \mathbf{n}$$

   The *Gaussian curvature* of the surface is given by the quantity

$$\kappa = \frac{LN - M^2}{EG - F^2}$$

Verify that the Gaussian curvature at any point on a sphere of radius R is $\frac{1}{R^2}$. Does this result remind you of how the curvature of certain curves are also constant?

Answer: Do the computations to find that $\kappa = 1/R^2$.

Section 16.6: Surface Integrals

V4: Surface Integrals. I can set up and compute surface integrals for scalar and vector valued functions.

Mechanics

1. Integrate $f(x, y, z) = yz$ over the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$.

   Answer: 0

2. Find the flux of the field $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ across the surface S which is the boundary of the solid half-cylinder $0 \leq z \leq \sqrt{1 - y^2}, 0 \leq x \leq 2$, with the outward orientation.

   Answer: $\dfrac{10\pi}{3}$

Applications

3. A fluid has constant density 870 kg/m^3 and flows with velocity $\mathbf{v} = \langle z, y^2, x^2\rangle$, where x, y, z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4, 0 \leq z \leq 1$.

   Answer: 0 kg/s

4. Consider a sheet of metal that has been molded into the shape of a paraboloid $z = x^2+y^2$, with $z \in [0, 1]$. Suppose also that its density is given by $\delta(x, y, z) = \sqrt{1 + 4z}$.

   (a) Compute the total mass of the sheet of metal.

   (b) Suppose you wanted to cut this sheet of metal with a horizontal cut so that the two pieces you obtain have the same mass. Where should you cut? What if you wanted to make a vertical cut?

   Answer:

   (a) 3π

   (b) Cut at height $z = \dfrac{\sqrt{13} - 1}{4}$.
   Any vertical cut through the z-axis will bisect the mass.

5. One of the premier trout fishing destinations in North America is Bow River, located in southern Alberta. There, marine biologists measured that the flow of rainbow trout near the northern tip of the river can be modeled by the vector field $\mathbf{F}(x, y, z) = \langle -x, -y, -z^4\rangle$. Excited by the flourishing aquatic life, you set up a hemispherical net just under the

surface of the water, given by $x^2 + y^2 + z^2 = 4$, with $z \leq 0$. Compute the approximate number of trout swimming **into** your net.

Answer: The trout are actually swimming out of the net at a rate of $\frac{32\pi}{3}$ trout per unit time.

Extensions

6. Give an example of a nontrivial vector field $\mathbf{F}(x, y, z)$ and a nonempty surface S so that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0$$

[Hint: This will be a much easier exercise to do after the material from next week!]

Answer: Many possible answers; one is $\mathbf{F}(x, y, z) = \langle 0, 0, z \rangle$ and S the cylinder $x^2 + y^2 = 1, 0 \leq z \leq 1$.

Section 16.7: Stokes' Theorem

V3: Generalizations of the FTC. I can state and apply Green's Theorem, Stokes' Theorem and the Divergence Theorem to solve problems in two and three dimensions. I can choose which theorem is appropriate for different integrals. I can compute curl and divergence of vector fields.

Mechanics

1. Consider the three surfaces: S the hemisphere given by $x^2+y^2+z^2=4$, for $z \geq 0$, P the portion of a paraboloid given by $z = 4-x^2-y^2$, for $z \geq 0$, and H the hyperboloid $z^2 = x^2+y^2-4$, for $z \leq 0$. Use Stokes' Theorem to explain why, if $\mathbf{F}$ is a smooth vector field in $\mathbb{R}^3$, we must have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_P (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = -\iint_H (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

   Answer: These three surfaces share the same boundary curve; the first two induce the same orientation on that curve, while the third induces the opposite orientation. By Stokes' Theorem, the surface integrals of the curl over these surfaces must be equal to the line integral of $\mathbf{F}$ around their common boundary curve (with appropriate signs for orientation). Thus, the integrals over S and P are equal, and the integral over H is the negative of those.

2. A particle moves along line segments from the origin to the points $(1,0,0)$, $(1,2,1)$, $(0,2,1)$, and back to the origin under the influence of the force field

$$\mathbf{F}(x,y,z) = z^2\mathbf{i} + 2xy\mathbf{j} + 2y^2\mathbf{k}$$

   Find the work done by the field on the particle.

   Answer: 3

3. Compute $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$ where S is the surface $z = e^{-x^2-y^2}$ for $z \geq \frac{1}{e}$, and $\mathbf{F}(x,y,z) = \langle -y, x, 1 \rangle$.

