MATH 2551 C/HP Notes

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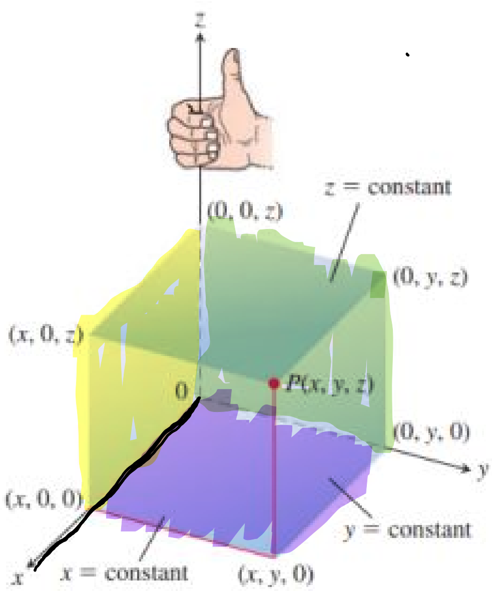
Spring 2025

Day 1 - Course Introduction and Cross Products

Pre-Lecture

12.1: Three-Dimensional Coordinates



- $\mathbb{R}^3$: (x,y,z)
- Right-handed system
- Coordinate planes
  - $x=0$ (the yz-plane)
  - $y=0$ (the xz-plane)
  - $z=0$ (the xy-plane)
- Eight octants
  - 1^{st} octant has $x,y,z \geq 0$

Day 1 - Lecture

MATH 2551 C/HP - Dr. H

Daily Announcements & Reminders:

- Introduce yourself to neighbors
- Make sure you can access Webwork & Ed Discussion
- Quiz 0 (practice only) tomorrow in studio
  - linear & single-var. calc topics
- Open Ed Discussion and answer warm-up poll

Goals for Today:

Sections 12.1, 12.3, 12.4

- Set classroom norms
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Class Values/Norms:

- Mistakes are a learning opportunity
- Mathematics is collaborative
- Make sure everyone is included
- Criticize ideas, not people
- Be respectful of everyone
- Stay organized
- Ask questions

Introduction to the Course

Big Idea: Extend differential & integral calculus.

What are some key ideas from these two courses?

Differential Calculus

Limits & continuity

Derivatives

Optimization

Visualizing graphs

Fundamental Theorem of Calculus

Integral Calculus

Integration Techniques

Polar/Parametric

Area under curves

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

$$f(x) = e^{\cos(3x)} + \ln(\sin(x))$$

Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

$$\mathbb{R} \to \mathbb{R}^3 : \quad \vec{r}(t) = \begin{bmatrix} t+1 \\ \cos(t) \\ t^2 + e^t \end{bmatrix}$$

$$\mathbb{R}^3 \to \mathbb{R} : \quad f(x,y,z) = x^2 y + \cos(z)$$

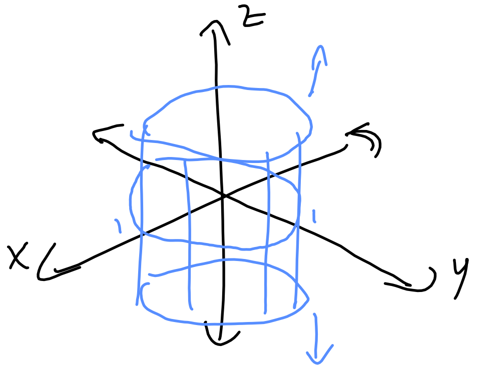
$$\mathbb{R}^2 \to \mathbb{R}^3 : \quad \vec{r}(s,t) = \begin{bmatrix} s+t \\ s^2 t \\ s+3t+\ln(t) \end{bmatrix}$$

linear algebra bridges the gap

Example 1. What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation

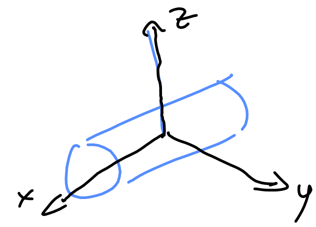
$$x^2 + y^2 = 1?$$

• circle (in xy-plane) $\longrightarrow$ • cylinder (z is free)

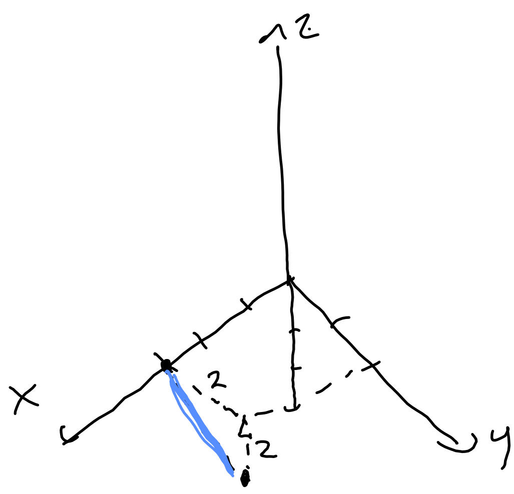


To get cylinder along x-axis

$y^2 + z^2 = 1$



Example 2. What is the distance from $(3, 2, -2)$ to the xy-plane? What is the distance to the x-axis?



dist from $(3,2,-2)$ to $z=0$: $|-2| = 2$

dist to x-axis is $\|(3,2,-2) - (3,0,0)\|$

$= \sqrt{0^2 + 2^2 + (-2)^2}$

$= \sqrt{8} = 2\sqrt{2}$

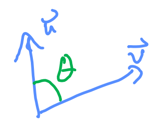
Section 12.3/4: Dot & Cross Products

Definition 3. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \dots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \dots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n = \vec{u}^T \vec{u} = [u_1 \dots u_n]\begin{bmatrix} u_1 \\ \vdots \\ v_n \end{bmatrix}$$

This product tells us about angle between vectors.

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$$



$u \cdot \vec{v} > 0 \iff \theta$ is acute

$u \cdot \vec{v} < 0 \iff \theta$ is obtuse

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

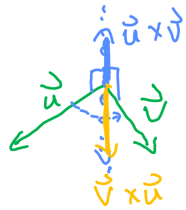
In particular, two vectors are **orthogonal** if and only if their dot product is 0.

Example 4. Are $\mathbf{u} = \langle 1, 1, 4 \rangle$ and $\mathbf{v} = \langle -3, -1, 1 \rangle$ orthogonal?

$$\vec{u} \cdot \vec{v} = (1)(-3) + (1)(-1) + 4(1) = -3 - 1 + 4 = 0$$

so $\vec{u}$ & $\vec{v}$ are orthogonal!

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.



$\vec{i} \times \vec{j} = \vec{k}$
$\vec{j} \times \vec{k} = \vec{i}$
$\vec{k} \times \vec{i} = \vec{j}$

1. Right Handed

$\Rightarrow$ "antisymmetric"
$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

2. Algebraically Product

$\rightarrow$ $\cancel{\vec{u} \times (\vec{v} \times \vec{w}) =} (\vec{u} \times \vec{v}) \times \vec{w}$

$\rightarrow$ $c(\vec{u} \times \vec{w}) = (c\vec{u}) \times \vec{w} = \vec{u} \times (c\vec{w})$

$\rightarrow$ $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{v} \times \vec{w})$

not $\vec{0}$

$\vec{u} \times (\vec{u} \times \vec{v}) \neq (\vec{u} \times \vec{u}) \times \vec{v} \leftarrow \vec{0}$

$\vec{i} = \langle 1, 0, 0 \rangle$
$\vec{j} = \langle 0, 1, 0 \rangle$
$\vec{k} = \langle 0, 0, 1 \rangle$

Definition 5. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- output is vector

- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| |\sin(\theta)|$

Example 6. Find $\langle 1, 2, 0\rangle \times \langle 3, -1, 0\rangle$.

$$\langle 1,2,0\rangle \times \langle 3,-1,0\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \vec{k}$$

$$= (0-0)\vec{i} - (0-0)\vec{j} + (-1-6)\vec{k}$$

$$= -7\vec{k}$$

$$= \langle 0, 0, -7\rangle$$

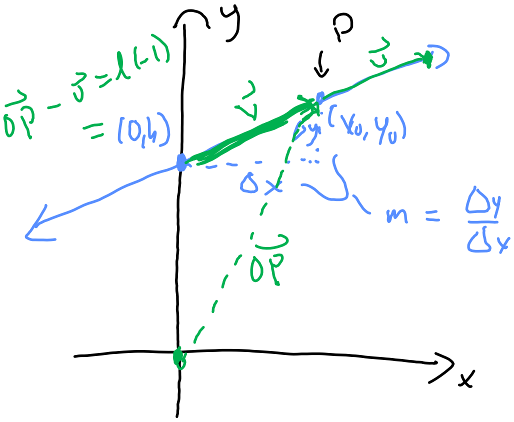
Day 2 - Lines, Planes, and Quadrics

Pre-Lecture

12.5: Lines

Lines in $\mathbb{R}^2$, a new perspective: $y = mx + b$ $\qquad y - y_0 = m(x - x_0)$

point $(0, b)$ $\qquad (x_0, y_0)$



direction : $m = \frac{\Delta y}{\Delta x}$

In $\mathbb{R}^3$: $\frac{\Delta y}{\Delta x}$ & $\frac{\Delta z}{\Delta x}$ & $\frac{\Delta z}{\Delta y}$ & ...

A line is all points of the form

$$\ell(t) = \vec{OP} + \vec{v} \cdot t \quad \text{where}$$

$\vec{P} = (x_0, y_0, z_0)$ on the line & $\vec{v}$ is a direction for the line.

Example 7. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.

Need: • point on the line : $P = (1, 0, 2)$

• direction vector :

$$\vec{v} = \vec{PQ} = \langle -2-1, 1-0, 1-2 \rangle$$
$$= \langle -3, 1, -1 \rangle$$

So a vector equation of the line is

$$\ell(t) = \langle -3, 1, -1 \rangle t + \langle 1, 0, 2 \rangle \qquad -\infty \le t \le \infty$$

or

$$\ell(t) = \langle -3t + 1, t, -t + 2 \rangle$$

Day 2 Lecture

Daily Announcements & Reminders:

- Solutions to Quiz 0 posted
- HW 12.2/12.3 due tonight at 10pm
- Next week attendance starts in studio
- Do warmup on Ed Discussion

Goals for Today:

Sections 12.5-12.6

- Apply the cross product to solve problems
- Learn the equations that describe lines, planes, and quadric surfaces in $\mathbb{R}^3$
- Solve problems involving the equations of lines and planes
- Sketch quadric surfaces in $\mathbb{R}^3$

Example 8. Find a set of parametric equations for the line through the point $(1, 10, 100)$ which is parallel to the line with vector equation

$$\mathbf{r}(t) = \langle 1, 4, -3 \rangle t + \langle 0, -1, 1 \rangle$$

$$\boxed{\begin{aligned} x(t) &= 1 + t \\ y(t) &= 10 + 4t \\ z(t) &= 100 - 3t \end{aligned}}$$

parametric equations

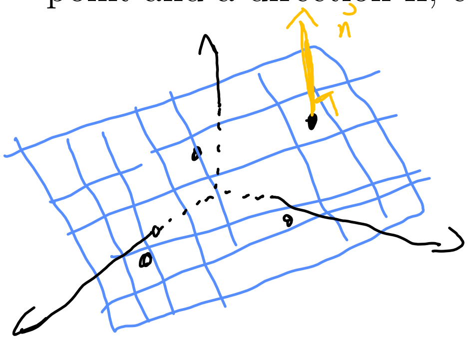
symmetric equations

$$x - 1 = \frac{y-10}{4} = \frac{z-100}{-3}$$

Section 12.5 Planes

Planes in $\mathbb{R}^3$

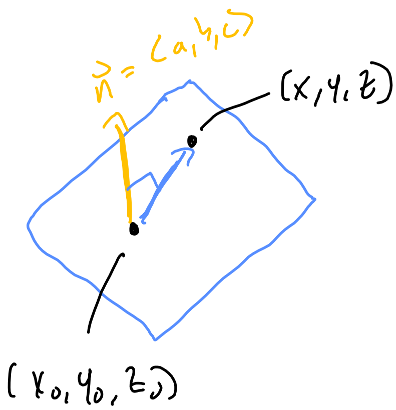
Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction $\mathbf{n}$, called the *normal vector*.



- 3 points
- 2 directions in plane & 1 point
- 1 direction $\perp$ plane & 1 point

Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

$$ax + by + cz = d$$



$$\langle x-x_0,\ y-y_0,\ z-z_0\rangle \cdot \langle a,b,c\rangle \overset{\perp}{=} 0$$

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

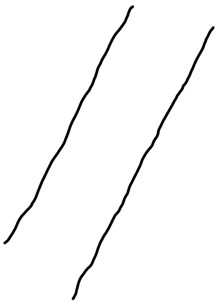
Std eqn of plane w/ $\vec{n} = \langle a,b,c\rangle$ through point (x_0,y_0,z_0)

Plane which is orthogonal to the line $\underbrace{\langle 1,4,-3\rangle}_{\text{parallel to } \vec{n}} t + \langle 1,10,100\rangle$ through the point $(1,1,1)$

is $\boxed{1(x-1) + 4(y-1) + (-3)(z-1) = 0}$

In $\mathbb{R}^3$, a pair of lines can be related in three ways:

parallel

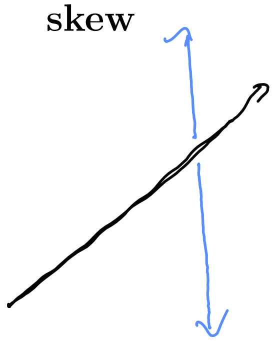


same direction

no intersection

parallel direction vectors

skew



not same direction

but no intersection

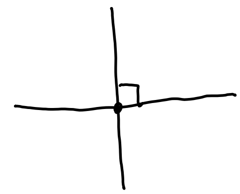
Q: How do we know if they are skew?

set up system!

$$\begin{aligned} x:&\quad t = 2s \\ y:&\quad t = -s \\ z:&\quad t+1 = -s \end{aligned}$$

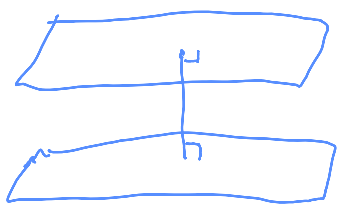
intersecting

↳ orthogonal

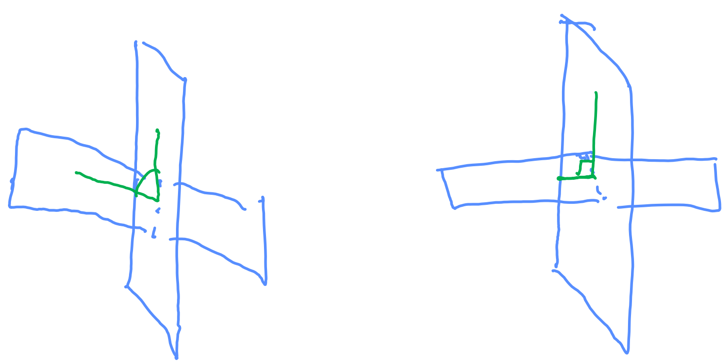


On the other hand, a pair of planes can be related in just two ways:

parallel



intersecting



orthogonal

$\iff$

$\vec{n}_1 \perp \vec{n}_2$

$\iff$

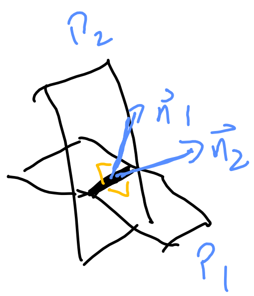
$\vec{n}_1 \cdot \vec{n}_2 = 0$

Example 9. Consider the planes $\overset{P_1}{y - z = -2}$ and $\overset{P_2}{x - y = 0}$. Show that the planes intersect and find an equation for the line passing through the point $\underline{P = (-8, 0, 2)}$ which is parallel to the line of intersection of the planes.