   Answer: 2π

Applications

4. In electrodynamics, *Faraday's Law* relates the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ via the equation $\operatorname{curl}(\mathbf{E}) = \frac{d}{dt}\mathbf{B}$. Integrating both sides with respect to any surface S yields

$$\iint_S \operatorname{curl}(\mathbf{E}) \cdot \mathbf{n}\, d\sigma = -\frac{d}{dt} \iint_S \mathbf{B} \cdot \mathbf{n}\, d\sigma$$

The integral on the right is called the *magnetic flux*. Applying Stokes' theorem to the left gives

$$\int_{\partial S} \mathbf{E} \cdot \mathbf{T}\, ds = -\frac{d}{dt} \iint_S \mathbf{B} \cdot \mathbf{n}\, d\sigma$$

The integral on the left is called the *electric potential*. Interpret the above equation to deduce *Faraday's Law of Induction:*

One may generate an _______________ by first causing a change in _______________.

This is the driving principle underpinning an AC current generator. It uses a system of rotating magnets (or related things) to constantly change the _______________, in order to generate _______________.

Answer: Blanks to be filled in are: "electric potential"; "magnetic flux", "magnetic flux"; "electric potential".

Extensions

5. Use Stokes' Theorem to evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle x^2z^2, y^2z^2, xyz\rangle$ and S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$, oriented upward.

   Answer: 0

6. In Section 16.3, you learned that a curl-free vector field $\mathbf{F}$ on a simply-connected domain is necessarily conservative (i.e., path-independent). In this exercise, you will justify this statement using Stokes' Theorem

   (a) Fix a curl-free vector field $\mathbf{F}$, and also fix any two points $A, B \in \mathbb{R}^3$ and let C_1 and C_2 be curves connecting A to B. For the moment, assume that C_1 and C_2 do not intersect. Use Stokes' Theorem to show that

   $$\int_{C_1 \cup -C_2} \mathbf{F} \cdot \mathbf{T}\, ds = 0$$

   where $-C_2$ represents the path C_2 traced backwards (i.e., from B to A). *[Hint: To apply Stokes' theorem, you need to pick a surface. Draw a picture of the situation to help you find this surface.]*

   (b) You have shown that line integrals along *simple* closed curves vanish. Of course, arbitrary choices of C_2 possibly intersect C_1! Nevertheless, assuming that C_2 can only intersect C_1 finitely many times, show that we still have that

   $$\int_{C_1 \cup -C_2} \mathbf{F} \cdot \mathbf{T}\, ds = 0$$

   [Hint: Draw a picture. Decompose the closed curve $C_1 \cup -C_2$ into a collection of simple closed curves.]

(c) Conclude that

$$\int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds$$

(d) Where did we use the simply-connected assumption?

Answer:

Section 16.8: The Divergence Theorem

V3: Generalizations of the FTC. I can state and apply Green's Theorem, Stokes' Theorem and the Divergence Theorem to solve problems in two and three dimensions. I can choose which theorem is appropriate for different integrals. I can compute curl and divergence of vector fields.

Mechanics

1. Consider the vector field $\mathbf{F}(x, y, z) = (z + y, x - z, x - y)$. Compute the flux out of the "closed" cone $z^2 = x^2 + y^2$ with $z \in [0, 1]$, capped off with a disc in the $z = 1$ plane in two ways: first directly using parameterizations, then by invoking the divergence theorem. Finally, verify that your answers agree.

   Answer: 0

2. Use the divergence theorem to compute the *inward* flux of the field

$$\mathbf{F}(x, y, z) = (2xz, 1 - 4xy^2, 2z - z^2)$$

   into the surface S bound by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane.

   Answer: $-\pi$

Applications

3. Suppose that a velocity field for a fluid is given by $\mathbf{F}(x, y, z) = (x/y, x/z, -z/y)$. Compute the flow rate of the fluid across the cube $[-3, 4] \times [1, 2] \times [2, 8]$.

   Note: Fluids with a velocity field $\mathbf{F}$ satisfying $\operatorname{div} \mathbf{F} = 0$ everywhere are called *incompressible*, since they are not allowed to "expand" or "shrink" at any point. Most liquids are only slightly compressible (i.e., $\operatorname{div} \mathbf{F} \approx 0$), so the theory of incompressible fluids is relatively robust for most applications.

   Answer: 0

Extensions

4. Consider the vector field $\mathbf{F}(x, y, z) = \left(\frac{1}{z}\cos(xz), y\sin(xz), z\right)$. Compute the flux through the lateral walls (i.e., excluding the top and bottom) of the cylinder $x^2 + y^2 = 1$, with $1 \leq z \leq 2$. *[Hint: This is a problem about a surface which isn't closed... why is it in the divergence theorem worksheet?]*

   Answer: 0

5. Consider the radial vector field

$$\mathbf{F}(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}\langle x, y, z\rangle$$

(a) Consider the region between two concentric spheres: $V = \{1 \leq x^2 + y^2 + z^2 \leq R^2\} \subset \mathbb{R}^3$, where R is any number bigger than 1. Calculate the *net* flux across the boundary of this region.