1) Intersect? $\vec{n}_1 = \langle 0, 1, -1 \rangle$, $\vec{n}_2 = \langle 1, -1, 0 \rangle$ not parallel, so P_1 is not $\parallel$ to P_2 and so they intersect

OR give (x, y, z) solving both eqns!
$(0, 0, 2)$ is on both

2) Line equation



- direction for line of intersection is

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \langle 0 - 1, -(0+1), (0-1) \rangle$$
$$= \langle -1, -1, -1 \rangle$$

- point is $(-8, 0, 2)$

$$\boxed{\ell(t) = \langle -1, -1, -1 \rangle t + \langle -8, 0, 2 \rangle \qquad t \in \mathbb{R}}$$

Section 12.6 Quadric Surfaces

Definition 10. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

You know several examples already:

- $x^2+y^2=1$ (circular cylinder)
- $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$ (sphere, radius $=r$, center (x_0,y_0,z_0))

The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections.

Example 11. Use cross-sections to sketch and identify the quadric surface $x = z^2+y^2$.

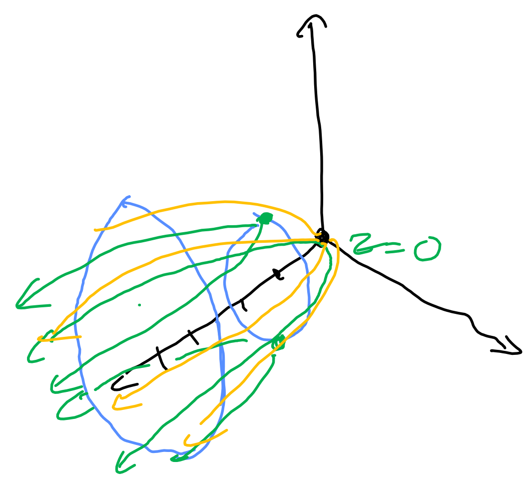
Fix values of x:

$x=0$: $0=z^2+y^2$

$x=1$: $1=z^2+y^2$

$x=4$: $4=z^2+y^2$

$x=-1$: $-1=z^2+y^2$



Fix value of z:

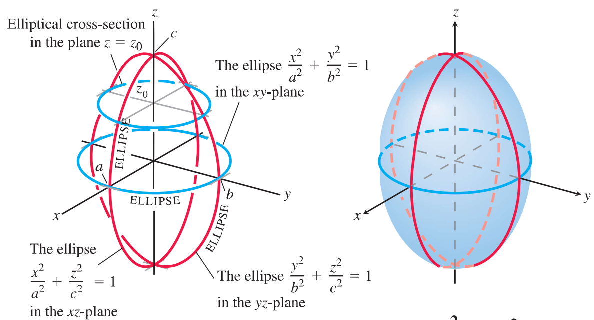
$z=0$: $x=y^2$

$z=1$: $x=1+y^2$

$z=-1$: $x=1+y^2$

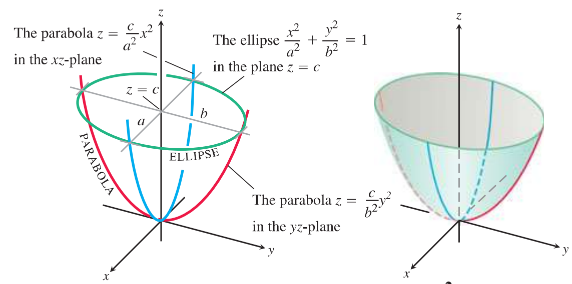
elliptical paraboloid

TABLE 12.1 Graphs of Quadric Surfaces



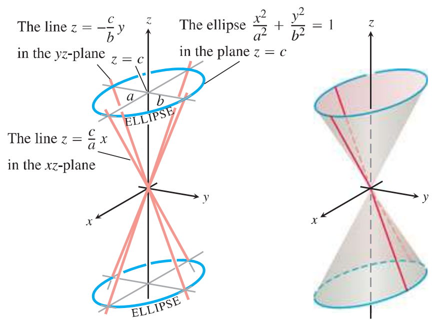
ELLIPSOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

has constant, no linear, all same sign



ELLIPTICAL PARABOLOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$

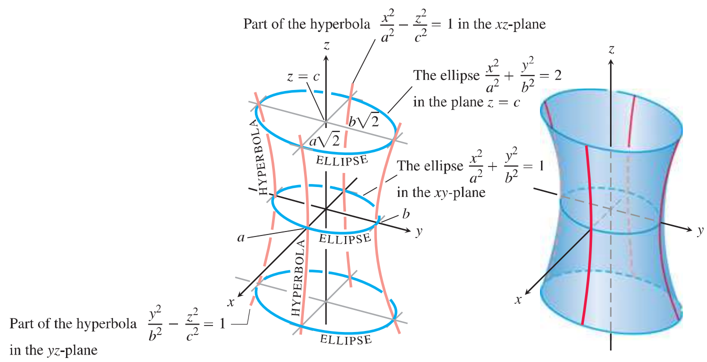
has linear term, same sign quadratics



ELLIPTICAL CONE $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

no constant, no linear term, one sign different

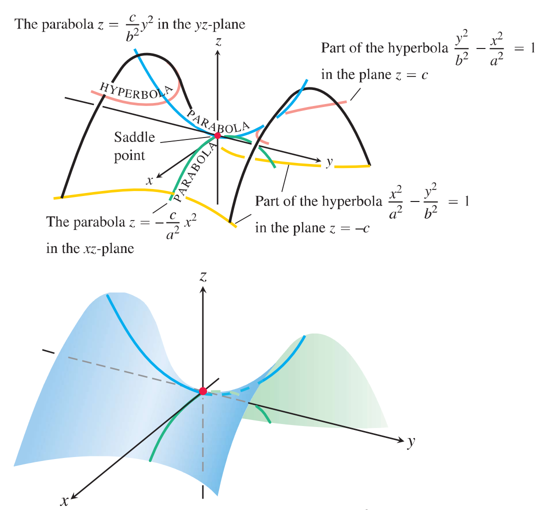
half cone: $z = \sqrt{x^2+y^2}$ or $z = -\sqrt{x^2+y^2}$

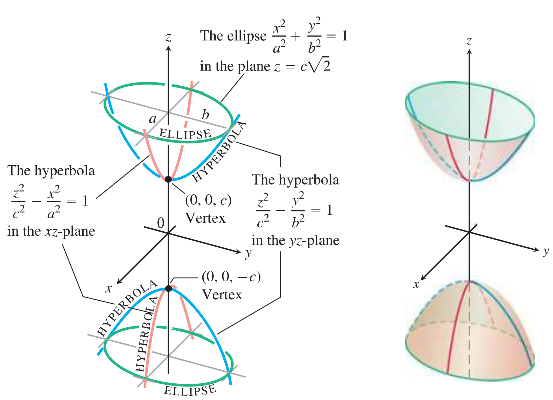


HYPERBOLOID OF ONE SHEET $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

have constant, no linear, 1 neg sign

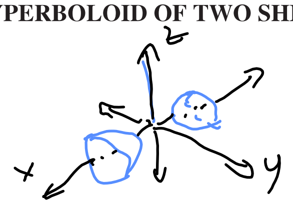
(last warmup rotated along z-axis





HYPERBOLOID OF TWO SHEETS $\quad \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

HYPERBOLIC PARABOLOID $\quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$



have constant,
no linear,
2 neg sign

has linear, diff. signs

Day 3 - Vector-Valued Functions & Calculus

Pre-Lecture

Section 13.1: Vector-Valued Functions

Last week, we used functions like

$$\underline{\ell(t) = \langle 2t+1, 3-t, t-1 \rangle, \quad -\infty \leq t \leq \infty}$$

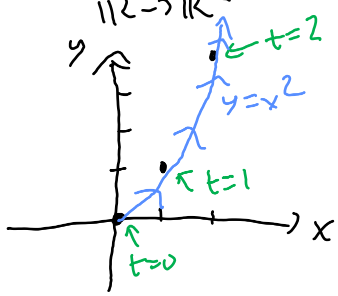
to produce lines in $\mathbb{R}^2$ and $\mathbb{R}^3$.

This is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

What happens when we change the component functions to be non-linear?

1) Let $\vec{r}(t) = \langle t, t^2 \rangle$, $t \geq 0$.

$\mathbb{R} \to \mathbb{R}^2$



$\vec{r}(0) = \langle 0, 0 \rangle$

$\vec{r}(1) = \langle 1, 1 \rangle$

$\vec{r}(2) = \langle 2, 4 \rangle$

$y(t) = [x(t)]^2$

with $x \geq 0$

3) In 3D: $\vec{r}(t) = \langle \cos(t), \sin(t), 1-\cos(t)-\sin(t) \rangle$

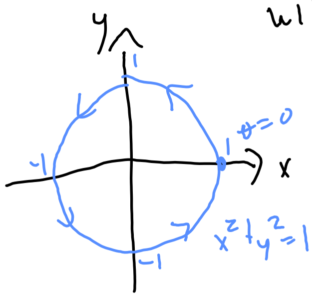
$0 \leq t \leq 2\pi$.

Same relation for x, y as #2!

$x(t)^2 + y(t)^2 = 1$

Also: $z(t) = 1 - x(t) - y(t)$

2) Find $\vec{r}(t) = \langle x(t), y(t) \rangle$ with this graph?

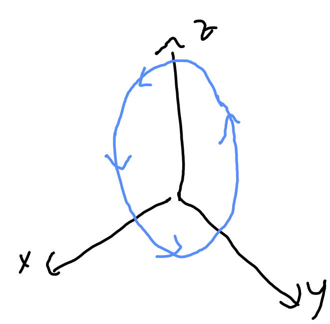


$\cos^2(\theta) + \sin^2(\theta) = 1$

so

let $\vec{r}(\theta) = \langle \cos(\theta), \sin(\theta) \rangle$

$0 \leq \theta \leq 2\pi$



Given a fixed curve C in space, producing a vector-valued function **r** whose graph is C is called <u>parameterizing</u> the curve C, and **r** is called a <u>parametrization</u> of C.

Day 3 Lecture

Daily Announcements & Reminders:

- HW 12.1, 12.4 due tonight
- Quiz 1 in studio tomorrow; 12.1-12.5
  - LO: G1
- Do warmup poll on Ed →

Goals for Today:

Sections 13.1, 13.2

- Introduce vector-valued functions
- Plot vector-valued functions and construct them from a graph
- Compute limits, derivatives, and tangent lines for vector-valued functions
- Compute integrals of vector-valued functions and solve initial value problems

Example 12. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?

- coil (1 rotation vs 2 rotation) (same)
  shorter vs taller ← coils closer together — yes!
- line (second steeper)

helix

- $\vec{r}_2(t)$ moves twice as fast as $\vec{r}_1(t)$
- with domains $\mathbb{R}$ instead these give the same curve

Domain matters

Check your intuition

Section 13.1: Calculus of Vector-Valued Functions

Unifying theme: Do what you already know, componentwise.

This works with limits:

Example 13. Compute $\lim_{t\to e}\langle t^2, 2, \ln(t)\rangle$.

$$= \left\langle \lim_{t\to e} t^2, \lim_{t\to e} 2, \lim_{t\to e} \ln(t) \right\rangle$$

$$= \langle e^2, 2, 1 \rangle$$

And with continuity: $\lim_{x\to a} f(x) = f(a)$ $\qquad \lim_{t\to a} \vec{r}(t) = \vec{r}(a)$

Example 14. Determine where the function $\mathbf{r}(t) = t\mathbf{i} - \dfrac{1}{t^2-4}\mathbf{j} + \sin(t)\mathbf{k}$ is continuous.

- $x(t) = t$ is continuous on $\mathbb{R}$
- $y(t) = -\dfrac{1}{t^2-4}$ is cts on $\mathbb{R}\setminus\{-2,2\} = (-\infty,-2)\cup(-2,2)\cup(2,\infty)$

  "all real #s except -2 and 2"
- $z(t) = \sin(t)$ is cts on $\mathbb{R}$

So $\vec{r}(t)$ is cts on $\mathbb{R} \cap (\mathbb{R}\setminus\{-2,2\}) \cap (\mathbb{R})$

$$= \boxed{\mathbb{R}\setminus\{-2,2\}}$$

And with derivatives:

Example 15. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$, find $\mathbf{r}'(t)$.

$\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle$
$= \langle 2 - t, 1\rangle$

$\vec{r}\,''(t) = \langle -1, 0\rangle$

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives velocity
- $\|\mathbf{r}'(t)\|$ gives speed
- $\mathbf{r}''(t)$ gives acceleration

• $\vec{r}\,'(t_0)$ is tangent to curve at $t = t_0$

Let's see this graphically

Example 16. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

In S.V.C: tangent line to $y = f(x)$ at $x = a$ is $y = f(a) + f'(a)(x - a)$

$\vec{r}(a)$ $\vec{r}\,'(a)$

Tangent line to $\vec{r}(t)$ at $t = a$
is $\ell(t) = \vec{r}\,'(a) \cdot t + \vec{r}(a)$
$\vec{r}\,'(a)(t - a) + \vec{r}(a)$

$\ell(t) = \langle 2-2, 1\rangle \cdot t + \langle 3, 1\rangle$
$= \langle 0, 1\rangle t + \langle 3, 1\rangle$

And with integrals:

Example 17. Find $\displaystyle\int_0^1 \langle t, e^{2t}, \sec^2(t)\rangle \, dt$.

$\int_a^b \vec{r}\,'(t)\,dt$ = displacement from t=a to t=b

$= \langle \int_0^1 t\,dt, \int_0^1 e^{2t}\,dt, \int_0^1 \sec^2(t)\,dt \rangle$

$= \langle \frac{1}{2}t^2 \big|_0^1, \frac{1}{2}e^{2t}\big|_0^1, \tan(t)\big|_0^1 \rangle$

$= \langle \frac{1}{2}, \frac{1}{2}(e^2-1), \tan(1) \rangle$

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 18. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$t \geq 0$

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t}\rangle \; m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0, 0, 0\rangle$, use calculus to reconstruct his flight path.

Start here next time

1) Find antiderivative (with + C)

$\vec{r}(t) = \langle 100\cos(2t) + C_1, 200\sin(t) + C_2,$
$400t - 400\ln|1+t| + C_3 \rangle$

$= \langle 100\cos(2t), 200\sin(t), 400(t - \ln|1+t|) \rangle + \vec{C}$

2) Apply I.C. to find C

$\vec{r}(0) = \langle 0,0,0 \rangle$

$= \langle 100, 0, 0\rangle + \vec{C}$

$\rightarrow \vec{C} = \langle -100, 0, 0 \rangle$

$\rightarrow$

$\boxed{\vec{r}(t) = \langle 100(\cos(2t)-1), 200\sin(t), 400(t - \ln(1+t)) \rangle}$

Day 4 - Geometry of Curves

Pre-Lecture

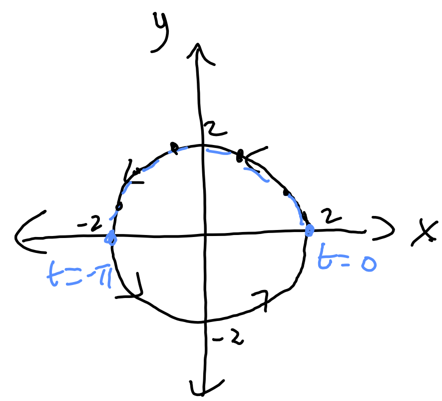
Section 13.3: Arc Length

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Our next goal is to be able to measure distance traveled or arc length.

Motivating problem: Suppose the position of a fly at time t is

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle,$$



where $0 \le t \le 2\pi$.

How far does the fly travel from $t = 0$ to $t = \pi$?

Algebra: dist = rate · time

might change!

rate = $\|\vec{r}\,'(t)\|$ = speed

dist $\approx \sum_{i=1}^{n} \|\vec{r}\,'(t_i)\| \Delta t \longrightarrow$ dist $= \int_a^b \|\vec{r}\,'(t)\|\, dt$

Here: dist $= \int_0^{\pi} \|\langle -2\sin(t), 2\cos(t)\rangle\|\, dt$

$= \int_0^{\pi} \sqrt{4\sin^2(t) + 4\cos^2(t)}\, dt$

$= \int_0^{\pi} 2\, dt$

$= \boxed{2\pi}$

length of a semicircle of radius 2 is $\frac{1}{2}(2\pi(2)) = 2\pi$ ✓

$\vec{r}\,'(t) \neq \vec{0}$ or undefined

Definition 19. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ from $t=a$ to $t=b$ that is traced out exactly once is

$$L = \int_a^b \|\vec{r}\,'(t)\|\, dt$$

Day 4 Lecture

Daily Announcements & Reminders:

- HW 12.5 due tonight
- No studio on Monday - holiday
- Do warmup on Ed ⟶

Goals for Today:

Sections 13.3, 13.4

- Compute arc lengths of curves using parameterizations
- Define and compute arc-length parameterizations
- Define, interpret, and compute the curvature of a curve
- Compute the unit tangent and principal unit normal vectors of a curve

Example 20. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1, 1, 1)$ to the point $(2, 4, 8)$.

$L = \int_a^b \|\vec{r}\,'(t)\|\, dt$

1) Endpoints? $t = a$ & $t = b$

At $t = 1$: $\vec{r}(1) = \langle 1, 1, 1 \rangle$

$t = 2$ $\vec{r}(2) = \langle 2, 4, 8 \rangle$

2) Speed? $\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{1 + (2t)^2 + (3t^2)^2}$

$= \sqrt{1 + 4t^2 + 9t^4}$

3) Substitute

$$\boxed{L = \int_1^2 \sqrt{1 + 4t^2 + 9t^4}\, dt}$$

Example 21. Find the distance traveled by a particle moving along the path

$$\mathbf{r}(t) = \langle \ln(t), \sqrt{2}t, \frac{1}{2}t^2 \rangle, \qquad t > 0$$

from $t = 1$ to $t = 2$.

Speed? $\vec{r}\,'(t) = \langle \frac{1}{t}, \sqrt{2}, t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{\frac{1}{t^2} + 2 + t^2}$

$$\text{dist} = \int_1^2 \sqrt{\frac{1}{t^2} + 2 + t^2}\, dt$$
$$= \int_1^2 \sqrt{\left(\frac{1}{t} + t\right)^2}\, dt$$
$$= \int_1^2 \frac{1}{t} + t\, dt$$
$$= \ln(t) + \frac{1}{2}t^2 \Big|_1^2$$
$$= (\ln 2 + 2) - (\ln(1) + \tfrac{1}{2}) = \boxed{\ln 2 + \frac{3}{2}}$$

When is $\vec{r}(t) = \langle \ln(\sqrt{2}), 2, 1 \rangle$?

$\ln(t) = \ln(\sqrt{2})$

$\sqrt{2}\,t = 2$

$\frac{1}{2}t^2 = 1$

↳ so $t = \sqrt{2}$

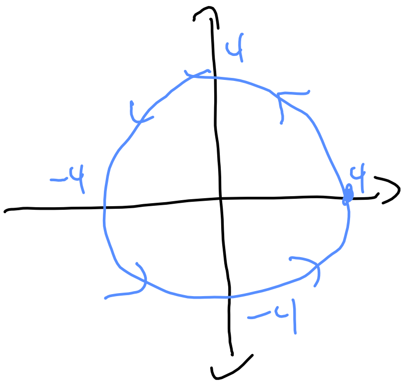
Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

$$s(t) = \int_{t_0}^{t} \|\vec{r}\,'(\tau)\|\, d\tau$$

We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s = 0$ and $s = 1$ would be exactly 1 unit of distance apart.

arc length parameterization $\Longleftrightarrow$ unit speed parameterization

Example 22. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle, 0 \le t \le 2\pi$.



1) Find $s(t) = \int_{t_0}^{t} \|\vec{r}\,'(T)\|\, dT$ ← maybe hard to integrate

Take $t_0 = 0$

$\vec{r}\,'(t) = \langle -4\sin(t), 4\cos(t)\rangle$

$\|\vec{r}\,'(t)\| = \sqrt{16\sin^2(t) + 16\cos^2(t)}$

$= \sqrt{16(\sin^2(t) + \cos^2(t))}$

$= 4$

$s(t) = \int_0^t 4\, dT = 4t$

2) Solve for $t = f(s)$: $s = 4t$ ← maybe hard to solve

$\Rightarrow t = \frac{s}{4}$

3) Substitute $t = f(s)$ in $\vec{r}(t)$:

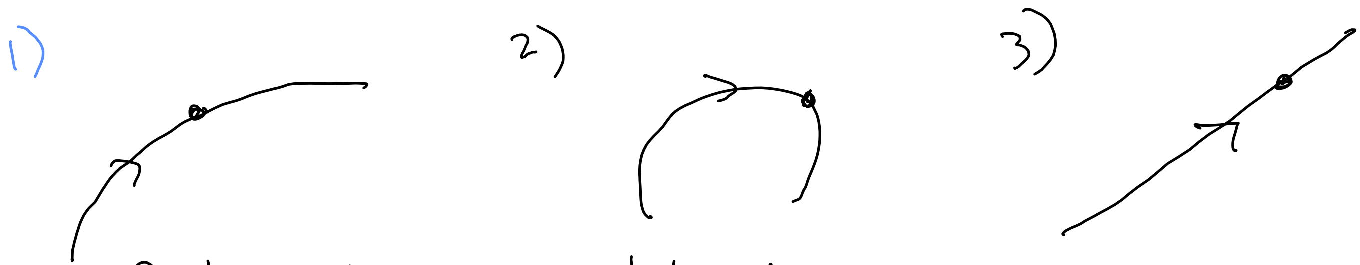
$\vec{r}_2(s) = \vec{r}(f(s)) = \vec{r}\left(\frac{s}{4}\right) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle$

$0 \le \frac{s}{4} \le 2\pi$

$\Leftrightarrow 0 \le s \le 8\pi$

13.4 - Curvature, Tangents, Normals

The next idea we are going to explore is the <u>curvature</u> of a curve in space along with two vectors that orient the curve.



Rank <u>curvature</u> from most to least:

First, we need the **unit tangent vector**, denoted **T**:

- In terms of an arc-length parameter s: $\vec{r}\,'(s)$
- In terms of any parameter t: $\vec{r}\,'(t)\,/\,\|\vec{r}\,'(t)\|$

This lets us define the **curvature**, $\kappa(s) = \|\vec{T}\,'(s)\|$

Example 23. Earlier, we found an arc length parameterization of the circle of radius 4 centered at $(0,0)$ in $\mathbb{R}^2$:

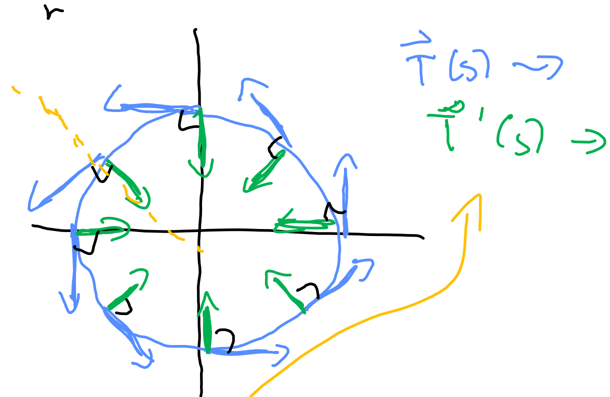
$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \le s \le 8\pi.$$

Use this to find $\mathbf{T}(s)$ and $\kappa(s)$.

- $\vec{T}(s) = \vec{r}\,'(s) = \left\langle -4\sin\left(\frac{s}{4}\right)\cdot\frac{1}{4},\ 4\cos\left(\frac{s}{4}\right)\cdot\frac{1}{4}\right\rangle$
  $= \left\langle -\sin\left(\frac{s}{4}\right), \cos\left(\frac{s}{4}\right)\right\rangle$

- $\kappa(s) = \|\vec{T}\,'(s)\|$
  $= \left\| \left\langle -\frac{1}{4}\cos\left(\frac{s}{4}\right), -\frac{1}{4}\sin\left(\frac{s}{4}\right)\right\rangle \right\|$
  $= \left|\frac{1}{4}\right| \cdot \underbrace{\left\| \left\langle -\cos\left(\frac{s}{4}\right), -\sin\left(\frac{s}{4}\right)\right\rangle \right\|}_{1 \text{ by Pythagorean Thm}}$
  $= \frac{1}{4}$

- every circle has constant curvature $\frac{1}{r}$



Question: In which direction is $\mathbf{T}$ changing?

This is the direction of the **principal unit normal**, $\mathbf{N}(s) =$ $\vec{T}\,'(s) / \|\vec{T}\,'(s)\|$

in direction of motion

$\|\vec{N}\| = 1$

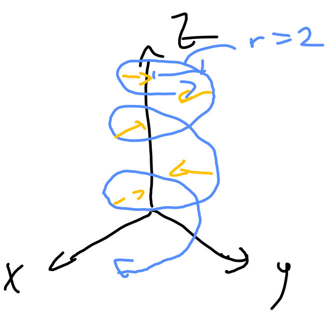
$\vec{T} \cdot \vec{N} = 0$

We said that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) = \vec{r}\,'(t) / \|\vec{r}\,'(t)\|$
- $\mathbf{N}(t) = \vec{T}'(t) / \|\vec{T}'(t)\|$
- $\kappa(t) = \|\vec{T}'(t)\| / \|\vec{r}\,'(t)\|$ or $\dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$ ← only in $\mathbb{R}^3$

↑ don't use if $\vec{T}'(t)$ is hard to compute

Example 24. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1\rangle$. ← $t \in \mathbb{R}$



$\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t), 1\rangle$

$\|\vec{r}\,'(t)\| = \sqrt{\underbrace{4\sin^2(t) + 4\cos^2(t)}_{=4} + 1}$

$= \sqrt{5}$

$\boxed{\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \left\langle -\dfrac{2}{\sqrt{5}}\sin(t), \dfrac{2}{\sqrt{5}}\cos(t), \dfrac{1}{\sqrt{5}}\right\rangle}$

$\vec{T}'(t) = \left\langle -\dfrac{2}{\sqrt{5}}\cos(t), \dfrac{-2}{\sqrt{5}}\sin(t), 0\right\rangle$

$\|T'(t)\| = \sqrt{\dfrac{4}{5}\cos^2(t) + \dfrac{4}{5}\sin^2(t) + 0} = \dfrac{2}{\sqrt{5}}$

$\boxed{\vec{N}(t) = \vec{T}'(t) / \|\vec{T}'(t)\| = \langle -\cos(t), -\sin(t), 0\rangle}$

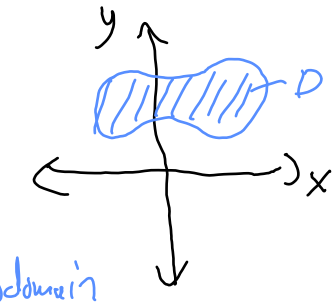
$\kappa(t) = \|\vec{T}'(t)\| / \|\vec{r}\,'(t)\| = \dfrac{2}{\sqrt{5}} \Big/ \sqrt{5} = \boxed{\dfrac{2}{5}}$

Day 5 - Functions of Multiple Variables

Pre-Lecture

Section 14.1: Functions of Multiple Variables

Definition 25. A function of two variables is a rule that assigns to each pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$.



name $\longrightarrow$ $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^2$

domain: the set of allowable inputs to f

codomain

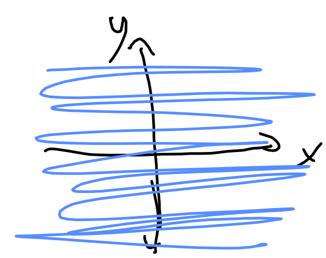
• the range: subset of codomain that are actually outputs

CAUTION: Domain cannot be an interval. $[0, 1)$ is not a domain for any fn. of 2 vars.

Example 26. Three examples are

$$f(x, y) = x^2 + y^2, \qquad g(x, y) = \ln(x + y), \qquad h(x, y) = \cancel{h(x,y)} = \sqrt{4 - x^2 - y^2}.$$

Example 27. Find the largest possible domains of f, g, and h.



$f(x,y) = x^2 + y^2 \to$ All of $\mathbb{R}^2 = \{(x,y) \in \mathbb{R}^2 \mid x, y \in \mathbb{R}\}$.