(b) Compute the outward flux of $\mathbf{F}$ over *any sphere* of radius R centered at the origin: $x^2 + y^2 + z^2 = R^2$.

(c) Generalize your answer to the previous problem:

The outward flux of $\mathbf{F}$ over *any smooth surface* containing the origin equals ________________. *[Note: this is essentially Gauss' Law for a point charge!]*

Answer:

(a) The field has zero divergence everywhere except at the origin, so by the divergence theorem the net flux across the boundary is 0.

(b) 4π

(c) 4π

Review for Exam 3

I1: Double & Triple Integrals. I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.

I2: Iterated Integrals. I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.

I3: Change of Variables. I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.

A1: Interpreting Derivatives. I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.

A2: Integral Applications. I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.

V1: Line Integrals. I can set up and evaluate scalar and vector field line integrals in two and three dimensions.

V2: Conservative Vector Fields. I can test for conservative vector fields and find potential functions. I can state and apply the Fundamental Theorem of Line Integrals.

V3: Generalizations of the FTC. I can state and apply Green's Theorem, Stokes' Theorem and the Divergence Theorem to solve problems in two and three dimensions. I can choose which theorem is appropriate for different integrals. I can compute curl and divergence of vector fields.

V4: Surface Integrals. I can set up and compute surface integrals for scalar and vector valued functions.

1. Set up an iterated integral in spherical coordinates for $\iiint_E z^2\ dV$ where E is the region between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=25$ and inside $z=-\sqrt{\frac{1}{3}(x^2+y^2)}$.

   Answer: $\displaystyle\int_0^{2\pi}\int_{2\pi/3}^{\pi}\int_2^5 \rho^4\cos^2(\phi)\sin(\phi)d\rho\ d\phi\ d\theta$

2. Set up an integral that computes the volume of the solid which is bounded above by the cylinder $z=4-x^2$, on the sides by the cylinder $x^2+y^2=4$, and below by the xy-plane using

   (a) Cartesian coordinates

   (b) cylindrical coordinates

   Which integral would you rather evaluate and why?

   Answer:

(a) $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2} dz\ dy\ dx$

(b) $\int_{0}^{2\pi}\int_{0}^{2}\int_{0}^{4-r^2\cos^2(\theta)} r\ dz\ dr\ d\theta$

3. Find an integral that computes the mass of the wire which lies along the curve $y^2 = x^3$ from $(0,0)$ to $(1,-1)$ and has density function $\rho(x,y) = 2xy^2$.

   Answer: One solution: $\int_0^1 2(t)(-t^{3/2})^2\sqrt{1+(\frac{3}{2}\sqrt{t})^2}\ dt.$

4. Show that the field $\mathbf{F} = 2x\mathbf{i} - y^2\mathbf{j} - \frac{4}{1+z^2}\mathbf{k}$ is conservative, find a potential function, and use it to compute the integral

$$\int_C 2x\ dx - y^2\ dy - \frac{4}{1+z^2}\ dz$$

   where C is any path from $(0,0,0)$ to $(3,3,1)$.

   Answer: $f(x,y,z) = x^2 - \frac{1}{3}y^3 - 4\arctan(z)$.

   Integral: $-\pi$.

5. Compute $\int_C (6y+x)\ dx + (y+2x)\ dy$ using any method, where C is the circle $(x-2)^2 + (y-3)^2 = 4$.

   Answer: $-4\cdot\pi(2)^2$

6. Find the flux of the field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$ through the portion of the sphere $x^2+y^2+z^2 = a^2$ in the first octant in the direction away from the origin.

   Answer: $\pi a^2/4$

7. Use Stokes' theorem to show that the circulation of the field $\mathbf{F} = \langle 2x, 2y, 2z\rangle$ around the boundary curve C of **any** smooth orientable surface S in $\mathbb{R}^3$ is 0.

   Answer: $\nabla\times\mathbf{F} = \langle 0-0, -(0-0), 0-0\rangle$

8. Find the outward flux of $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\sqrt{x^2+y^2+z^2}$ through the boundary S of the "thick sphere" D given by the points satisfying $1 \le x^2+y^2+z^2 \le 4$.

   Answer: 12π

Final Exam Review

1. Find the equation of the plane through $(1, -1, 3)$ parallel to the plane $3x + y + z = 7$. Is there a unique plane through $(1, -1, 3)$ which is perpendicular to the plane $3x+y+z = 7$. Explain why or why not.

   Answer: $3x + y + z = 5$. There is not a unique plane because there is not a unique normal direction perpendicular to $\langle 3, 1, 1\rangle$.

2. Find the point on the curve

   $$\mathbf{r}(t) = (5\sin(t))\mathbf{i} + (5\cos(t))\mathbf{j} + 12t\mathbf{k}$$

   at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing parameter t.