$f(1,2) = 1^2 + 2^2 = 5$

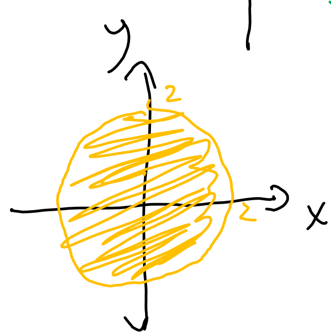
$g(x,y) = \ln(x+y)$. Domain is $\{(x,y) \in \mathbb{R}^2 \mid x+y > 0\}$

or $\{(x,y) \in \mathbb{R}^2 \mid y > -x\}$

$h(x,y) = \sqrt{4 - x^2 - y^2}$ Domain is

$\{(x,y) \in \mathbb{R}^2 \mid 4 - x^2 - y^2 \geq 0\}$

$\iff \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$



Day 5 Lecture

Daily Announcements & Reminders:

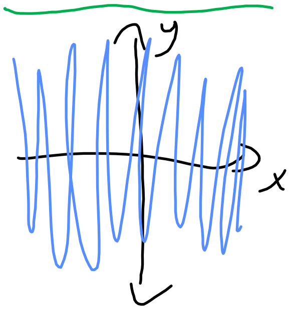
- HW 12.6 & 13.1 due tonight
- Quiz 2 in studio tomorrow if normal operations
  – 12.6, 13.1, 13.2 / L.O. G2/G4
- Do warmup on Ed ⟶

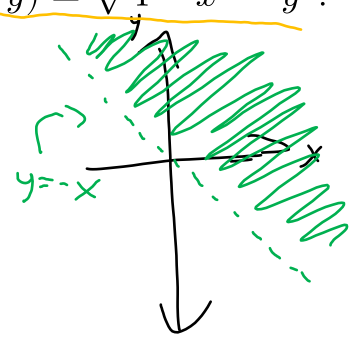
Goals for Today:

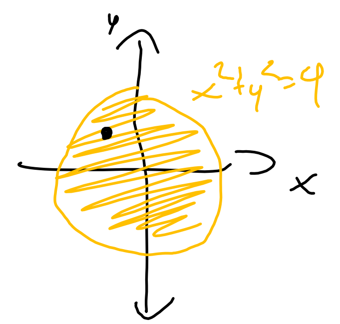
Sections 13.4-14.1

- Compute the unit tangent and principal unit normal vectors of a curve
- Give examples of functions of multiple variables
- Find the domain of functions of two variables
- Introduce and sketch traces and contours of functions of two variables
- Graph functions of two variables

In the pre-lecture video, we discussed the domains of the functions $f(x, y) = x^2 + y^2$, $g(x, y) = \ln(x + y)$, and $h(x, y) = \sqrt{4 - x^2 - y^2}$.



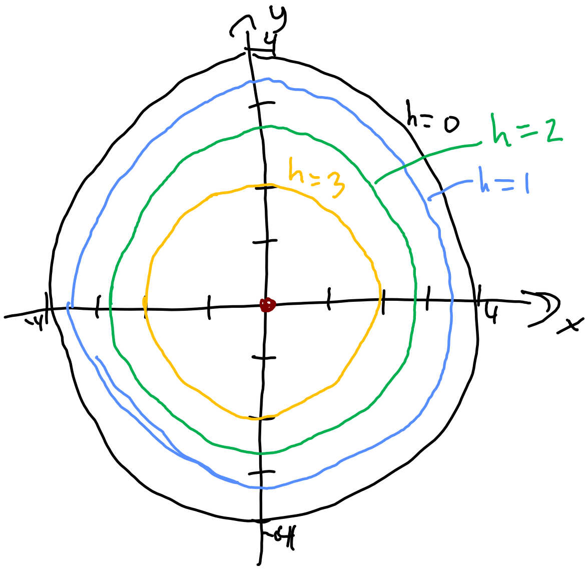




Definition 28. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Here are the graphs of the three functions above.

Example 29. Suppose a small hill has height $z = h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x, y). How could we draw a picture that represents the hill in 2D?



Domain: $z \geq 0$: $x^2 + y^2 \leq 16$

- Draw curves of all pts in domain with a fixed height

h=0 m: $0 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow x^2 + y^2 = 16$

h=1 m: $1 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow 4 = 16 - x^2 - y^2$, $x^2 + y^2 = 12$

h=2 m: $2 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow x^2 + y^2 = 8$

h=3 m: $3 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow x^2 + y^2 = 4$

h=4 m: $4 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow x^2 + y^2 = 0$

In 3D, it looks like this.

Definition 30. The contours (also called level curves or level sets) of a function f of two variables are the curves with equations $k = f(x, y)$, where k is a constant (in the range of f). A plot of contours for various values of z is a contour plot/map (or level curve plot).

Some common examples of these are:

- topographical maps
- field lines / equipotential lines
- blueprint / slicing for 3D printing
- heat / weather maps
- electron density plot

$\{(x,y) \in \mathbb{R}^2 \mid x,y \in \mathbb{R}\} = \mathbb{R}^2$

Example 31. Create a contour diagram of $f(x, y) = x^2 - y^2$

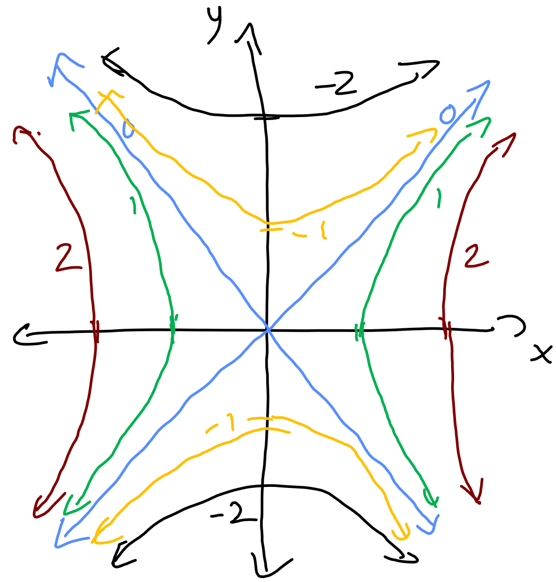
$0 = x^2 - y^2 \Rightarrow x^2 = y^2 \Rightarrow x = y$
$x = -y$

$1 = x^2 - y^2 \rightarrow$ hyperbola, $x \neq 0$
$y = 0 \rightarrow x = \pm 1$

$-1 = x^2 - y^2$
$\Leftrightarrow 1 = y^2 - x^2 \rightarrow$ hyperbola, $y \neq 0$
$x = 0 \rightarrow y = \pm 1$

$4 = x^2 - y^2 \rightarrow$ hyperbola, $x \neq 0$

$-4 = x^2 - y^2$



Example 32. Create a contour diagram of $g(x, y) = \sqrt{16 - 4x^2 - y^2}$.

Definition 33. The ______________ of a surface are the curves of ______________________ of the surface with planes parallel to the __________________________.

Example 34. Use the traces and contours of $z = f(x, y) = 4 - 2x - y^2$ to sketch the portion of its graph in the first octant.

Let's check our work: https://tinyurl.com/math2551-2var-graph

Day 6 - Functions of Multiple Variables & Limits

Pre-Lecture

Section 14.1: Traces & Graphs

Definition 32. The ___traces___ of a surface ($z=f(x,y)$) are the curves of ___intersection___ of the surface with planes parallel to the ___xz & yz-planes___.

Example 33. Find the traces of the surface $z = x^2 - y^2$.

Traces parallel to the yz-plane: $x=k$

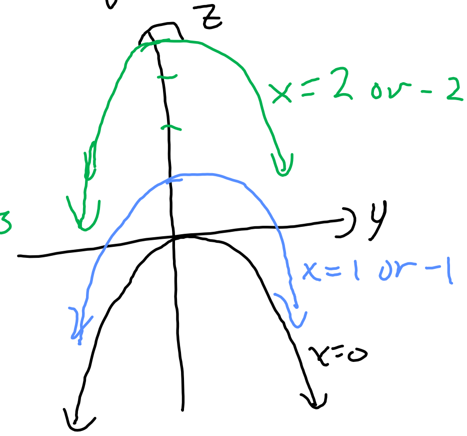
If $x=0$, get $z=-y^2$; downward parabola at origin

If $x=1$, get $z=1-y^2$; same parabola shifted up 1 unit

If $x=2$, get $z=4-y^2$; same parabola ↑ 4 units

If $x=-1$, get $z=1-y^2$; same picture as $x=1$

If $x=-2$, get $z=4-y^2$;



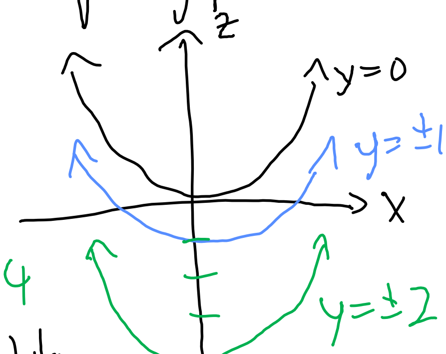
$x=k$; $z=k^2-y^2$; downward parabolas shifted up by k^2

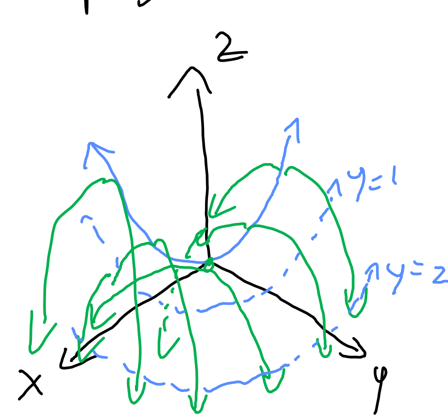
Parallel to xz-plane: $y=k$

If $y=0$, get $z=x^2$; upward opening parabola

If $y=\pm 1$, get $z=x^2-1$
same parabola shifted ↓ 1

If $y=\pm 2$, get $z=x^2-4$
same parabola shifted ↓ 4





$y=k$: $z=x^2-k^2$, upward parabola shifted ↓ by k^2

Day 6 Lecture

Daily Announcements & Reminders:

- HW 13.2 due tonight
- Quiz 1 still being graded, posted soon
- Quiz 2 posted under Modules → Solutions for practice
- Exam 1 is in a week and a half, 2/4
  - Canvas announcement soon w/details
- Do warmup on Ed

Goals for Today:

Sections 14.1-14.2

- Introduce and sketch traces and contours of functions of two variables
- Graph functions of two variables
- Find level surfaces of functions of three variables
- Evaluate limits of functions of two variables

domain: $\{(x,y) \mid 4x^2+y^2 \leq 16\}$

Example 34. Create a contour diagram of $g(x, y) = \sqrt{16 - 4x^2 - y^2}$.

$0 = \sqrt{16-4x^2-y^2}$

$16 = 4x^2+y^2$

$1 = \frac{x^2}{4} + \frac{y^2}{16}$

$z = \sqrt{7}$

$\sqrt{7} = \sqrt{16-4x^2-y^2}$

$7 = 16-4x^2-y^2$

$4x^2+y^2 = 9$

$\frac{x^2}{9/4} + \frac{y^2}{9} = 1$

$z = \sqrt{12}$

$4x^2+y^2 = 4$

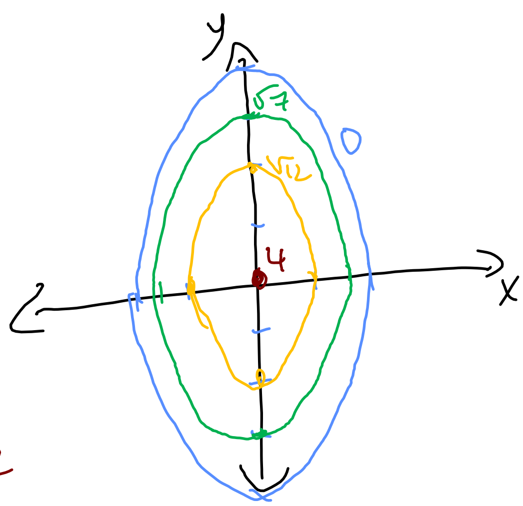
$x^2 + \frac{y^2}{4} = 1$

$z = 4$

$4 = \sqrt{16-4x^2-y^2}$

$16 = 16-4x^2-y^2$

$4x^2+y^2 = 0$



$z = \sqrt{16-y^2}$

$z^2 = 16-y^2$

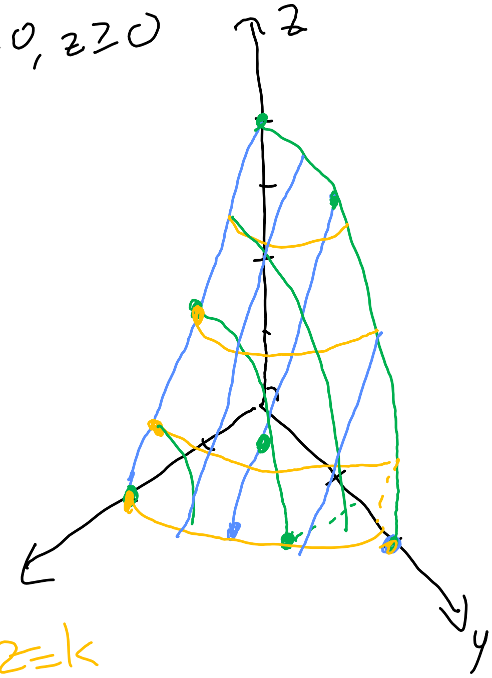
$z^2+y^2 = 16$

$z = \sqrt{16-4x^2}$

$z^2+4x^2 = 16$

Example 35. Use the traces and contours of $z = f(x, y) = 4 - 2x - y^2$ to sketch the portion of its graph in the first octant.

$\to x \geq 0, y \geq 0, z \geq 0$



y-traces: $y = k$

$y=0$: $z = 4 - 2x$

$y=1$: $z = 4 - 2x - 1$

$z = 3 - 2x$

$y = 2$: $z = 4 - 2x - 4$

$z = -2x$

x-traces: $x = k$

$x=0$: $z = 4 - y^2$

$x=1$: $z = 4 - 2 - y^2$

$z = 2 - y^2$

$x=2$: $z = -y^2$

contours: $z = k$

$z=0$: $0 = 4 - 2x - y^2$

$x = 2 - \frac{1}{2}y^2$

$z=1$: $1 = 4 - 2x - y^2$

$x = \frac{3}{2} - \frac{1}{2}y^2$

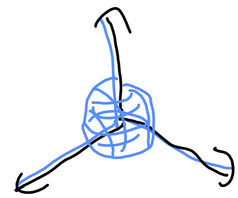
$z=2$: $2 = 4 - 2x - y^2$

$x = 1 - \frac{1}{2}y^2$

Let's check our work: https://tinyurl.com/math2551-2var-graph

Definition 36. A <u>function of three variables</u> is a rule that assigns to each <u>triple</u> of real numbers (x, y, z) in a set D a <u>unique real number</u> denoted by $f(x, y, z)$.

$\mathbb{R}^3$



$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

domain

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 37. Describe the domain of the function $f(x, y, z) = \dfrac{1}{4 - x^2 - y^2 - z^2}$.

Domain: $\{(x, y, z) \in \mathbb{R}^3 \mid 4 - x^2 - y^2 - z^2 \neq 0\}$

$\Leftrightarrow x^2 + y^2 + z^2 \neq 4$

$\Leftrightarrow$ All of $\mathbb{R}^3$ except for sphere of radius 2 centered at the origin.

(After class)

Range: $k = \dfrac{1}{4 - x^2 - y^2 - z^2}$

$4 - x^2 - y^2 - z^2 = \dfrac{1}{k}$

$x^2 + y^2 + z^2 = 4 - \dfrac{1}{k}$

radius

$\sqrt{4 - \frac{1}{k}} \geq 0$

$4 - \frac{1}{k} \geq 0$

$4 \geq \frac{1}{k}$

$k > 0$: $4k \geq 1$, $k \geq \frac{1}{4}$

$k < 0$: $4k \leq 1$, $k \leq \frac{1}{4}$

$\boxed{[\tfrac{1}{4}, \infty) \cup (-\infty, 0)}$

Example 38. Describe the level surfaces of the function $g(x, y, z) = 2x^2 + y^2 + z^2$.

$0 = 2x^2 + y^2 + z^2 \to (0, 0, 0)$

$1 = 2x^2 + y^2 + z^2 \to$ ellipsoid

$k > 0$: $k = 2x^2 + y^2 + z^2 \to$ ellipsoid

$k < 0$: $-1 = 2x^2 + y^2 + z^2$

no solutions

Section 14.2 Limits & Continuity

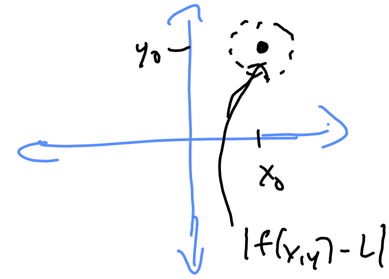
Definition 39. What is a limit of a function of two variables?

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$

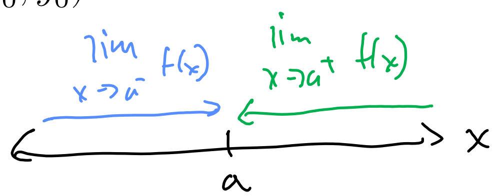
if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

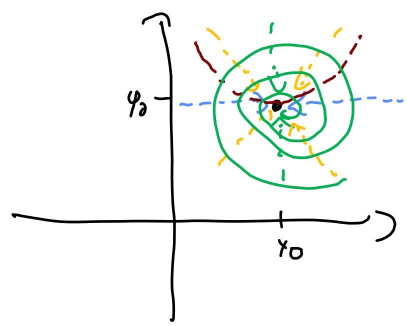


small if (x,y) is in circle

We won't use this definition much: the big idea is that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if and only if $f(x,y)$ approaches L regardless of how we approach (x_0, y_0).



$\Longrightarrow$



$\lim_{x \to a} f(x)$

Definition 40. A function $f(x, y)$ is **continuous** at (x_0, y_0) if

1. $f(x_0, y_0)$ exists
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

Key Fact: Adding, subtracting, multiplying, dividing, or composing two continuous functions results in another continuous function.

- $f(x,y) = x$ & $f(x,y) = y$ are cts

$\Rightarrow f(x,y) = x+y$ is cts, $g(x,y) = x^7 + 3x^3y^2 + y^{57}$ is cts

$h(x,y) = e^{x^2+y^2}$ is cts : $(x,y) \mapsto x^2+y^2 \mapsto e^{x^2+y^2}$
$u \mapsto e^u$

Example 41. Evaluate $\displaystyle\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$, if it exists.

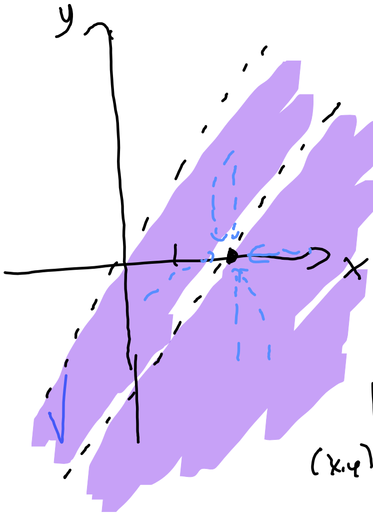
Next time

Day 7 - Limits & Partial Derivatives

Pre-Lecture

Section 14.2: Limits - Algebraic Techniques

Example 41. Evaluate $\lim_{(x,y)\to(2,0)} \dfrac{\sqrt{2x-y}-2}{2x-y-4}$, if it exists.



Domain: $2x-y \geq 0$ & $2x-y-4 \neq 0$

$y \leq 2x$ $\quad$ $y \neq 2x-4$

$$\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4} = \frac{\sqrt{4-0}-2}{4-0-4} = \frac{0}{0} \text{; indeterminate}$$

$$\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4} \cdot \frac{\sqrt{2x-y}+2}{\sqrt{2x-y}+2}$$

$$= \lim_{(x,y)\to(2,0)} \frac{\cancel{(2x-y-4)}}{\cancel{(2x-y-4)}(\sqrt{2x-y}+2)}$$

Note: We can cancel $2x-y-4$ b/c points where this is 0 are not domain of f!

$$= \lim_{(x,y)\to(2,0)} \frac{1}{\sqrt{2x-y}+2}$$

$$= \frac{1}{\sqrt{4-0}+2} = \boxed{\frac{1}{4}}$$

Other techniques: factoring, dividing by highest/lowest terms

IMPORTANT: L'Hospital's is usually not usable for functions of ≥ 2 vars

Day 7 Lecture

Daily Announcements & Reminders:

- HW 13.3/13.4 due tonight
- Quiz 3 tomorrow; 13.3, 13.4, 14.1
  - L.O. G3/G4
- Exam 1 next Tue, 75 min, material through today
  - Canvas announcement today
- Do warmup poll on Ed

Goals for Today: Sections 14.2, 14.3

- Evaluate limits of functions of two variables
- Show that a limit does not exist using the two-path test
- Determine the set of points where a function is continuous
- Start to understand how we can measure how a function of two variables is changing

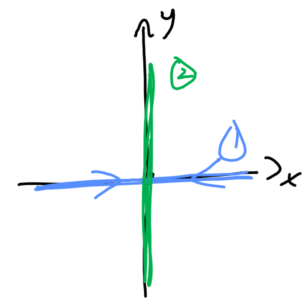
Limits of Functions of 2 vars

- Always check plugging in point
- If 0/0, try algebra
  - factoring
  - rationalizing
  - NOT L'Hospital's

Sometimes, life is harder in $\mathbb{R}^2$ and limits can fail to exist in ways that are very different from what we've seen before.

Big Idea: Limits can behave differently along different paths of approach

Example 42. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$, if it exists. Here is its graph.



① Along x-axis: $y=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+0^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$$

② Along y-axis: $x=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{0+y^2} = \lim_{y\to 0} \frac{0}{y^2} = \lim_{y\to 0} 0 = 0$$

Because the limit along the x-axis is different from the limit along the y-axis, this limit does not exist.

This idea is called the **two-path test:**

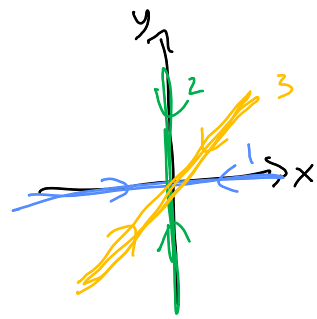
If we can find two paths to (x_0, y_0) along which the limit of $f(x,y)$ takes on two different values, then $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

CAUTION: The converse is false. Finding 2 paths where the limit is the same does not ensure existence.

Example 43. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+y^2} \Rightarrow \frac{0}{0+0} = 0/0, \text{ no factoring}$$

does not exist. Goal: Find 2 paths through $(0,0)$ w/ diff. values for limit



① Along $x=0$:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{0+y^2} = 0$$

need another!

② Along $y=0$:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{0}{2x^2+0} = 0$$

③ Along $y=x$:

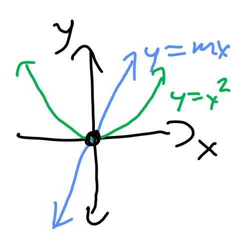
$$\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{2x^2+x^2} = \lim_{x\to 0} \frac{x^2}{3x^2} = \lim_{x\to0} \frac{1}{3} = \frac{1}{3}$$

Different values, so by the Two Path Test, the limit DNE.

Example 44. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} \to 0/0$$

does not exist.



① Along $y=mx$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,mx)\to(0,0)} \frac{\cancel{x^2}(mx)}{x^4+(mx)^2} = \lim_{x\to0} \frac{mx}{x^2+m^2}$$

$$= \frac{0}{0+m^2} = 0$$

② Along $y=x^2$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,x^2)\to(0,0)} \frac{x^2\cdot x^2}{x^4+(x^2)^2} = \lim_{x\to0} \frac{x^4}{x^4+x^4} = \frac{1}{2}$$

These values are different, so by the Two Path Test this limit does not exist.

14.3 Partial Derivatives

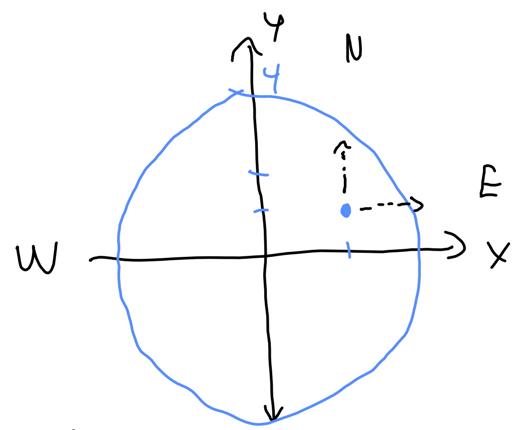
Goal: Describe how a function of two (or three, later) variables is changing at a point (a, b).

Example 45. Let's go back to our example of the small hill that has height

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point (x, y). If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive y-direction), at what rate will our height change? What if we walk due east (the positive x-direction)?

Let's investigate graphically.



Walking north: $(x,y) = (2, y)$

so $h(2,y) = 4 - 1 - \frac{1}{4}y^2 = 3 - \frac{1}{4}y^2$

- Find rate of change of this slice as y increases:

$$\frac{d}{dy}\left(h(2,y)\right)\Big|_{y=1} = \frac{d}{dy}\left(3 - \frac{1}{4}y^2\right)\Big|_{y=1} = -\frac{1}{2}y\Big|_{y=1} = -\frac{1}{2} \ \frac{\text{m height}}{\text{m North}}$$

- The slope of the hill in the positive y-direction at $(2, 1, \frac{11}{4})$ is $-\frac{1}{2}$

This is the partial derivative of h w.r.t. y at $(2,1)$.

- East: (partial derivative of h w.r.t. x at $(2,1)$)

$$\frac{d}{dx}\left(h(x,1)\right)\Big|_{x=2} = \frac{d}{dx}\left(\frac{15}{4} - \frac{1}{4}x^2\right)\Big|_{x=2} = \left(-\frac{1}{2}x\right)\Big|_{x=2} = -1 \ \text{m height/m E}$$

Day 8 - Derivatives & Linear Approximation

Pre-Lecture

General Partial Derivatives

Definition 46. If f is a function of two variables x and y, its partial derivatives are the functions f_x and f_y defined by

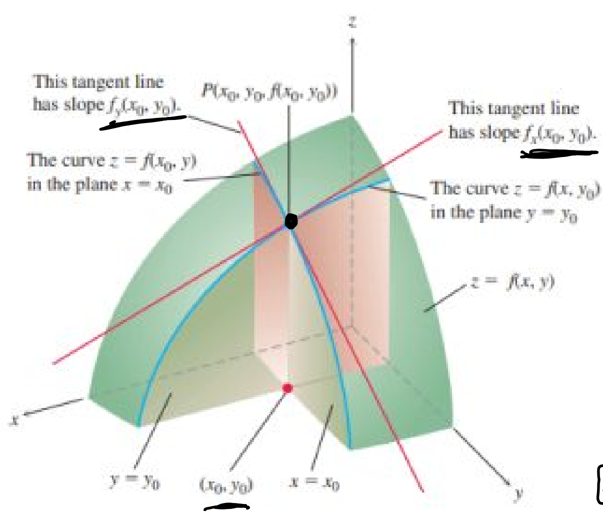
$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notations:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f)$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(f)$$

Interpretations:



Example 47. Find the partial derivatives f_x and f_y for

$$f(x,y) = 5x^2 + 2xy + 3y^3.$$

To find $f_x(x,y)$: treat y's as constants & derive w.r.t. x

$$f_x(x,y) = \frac{\partial}{\partial x}(5x^2) + \frac{\partial}{\partial x}(\underbrace{2xy}_{2y\cdot x}) + \frac{\partial}{\partial x}(3y^3)$$

$$= 10x + 2y + 0$$

$$f_y(x,y) = \frac{\partial}{\partial y}(5x^2) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial y}(3y^3)$$

$$= 0 + 2x + 9y^2$$

To get interpretation, plug in a point:

E.g. $f_x(1,1) = 10+2 = 12$, so rate of change of f at $(1,1)$ in the x-direction is 12.

The slope of the tangent line at $(1,1,10)$ to the curve $z = 5x^2 + 2x + 3$ in the plane $y=1$ is 12.

Day 8 Lecture

Daily Announcements & Reminders:

- HW 14.1 due tonight, 14.2 & Vectors/Functions Review due Tuesday
- Quiz 3 will be graded by Monday
- Exam 1 on Tuesday; see Canvas
- Do warmup problem on Ed

Goals for Today:

Section 14.3, 14.6

- Learn how to compute partial derivatives of functions of multiple variables
- Learn how to compute higher-order partial derivatives
- Understand Clairaut's theorem
- Define the total derivative
- Learn how to find a linear approximation of a differentiable function of multiple variables

Example 48. Find the partial derivatives of the functions below.

(a) $f(x, y) = 3x^2y + x - 2y$

Poll

$f_x(x,y) = 6xy + 1 - 0$

$f_y(x,y) = 3x^2 + 0 - 2$

(b) $g(x, y) = \sqrt{5x - y} = (5x-y)^{1/2}$

Chain Rule!