   Answer: $(0, 5, 24\pi)$

3. Find the domain and range of $f(x, y) = \sqrt{x^2 - y}$ and identify its level curves.

   Answer: Domain $\{(x, y) \mid y \leq x^2\}$ Range $[0, \infty)$ Level curves are the parabolas $y = x^2 - c^2$ for all $c \geq 0$.

4. Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^2 - y}$ or show this limit does not exist.

   Answer: The limit does not exist

5. Let $f(x, y, z) = xy + 2yz - 3xz$. Find the tangent plane to the surface $f(x, y, z) = 1$ at $(1, 1, 0)$ and the linearization $L(x, y, z)$ at $(1, 1, 0)$.

   Answer: Tangent plane: $(x - 1) + (y - 1) - z = 0$ Linearization: $L(x, y, z) = 1 + (x - 1) + (y - 1) - z$

6. At the point $(1, 2)$, the function $f(x, y)$ has a derivative of 2 in the direction toward $(2, 2)$ and a derivative of -2 in the direction toward $(1, 1)$. Find $\nabla f(1, 2)$ and the derivative of f at $(1, 2)$ in the direction toward the point $(4, 6)$.

   Answer: $\nabla f(1, 2) = \langle 2, 2\rangle$, $Df_{\mathbf{u}}(1, 2) = 14/5$

7. Find the value of the derivative of $f(x, y, z) = xy+yz+xz$ with respect to t on the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), \cos(2t)\rangle$ at $t = 1$.

   Answer: $-\sin^2(1)-\sin(1)\cos(1)+\cos^2(1)+\cos(1)\cos(2)-2\cos(1)\sin(2)-2\sin(1)\sin(2)$

8. Find the local minima, local maxima, and saddle points of the function $f(x, y) = x^4 - 8x^2 + 3y^2 - 6y$.

   Answer: $(0, 1)$ saddle point, $(2, 1), (-2, 1)$ local minimum

9. Find the extreme values of $f(x, y) = 4xy - x^4 - y^4 + 16$ on the triangular region bounded below by the line $y = -2$, above by the line $y = x$, and on the right by the line $x = 2$.

   Answer: Min of -32 at $(2, -2)$ and max of 18 at $(1, 1)$

10. Find the extreme values of $f(x, y) = xy$ on the circle $x^2 + y^2 = 1$.

    Answer: Min of $-1/2$ at $(1/\pm\sqrt{2}, 1/\pm\sqrt{2})$ and max of $1/2$ at $(1/\pm\sqrt{2}, 1/\mp\sqrt{2})$.

11. Sketch the region of integration and reverse the order of integration for the integral

$$\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\ dx\ dy.$$

Answer: $\displaystyle\int_{-3}^{3} \int_0^{\sqrt{9/4-x^2/4}} y\ dy\ dx$

12. Evaluate the integral

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2\ dx\ dy}{(1+x^2+y^2)^2}$$

    by changing to polar coordinates.

    Answer: π

13. Find the centroid of the region bounded by the lines $x = 2, y = 2$, and the hyperbola $xy = 2$ in the xy-plane.

    Answer: $(\bar{x}, \bar{y}) = \left(\frac{1}{2 - \ln(4)}, \frac{1}{2 - \ln(4)}\right)$

14. Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

    Answer: $V = \frac{\pi}{6}(8\sqrt{2} - 7)$

15. Use the transformation $u = 3x + 2y, v = x + 4y$ to evaluate the integral

$$\iint_R (3x^2 + 14xy + 8y^2)\ dx\ dy$$

    where R is the region in the first quadrant bounded by the lines $y = (-3/2)x + 1, y = (-3/2)x + 3, y = -(1/4)x$, and $y = -(1/4)x + 1)$.

    Answer: $\frac{64}{5}$

16. Evaluate the integral $\int_C y^2\ dx + x^2\ dy$ where C is the circle $x^2 + y^2 = 4$.

    Answer: 0

17. Find the outward flux of $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ across the boundary of the cube cut from the first octant by the planes $x = 1, y = 1, z = 1$.

    Answer: 3

18. Find the work done by $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)^{3/2}}$ over the plane curve $\mathbf{r}(t) = \langle e^t \cos(t), e^t \sin(t)\rangle$ from the point $(1, 0)$ to the point $(e^{2\pi}, 0)$.

    Answer: $1 - e^{-2\pi}$

19. Find the flux of the field $\mathbf{F} = \langle 2xy + x, xy - y\rangle$ outward across the boundary of the square bounded by $x = 0, x = 1, y = 0, x = 1$.

    Answer: $\frac{3}{2}$

20. Find the flux of $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$ across the upper cap cut from the sphere $x^2 + y^2 + z^2 = 25$ by the plane $z = 3$, oriented away from the xy-plane.

    Answer: $\frac{208\pi}{5}$