$g_x(x,y) = \frac{1}{2}(5x-y)^{-1/2} \cdot \underbrace{\frac{\partial}{\partial x}(5x-y)}_{5-0} = \frac{5}{2}(5x-y)^{-1/2}$

$g_y(x,y) = \frac{1}{2}(5x-y)^{-1/2} \cdot \underbrace{\frac{\partial}{\partial y}(5x-y)}_{0-1} = -\frac{1}{2}(5x-y)^{-1/2}$

Question: How would you define the second partial derivatives?

Take partial derivatives of the partial derivatives?

Notation:

• inside to outside

"pure"

$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$

$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

"mixed"

$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$

Example 49. Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} of the function $f(x, y) = \sqrt{5x - y}$

$f_x = \frac{5}{2}(5x-y)^{-1/2}$ $\qquad$ $f_y = -\frac{1}{2}(5x-y)^{-1/2}$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{5}{2}(5x-y)^{-1/2}\right) = -\frac{5}{4}(5x-y)^{-3/2}\cdot 5$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{5}{2}(5x-y)^{-1/2}\right) = -\frac{5}{4}(5x-y)^{-3/2}\cdot(-1)$$

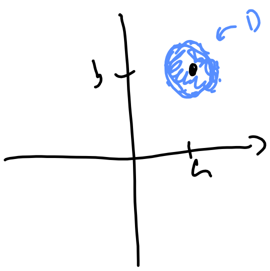
$$f_{yx} = \frac{\partial}{\partial x}\left(-\frac{1}{2}(5x-y)^{-1/2}\right) = \frac{1}{4}(5x-y)^{-3/2}\cdot 5$$

$$f_{yy} = \frac{\partial}{\partial y}\left(-\frac{1}{2}(5x-y)^{-1/2}\right) = \frac{1}{4}(5x-y)^{-3/2}\cdot(-1)$$

What do you notice about f_{xy} and f_{yx} in the previous example? $f_{xy} = f_{yx}$

Theorem 50 (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions $f, f_x, f_y, f_{xy}, f_{yx}$ are all continuous on D, then*

$f_{xy} = f_{yx}$ at (a,b).



• If all 1st, 2nd, 3rd order partial deriv. are cts on D,

then $f_{xxy} = f_{xyx} = f_{yxx}$ not equal to each other $f_{xyy} = f_{yxy} = f_{yyx}$

See ED post for non-example.

Example 51. What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

3 ?

$$f_x = \frac{\partial}{\partial x}(2xyz - z^2y) = 2yz - 0 = 2yz$$

$$f_y = \frac{\partial}{\partial y}(2xyz - z^2y) = 2xz - z^2$$

$$f_z = \frac{\partial}{\partial z}(2xyz - z^2y) = 2xy - 2yz$$

Example 52. How many rates of change should the function $f(s,t) = \begin{bmatrix} s^2 + t \\ 2s - t \\ st \end{bmatrix}$ $\begin{matrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{matrix}$ have? Compute them.

6

2

$x_s = 2s$ $\quad x_t = 1$

$y_s = 2$ $\quad y_t = -1$

$z_s = t$ $\quad z_t = s$

$$f_s = \begin{bmatrix} 2s \\ 2 \\ t \end{bmatrix} \qquad f_t = \begin{bmatrix} 1 \\ -1 \\ s \end{bmatrix}$$

$f: \mathbb{R}^2 \to \mathbb{R}^3$

$n = 2 \quad m = 3$

$$\begin{array}{cc} s & t \\ \| & \| \\ \text{col } 1 & \text{col } 2 \end{array}$$

$$Df = \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix} \begin{array}{l} \text{row } 1 \to x \\ \text{row } 2 \to y \\ \text{row } 3 \to z \end{array}$$

Total Derivatives

How might we **organize** this information? A matrix

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(\underbrace{x_1, \dots, x_n}) = \left.\begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{bmatrix}\right\}$ m rows,

we have ___n___ inputs, ___m___ outputs, and ___n·m___ partial derivatives, which we can use to form the **total derivative**.

This is a ___linear map___ map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an ___n×m matrix___, with one column per input and one row per output.

It has the formula $Df_{ij} = \frac{\partial}{\partial x_j}(f_i)$

Example 53. Find the total derivatives of each function:

a) $f(x) = x^2 + 1$ $\quad f: \mathbb{R} \to \mathbb{R}$

Df is 1×1

$Df = [2x]$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ $\quad \vec{r}: \mathbb{R} \to \mathbb{R}^3$ $\quad D\vec{r}$ is 3×1

$$D\vec{r}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{bmatrix} = \vec{r}\,'(t)$$

c) $f(x, y) = \sqrt{5x - y}$ $\quad f: \mathbb{R}^2 \to \mathbb{R}$ $\quad 1 \times 2$ matrix

$$Df(x,y) = [\, f_x \quad f_y \,] = \left[\tfrac{5}{2}(5x-y)^{-1/2} \quad -\tfrac{1}{2}(5x-y)^{-1/2} \right]$$

d) $f(x, y, z) = 2xyz - z^2 y$ $\quad f: \mathbb{R}^3 \to \mathbb{R}$ $\quad 1 \times 3$ matrix

$$Df(x,y,z) = [\, \underbrace{2yz}_{f_x} \quad \underbrace{2xz - z^2}_{f_y} \quad \underbrace{2xy - 2yz}_{f_z} \,]$$

e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$ $\quad f: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad 3 \times 2$ matrix

$$D\vec{f}(s,t) = \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix}$$

Section 14.6: Linear Approximation

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.) $\mathbb{R}^2 \to \mathbb{R}^2$: $Df(a,b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to$ near (a,b) f looks $x \mapsto y$, $y \mapsto x$

$Df(c,d) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \to$ near (c,d) f stretches x by 2, leaves y fixed

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \ldots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable:

L $f(x) = f(a) + f'(a)(x - a)$. $\Leftarrow$ Warmup: $f = \cos x$, $a = \frac{\pi}{2}$: $L(x) = 0 + (-\sin(\frac{\pi}{2}))(x - \frac{\pi}{2})$
$= -(x - \frac{\pi}{2})$

Theme: Our ideas from single variable calc work if we replace single variable derivatives w/ total derivatives.

Definition 54. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \ldots, a_n)$ is

$$L(\mathbf{x}) = f(\vec{a}) + Df(\vec{a})\underbrace{(\vec{x} - \vec{a})}_{\begin{bmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{bmatrix}}$$

Example 55. Find the linearization of the function $f(x,y) = \sqrt{5x-y}$ at the point $(1,1)$. Use it to approximate $f(1.1, 1.1)$.

$$L(x,y) = f(a,b) + \underbrace{Df(a,b)}_{(1\times 2)} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

1) Find linearization:

$$f(1,1) = \sqrt{5-1} = 2$$

$$Df(x,y) = \left[\frac{5}{2}(5x-y)^{-1/2} \quad -\frac{1}{2}(5x-y)^{-1/2} \right]$$

$$Df(1,1) = \left[\frac{5}{4} \quad -\frac{1}{4} \right]$$

$$L(x,y) = 2 + \left[\frac{5}{4} \quad -\frac{1}{4} \right] \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$= 2 + \frac{5}{4}(x-1) - \frac{1}{4}(y-1)$$

$$= f(1,1) + f_x(1,1) \cdot \underbrace{(x-1)}_{\Delta x} + f_y(1,1) \underbrace{(y-1)}_{\Delta y}$$

2) Approximate:

$$f(1.1, 1.1) \approx L(1.1, 1.1) = 2 + \frac{5}{4}(1.1-1) - \frac{1}{4}(1.1-1)$$

$$= 2 + \frac{5}{4}(.1) - \frac{1}{4}(.1)$$

$$= 2.1$$

Question: What do you notice about the equation of the linearization?

It's graph is a plane. In particular, it is tangent plane to $z = \sqrt{5x-y}$ at $(1,1,2)$.

Day 9 - Chain Rule and Directional Derivatives

Pre-Lecture

Differentiability

Definition 56. We say $f : \mathbb{R}^2 \to \mathbb{R}$ is **differentiable** at (a,b) if its linearization is a good approximation of f near (a,b).

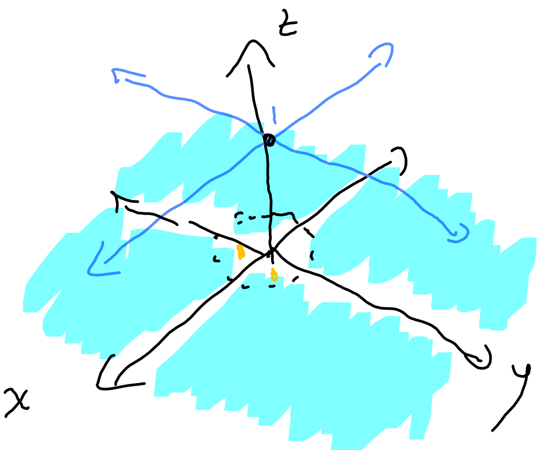
$L(x,y) = f(a,b) + Df(a,b)\begin{bmatrix} x-a \\ y-b \end{bmatrix}$

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|(x,y)-(a,b)\|} = 0.$$

error in using L to approx f.

In particular, if f is differentiable at (a,b) if it has a unique tangent plane at (a,b).

Example 57. Determine if $f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$ is differentiable at $(0,0)$.



At $(0,0)$: $f_x(0,0) = 0 = f_y(0,0)$

$Df(0,0) = [0 \ \ 0]$

$L(x,y) = f(0,0) + Df(0,0)\begin{bmatrix} x-0 \\ y-0 \end{bmatrix}$

$= 1 + [0 \ \ 0]\begin{bmatrix} x \\ y \end{bmatrix}$

$= 1$

But $L=1$ is a bad approx!

Along axes: limit $\to 0$ | Everywhere else limit $\to -\infty$

Two tangent planes for f near $(0,0)$: $z=0$ & $z=1$

- Partial Derivatives without differentiability can happen!
- Differentiable $\Rightarrow$ Continuous, Chain Rule, Extreme Value Thm, Optimization is good

Day 9 Lecture

Daily Announcements & Reminders:

- HW 14.3 due tonight
- Exam 1 grades back next T evening
- Do warmup poll on Ed

$Dg(x,y,z) = [2xyz \quad x^2z-1 \quad x^2y-1]$

so $Dg(5,-2,3) = [-60 \quad 74 \quad -51]$

Goals for Today:

Sections 14.4-14.5

- Learn the definition of differentiability for functions of multiple variables
- Learn the Chain Rule for derivatives of functions of multiple variables
- Be able to compute implicit partial derivatives
- Introduce the directional derivative of a function of multiple variables

Section 14.4 - Chain Rule

Recall the Chain Rule from single variable calculus:

$$\frac{d}{dx}\left(f(g(x))\right) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}$$

Similarly, the **Chain Rule** for functions of multiple variables says that if $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable functions then

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R}^p \arrow[r, "f"] & \mathbb{R}^m \end{tikzcd}$$

$$\underbrace{D(f(g(\mathbf{x})))}_{m \times n} = \underbrace{Df(g(\mathbf{x}))}_{m \times p}\underbrace{Dg(\mathbf{x})}_{p \times n}.$$

$g \circ f$! $\quad p \times n \cdot m \times p$

Example 58. Suppose we are walking on our hill with height $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t+1, 2-t^2 \rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

<u>Want</u>: $h'(1) = Dh(1) = Dh(\vec{r}(1))\, D\vec{r}(1)$ since $h(t) = h(\vec{r}(t))$

<u>Need</u>: $Dh(x,y)$, $\vec{r}(1)$, $D\vec{r}(t)$

$$Dh(x,y) = \begin{bmatrix} -\frac{1}{2}x & -\frac{1}{2}y \end{bmatrix} \qquad \vec{r}(1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad D\vec{r}(t) = \begin{bmatrix} 1 \\ -2t \end{bmatrix}$$

$$h'(1) = Dh(1) = \begin{bmatrix} -\frac{1}{2}x & -\frac{1}{2}y \end{bmatrix}\Big|_{(2,1)} \cdot \begin{bmatrix} 1 \\ -2t \end{bmatrix}_{t=1}$$

$$= \begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -1 + 1 = 0$$

Example 59. Use the Chain Rule to compute the rate of change of the composite function $h(t) = g(f(t))$ at $t = 2$, where

$$f(t) = \begin{bmatrix} \frac{1}{4}t^2 + 2t \\ \sin(\pi t) - t \\ t + 1 \end{bmatrix} \qquad g(x,y,z) = x^2yz - y - z.$$

Poll

$$Dg(x,y,z) = \begin{bmatrix} 2xyz & x^2z - 1 & x^2y - 1 \end{bmatrix}$$

$$f(2) = (5,-2,3) \qquad Dg(5,-2,3) = \begin{bmatrix} -60 & 74 & -51 \end{bmatrix}$$

$$Df(t) = \begin{bmatrix} \frac{1}{2}t + 2 \\ \pi\cos(\pi t) - 1 \\ 1 \end{bmatrix}$$

Find $Dh(2) = Dg(f(2))\, Df(2)$

$$= \begin{bmatrix} -60 & 74 & -51 \end{bmatrix} \begin{bmatrix} 3 \\ \pi - 1 \\ 1 \end{bmatrix}$$

$$Df(2) = \begin{bmatrix} 3 \\ \pi - 1 \\ 1 \end{bmatrix}$$

$$= -180 + 74\pi - 74 - 51$$

$$= -305 + 74\pi$$

$F(u,v)$

Example 60. Suppose that $W(s,t) = F(u(s,t), v(s,t))$, where F, u, v are differentiable functions and we know the following information.

$$u(1,0) = 2 \qquad v(1,0) = 3$$

$$u_s(1,0) = -2 \qquad v_s(1,0) = 5$$

$$u_t(1,0) = 6 \qquad v_t(1,0) = 4$$

$$F_u(2,3) = -1 \qquad F_v(2,3) = 10$$

$$\begin{tikzcd} \overset{(s,t)}{\mathbb{R}^2} \arrow[r, "g"] \arrow[rr, bend right, "W"'] & \overset{(u,v)}{\mathbb{R}^2} \arrow[r, "F"] & \mathbb{R} \end{tikzcd}$$

$$g(s,t) = \begin{bmatrix} u(s,t) \\ v(s,t) \end{bmatrix} \qquad \text{e.g. } g(s,t) = \begin{bmatrix} \sin(s)+t \\ e^s + 2t \end{bmatrix}$$

Find $W_s(1,0)$ and $W_t(1,0)$.

$$\Rightarrow DW(s=1,t=0) = DF(g(s,t))\Big|_{(u,v)=g(1,0)} \cdot D\vec{g}(s,t)\Big|_{(s,t)=(1,0)}$$

$$= \begin{bmatrix} F_u(u,v) & F_v(u,v) \end{bmatrix}\Big|_{(u,v)=g(1,0)} \begin{bmatrix} u_s(s,t) & u_t(s,t) \\ v_s(s,t) & v_t(s,t) \end{bmatrix}\Bigg|_{(s,t)=(1,0)}$$

$$= \begin{bmatrix} F_u(2,3) & F_v(2,3) \end{bmatrix} \begin{bmatrix} u_s(1,0) & u_t(1,0) \\ v_s(1,0) & v_t(1,0) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 10 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 5 & 4 \end{bmatrix}$$

$$\begin{bmatrix} W_s(1,0) & W_t(1,0) \end{bmatrix} = \begin{bmatrix} 52 & 34 \end{bmatrix}$$

Application to Implicit Differentiation: If $F(x, y, z) = c$ is used to *implicitly* define z as a function of x and y, then the chain rule says:

$x^2 + y^2 + z + \ln(z) = 4$

$F(g(x,y)) = c$ $\quad g(x,y) = \begin{bmatrix} x \\ y \\ z(x,y) \end{bmatrix}$

$F_x + z_x F_z = 0$

$F_y + z_y F_z = 0$

$DF(g(x,y))\, Dg(x,y) = \vec{0}$

$$\boxed{z_x = -\frac{F_x}{F_z} \qquad z_y = \frac{-F_y}{F_z}}$$

$$\begin{bmatrix} F_x & F_y & F_z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ z_x & z_y \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Example 61. Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the sphere $x^2 + y^2 + z^2 = 4$.

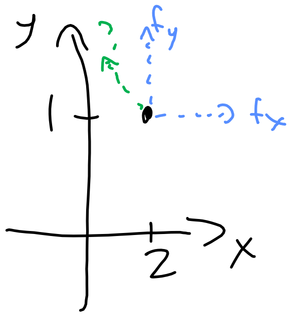
$F(x,y,z) = x^2 + y^2 + z^2$

$$\frac{\partial z}{\partial x} = -\frac{2x}{2z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = \frac{-2y}{2z} = -\frac{y}{z}$$

Section 14.5 - Directional Derivatives

Example 62. Recall that if $z = f(x, y)$, then f_x represents the rate of change of z in the x-direction and f_y represents the rate of change of z in the y-direction. What about other directions?



Let's go back to our hill example again, $\overset{f}{\cancel{h}}(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$. How could we figure out the rate of change of our height from the point $(2, 1)$ if we move in the direction $\langle -1, 1 \rangle$?

$h(2,1) = 11/4$ $\quad Dh(2,1) = \begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix}$

1) Normalize direction to unit length: $\hat{u} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

2) $$D_{\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} h(2,1) = \lim_{h \to 0} \frac{f\left((2,1) + h\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle\right) - f(2,1)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(2 - \frac{h}{\sqrt{2}}, 1 + \frac{h}{\sqrt{2}}\right) - 11/4}{h}$$

$$= \frac{1}{2\sqrt{2}}$$

Definition 63. The ____________________________ of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(\mathbf{p}) =$$

if this limit exists.

E.g. for our hill example above we have:

Note that $D_{\mathbf{i}} f =$ $D_{\mathbf{j}} f =$ $D_{\mathbf{k}} f =$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \Rightarrow 0 = \lim_{x \to a} \frac{\boxed{f(x)} - \boxed{f(a) - f'(a)(x-a)}}{x - a}$$

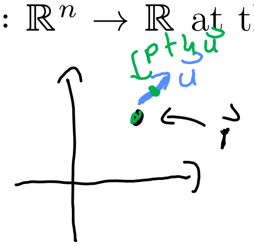
Day 10 - Directional Derivatives, Gradients, Tangent Planes

Pre-Lecture

Section 14.5 - Directional Derivatives

Definition 64. The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(\mathbf{p}) = \lim_{h\to 0} \frac{f(\vec{p}+h\vec{u})-f(\vec{p})}{h}$$



if this limit exists.

Note that $D_{\mathbf{i}}f =$ f_x $\quad D_{\mathbf{j}}f =$ f_y $\quad D_{\mathbf{k}}f =$ f_z

In practice, we want to avoid using this limit definition!

Note: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

$$D_{\mathbf{u}}f(\mathbf{p}) = \underbrace{Df(\vec{p})}\,\underbrace{\vec{u}}$$

matrix-vector product

- partial deriv's form a basis for all directional derivatives

Example 65. Compute the rate of change of $f(x,y) = e^{xy}$ at the point $(1,2)$ in the direction $\mathbf{u} = \langle 1/\sqrt{5}, 2/\sqrt{5}\rangle$.

directional derivative

$$D_{\vec{u}}f(1,2) = \lim_{h\to 0} \frac{e^{\left(1+\frac{h}{\sqrt{5}}\right)\left(2+\frac{2h}{\sqrt{5}}\right)} - e^2}{h}$$

OR $\quad Df = \begin{bmatrix} ye^{xy} & xe^{xy}\end{bmatrix}$

$$Df(1,2) = \begin{bmatrix} 2e^2 & e^2 \end{bmatrix}$$

$$D_{\vec{u}}f(1,2) = \begin{bmatrix} 2e^2 & e^2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5}\end{bmatrix}$$

$$= \frac{2}{\sqrt{5}}e^2 + \frac{2}{\sqrt{5}}e^2 = \boxed{\frac{4}{\sqrt{5}}e^2}$$

Day 10 Lecture

Daily Announcements & Reminders:

- HW 14.4 due tonight
- Quiz 4 tomorrow on 14.3 & 14.4
  - L.O. D1
- Exam grades back tonight or tomorrow morning
- Fill out check in Survey (Canvas announcement)
- Do warmup on Ed
- Office hours on Th → 8:00–9:00 am in Skiles 218C
  for next 3 weeks

Goals for Today: Sections 14.4-14.6

- Learn to compute the rate of change of a multivariable function in any direction
- Investigate the connection between the gradient vector and level curves/surfaces
- Discuss tangent planes to surfaces, how to find them, and when they exist

Section 14.5: Gradients

Definition 66. If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient of f at $\mathbf{p} \in \mathbb{R}^n$ is the vector function ∇f (or grad f) defined by

$$\nabla f(\mathbf{p}) = \langle f_{x_1}(\vec{p}), f_{x_2}(\vec{p}), \ldots, f_{x_n}(\vec{p})\rangle$$

$$= Df(p)^T$$

We saw: for f diff'able we have

$$D_{\vec{u}} f(\vec{p}) = Df(\vec{p})\vec{u} = \nabla f(\vec{p}) \cdot \vec{u}$$

Example 67. Find the gradient vector and the directional derivative of each function at the given point $\mathbf{p}$ in the direction of the given vector $\mathbf{u}$.

a) $f(x, y) = \ln(x^2 + y^2)$, $\mathbf{p} = (-1, 1)$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

1) $\nabla f = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$

2) At $\vec{p} = (-1,1)$: $\nabla f(-1,1) = \left\langle \frac{-2}{2}, \frac{2}{2} \right\rangle = \langle -1, 1 \rangle$

3) Find directional deriv:

$$D_{\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle} f(-1,1) = \nabla f(-1,1) \cdot \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle = \langle -1, 1 \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$$
$$= \boxed{\frac{-3}{\sqrt{5}}} \quad \frac{\text{m height}}{\text{m in } \vec{u} \text{ direction}}$$

Check unit vector

$$\sqrt{\frac{1}{5} + \frac{4}{5}} = 1$$

b) $g(x, y, z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = (1, 2, 1)$, $\mathbf{u}$ the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

Find gradient)
$$\nabla g = \langle 2x + 4y^2, 8xy, 2z \rangle$$
$$\nabla g(1,2,1) = \langle 18, 16, 2 \rangle$$

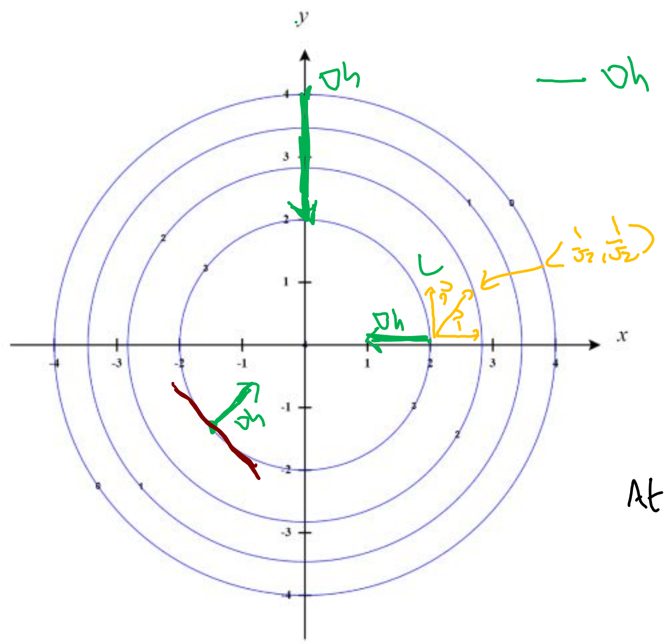
Check unit vector: $\|\langle 1, 2, -1 \rangle\| = \sqrt{1+4+1} = \sqrt{6}$

so $\vec{u} = \frac{\langle 1, 2, -1 \rangle}{\sqrt{6}}$

Compute dir. deriv:
$$D_{\vec{u}}\, g(1,2,1) = \langle 18, 16, 2 \rangle \cdot \frac{1}{\sqrt{6}} \langle 1, 2, -1 \rangle$$
$$= \frac{1}{\sqrt{6}}(18 + 32 - 2) = \boxed{\frac{48}{\sqrt{6}}}$$

Example 68. If $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, the contour map is given below. Find and draw ∇h on the diagram at the points $(2,0), (0,4)$, and $(-\sqrt{2}, -\sqrt{2})$. At the point $(2,0)$, compute $D_{\mathbf{u}}h$ for the vectors $\mathbf{u}_1 = \mathbf{i}, \mathbf{u}_2 = \mathbf{j}, \mathbf{u}_3 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

$\nabla h = \langle -\frac{1}{2}x, \frac{-1}{2}y \rangle = \begin{bmatrix} -\frac{1}{2}x \\ -\frac{1}{2}y \end{bmatrix} = -\frac{1}{2}x\vec{i} - \frac{1}{2}y\vec{j}$



(a,b)	$\nabla h(a,b)$
$(2,0)$	$\langle -1, 0 \rangle$
$(0,4)$	$\langle 0, 2 \rangle$
$(-\sqrt{2}, -\sqrt{2})$	$\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

At $(2,0)$:

$\vec{u}$	$D_{\vec{u}}h(2,0)$
$\vec{i}$	-1
$\vec{j}$	0
$\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$	$-\frac{1}{\sqrt{2}}$

- ∇h points inward here
- ∇h is longer further from $(0,0)$

- At $\vec{p}$, $\nabla f(\vec{p})$ is $\perp$ to level set containing $\vec{p}$.

- $D_{\vec{u}}f = 0$ in the direction along (i.e. tangent to) a level curve
- $-\nabla f(\vec{p})$ points in the direction of greatest decrease of f at $\vec{p}$ and that rate $-\|\nabla f(\vec{p})\|$

- $\nabla f(\vec{p})$ points in the direction of greatest increase of f at $\vec{p}$
- $\|\nabla f(\vec{p})\|$ is the greatest rate of increase of f at $\vec{p}$

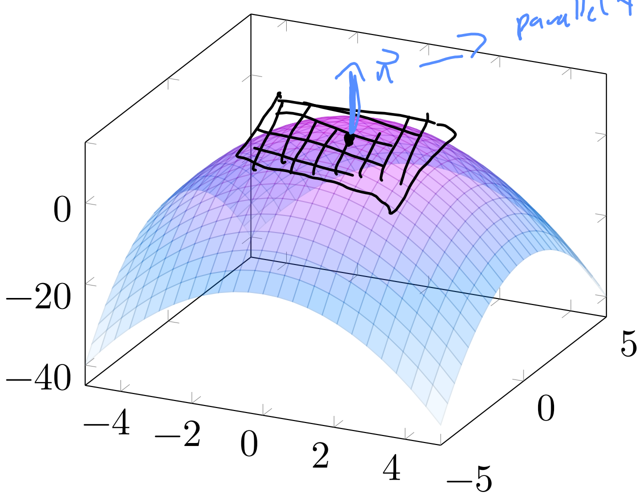
Note that the gradient vector is orthogonal to level curves. of $f(x,y)$

Similarly, for $f(x,y,z)$, $\nabla f(a,b,c)$ is orthogonal to level surfaces

Section 14.6 - Tangent Planes

Suppose S is a surface with equation $F(x, y, z) = k$. How can we find an equation of the tangent plane of S at $P(x_0, y_0, z_0)$?

$\vec{n}$ → parallel to ∇F



$x^2 + y^2 + z = 10, P = (-1, 3, 0)$

Plane:

- point: $(x_0, y_0, z_0) = \vec{p}$
- normal: gradient of F at $\vec{p}$

1) Identify F defining surface:

$F(x,y,z) = x^2 + y^2 + z$

$\underbrace{x^2 + y^2 + z - 10}_{F(x,y,z) = x^2+y^2+z-10} = 0$

2) Find point: $(-1, 3, 0)$

3) Find normal: $\nabla F = \langle 2x, 2y, 1 \rangle$

$\nabla F(-1,3,0) = \langle -2, 6, 1 \rangle$

4) Plane equation: $-2(x+1) + 6(y-3) + 1\cdot(z-0) = 0$

Example 69. Find the equation of the tangent plane at the point $(-2, 1, -1)$ to the surface given by

$$z = 4 - x^2 - y$$

1) Move variables to same side:

$$x^2 + y + z = 4$$

so $F(x,y,z) = x^2 + y + z$

2) Point: $(-2,1,-1)$

3) Normal: $\nabla F = \langle 2x, 1, 1 \rangle$

$DF(-2,1,-1) = \langle -4, 1, 1 \rangle$

4) Tangent plane: $\boxed{-4(x+2) + 1(y-1) + 1(z+1) = 0}$

$(a,b) = (-2,1)$ $f(a,b) = -1$

$L(x,y) = f(a,b) + Df(a,b)\begin{bmatrix} x-a \\ y-b \end{bmatrix}$

$Df = \begin{bmatrix} -2x & -1 \end{bmatrix}$

$Df(2,1) = \begin{bmatrix} 4, -1 \end{bmatrix}$

$z = -1 + \begin{bmatrix} 4 & -1 \end{bmatrix}\begin{bmatrix} x+2 \\ y-1 \end{bmatrix}$

$\boxed{z = -1 + 4(x+2) - (y-1)}$

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

$$0 = \underbrace{f(x,y) - z}_{F}$$

$$\nabla F = \langle f_x(x,y), f_y(x,y), -1 \rangle$$

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

This should look familiar: it's linearization

Day 11 - Optimization: Local & Global

Pre-Lecture

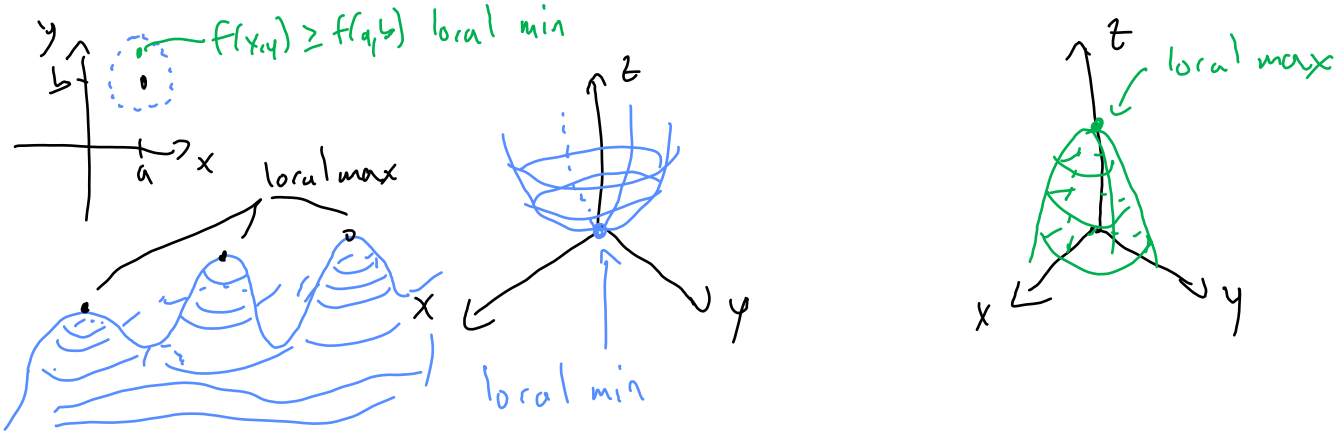
Section 14.7 - Local Extreme Values

Last time: If $f(x, y)$ is a function of two variables, we said $\nabla f(a, b)$ points in the direction of greatest change of f.

What does it mean if $\nabla f(a, b) = \langle 0, 0 \rangle$?

Definition 70. Let $f(x, y)$ be defined on a region containing the point (a, b). We say

- $f(a, b)$ is a local minimum value of f if $f(a, b) \le f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)
- $f(a, b)$ is a local maximum value of f if $f(a, b) \ge f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)



In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $z = x^2 - y^2$ (a hyperbolic paraboloid!) near $(0, 0)$.

This is called a saddle point : $\exists 1$ direction of increase at point, ≥ 1 direction of decrease

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e. $Df = [0 \ 0]$ $\nabla f = \langle 0, 0 \rangle$

Definition 71. If $f(x, y)$ is a function of two variables, a point (a, b) in the domain of f with $Df(a, b) =$ $[0 \quad 0]$ or where $Df(a, b)$ undefined is called a critical points of f.

- If f has a local min or local max at (a,b) then (a,b) is a critical point!

Example 72. Find the critical points of the function $f(x, y) = x^3 + y^3 - 3xy$.

Solve $Df = [3x^2-3y \quad 3y^2-3x] = [0 \quad 0]$

$$\begin{cases} 3x^2-3y=0 \\ 3y^2-3x=0 \end{cases} \to \begin{cases} x^2=y \\ y^2=x \end{cases}$$

$\to x^4 = x$

$\to x(x^3-1)=0$

$x=0$ or $x=1$

$y=x^2$

so $y=0$ $\quad y=1$

crit points: $(0,0)$ & $(1,1)$

Day 11 Lecture

Daily Announcements & Reminders:

- HW 14.5 Due tonight
- Exam 1 grades released.
  - regrades open until noon next W
- Do warmup on Ed

Goals for Today:

Section 14.7

- Define local & global extreme values for functions of two variables
- Learn how to find local extreme values for functions of two variables
- Learn how to classify critical points for functions of two variables
- Learn how to find global extreme values on a closed & bounded domain

Example 73. Which of the following functions have a critical point at $(0,0)$?

(0,0) not in domain

$$f(x,y) = 3x+y^3+2y^2 \quad g(x,y) = \cos(x)+\sin(y) \quad h(x,y) = \frac{4}{x^2+y^2} \quad k(x,y) = x^2+y^2$$

Poll

$Df = [3 \quad 3y^2+4y]$

$Dg = [-\sin(x) \quad \cos(y)]$

$Dg(0,0) = [0 \quad 1]$

$Dh = \left[\frac{-8x}{(x^2+y^2)^2} \quad \frac{-8y}{(x^2+y^2)^2} \right]$

$Dh(0,0)$ DNE

$Dk = [2x \quad 2y]$

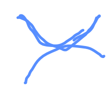
$D(0,0) = [0 \quad 0]$

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix** of $f(x,y)$ at (a,b):

$$\overset{D^2 f(a,b)}{Hf(a,b)} = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

Theorem 74 (2nd Derivative Test)**.** *Suppose* (a,b) *is a critical point of* $f(x,y)$ *and* f *has continuous second partial derivatives. Then we have:*

- *If* $\det(Hf(a,b)) > 0$ *and* $f_{xx}(a,b) > 0$, $f(a,b)$ *is a local minimum*
- *If* $\det(Hf(a,b)) > 0$ *and* $f_{xx}(a,b) < 0$, $f(a,b)$ *is a local maximum*
- *If* $\det(Hf(a,b)) < 0$, f *has a saddle point at* (a,b)



- *If* $\det(Hf(a,b)) = 0$, *the test is inconclusive.*

[Advanced] More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at $\mathbf{p}$ then

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, f is concave up in every direction from $\mathbf{p}$ and so has a local minimum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, f is concave down in every direction from $\mathbf{p}$ and so has a local maximum at $\mathbf{p}$.
- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, f is concave up in some directions from $\mathbf{p}$ and concave down in others, so has neither a local minimum or maximum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, f may have either a local minimum or neither at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, f may have either a local maximum or neither at $\mathbf{p}$.

(for 2×2): $\lambda_1 \lambda_2 = \det A$ $\qquad$ $\lambda_1 + \lambda_2 = \operatorname{tr}(A) = a_{11} + a_{22}$

Example 75. Classify the critical points of $f(x,y) = x^3 + y^3 - 3xy$ from Example 72.

$Df = [3x^2 - 3y \quad 3y^2 - 3x]$ crit pts: $(0,0)$ & $(1,1)$

1) Find Hf:

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$$

2) Apply 2nd Deriv Test.

At $(0,0)$: $Hf(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$

$\det(Hf(0,0)) = 0 - (-3)(-3) = -9 < 0$

So by 2nd Deriv. Test, f has a saddle point at $(0,0)$.

At $(1,1)$: $Hf(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$

$\det(Hf(1,1)) = 36 - 9 = 27 > 0$ & $f_{xx}, f_{yy} > 0$

So by 2nd Deriv Test, f has a local min at $(1,1)$.

Example 76. Find and classify the critical points of $f(x, y) = x^2 y + y^2 + xy$.

1) Take derivatives:

$$Df = \begin{bmatrix} 2xy + y & x^2 + 2y + x \end{bmatrix}$$

$$Hf = \begin{bmatrix} 2y & 2x+1 \\ 2x+1 & 2 \end{bmatrix}$$

2) Find crit pts:

$$\begin{cases} 2xy + y = 0 & ① \\ x^2 + 2y + x = 0 & ② \end{cases} \longrightarrow y(2x+1) = 0$$

Case 1: $y = 0$

$② \Rightarrow x^2 + x = 0$

$x(x+1) = 0$

$x = 0$ or $x = -1$

Case 2: $2x + 1 = 0$

$x = -\frac{1}{2}$

$② \Rightarrow \frac{1}{4} + 2y - \frac{1}{2} = 0$

$2y = \frac{1}{4}$

$y = \frac{1}{8}$

crit pts: $(0,0)$, $(-1,0)$, $\left(-\frac{1}{2}, \frac{1}{8}\right)$

3) Use 2nd Deriv Test: $Hf = \begin{bmatrix} 2y & 2x+1 \\ 2x+1 & 2 \end{bmatrix}$

At $(0,0)$: $Hf = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \det(Hf(0,0)) = -1 < 0$

$\Rightarrow f$ has a saddle point at $(0,0)$

At $(-1,0)$: $Hf = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} \rightarrow \det(Hf(-1,0)) = -1 < 0$

$\Rightarrow f$ has a saddle point at $(-1,0)$

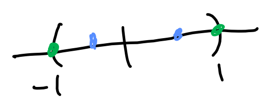
At $\left(-\frac{1}{2}, \frac{1}{8}\right)$: $Hf = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \det\left(Hf\left(-\frac{1}{2}, \frac{1}{8}\right)\right) = \frac{1}{2} > 0$ & $f_{xx}, f_{yy} > 0$

$\Rightarrow f$ has a local min at $\left(-\frac{1}{2}, \frac{1}{8}\right)$.

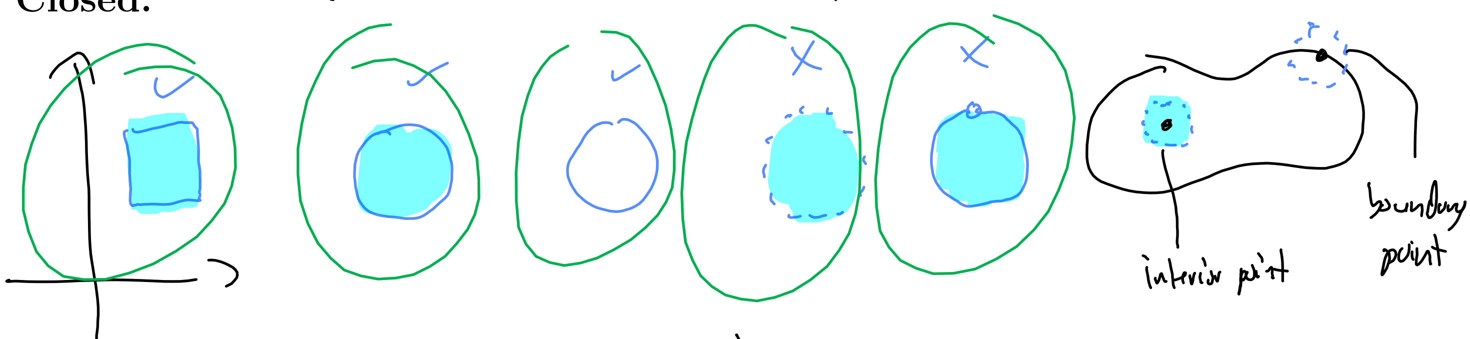
Two Local Maxima, No Local Minimum: The function $g(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2 + 2$ has two critical points, at $(-1,0)$ and $(1,2)$. Both are local maxima, and the function never has a local minimum!

A global maximum of $f(x,y)$ is like a local maximum, except we must have $f(a,b) \geq f(x,y)$ for **all** (x,y) in the domain of f. A global minimum is defined similarly.

Theorem 77 (Extreme Value Theorem)**.** *On a closed & bounded domain, any continuous function $f(x,y)$ attains a global minimum & maximum.*



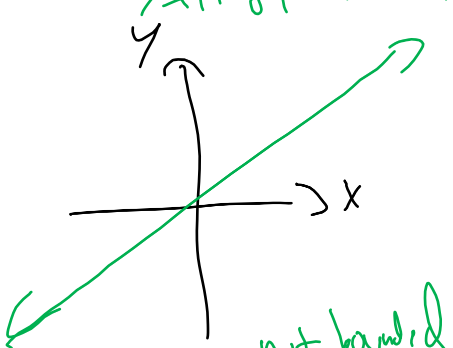
Closed: The set contains all of its boundary points.

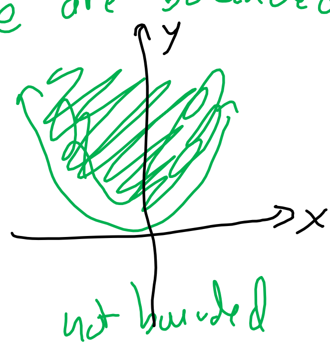


Bounded: The set fits in a big enough circle

$\mathbb{R}^2$ closed
not bounded

All of the above are bounded





Strategy for finding global min/max of $f(x, y)$ on a closed & bounded domain R

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 78. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

Day 12 - Optimization: Global & Constrained

Pre-Lecture

Section 14.8 - Constrained Optimization

Goal: Maximize or minimize $f(x, y)$ subject to a *constraint*, $g(x, y) = c$.

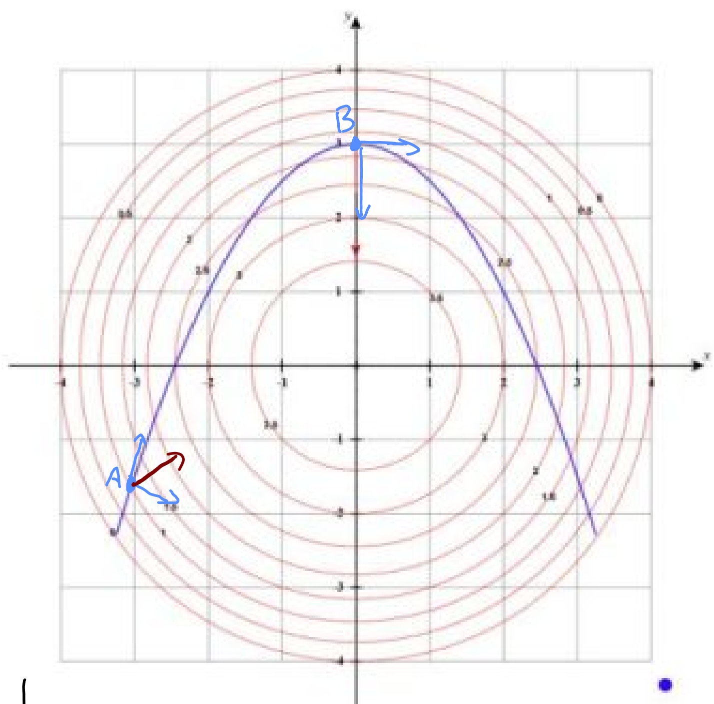
Example 78. A new hiking trail has been constructed on the hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the xy-plane. What is the highest point on the hill on this path?

Objective function:

$h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$

Constraint equation:

$y + \frac{1}{2}x^2 = 3$ | $g(x,y) = y + \frac{1}{2}x^2$



Goal: Locate points with no rate of change along the constraint curve

At A: rate of change is > 0

At B: rate of change is $= 0$

Find pts where dir. deriv. in $\vec{T}$ direction to constraint is 0

$\Longleftrightarrow \vec{T} \perp \nabla h$ OR $\nabla h = \vec{0}$

$\Longleftrightarrow \vec{N} \parallel \nabla h$ OR $\nabla h = \vec{0}$

$\Longleftrightarrow \nabla h \parallel \nabla g$ or $\nabla h = \vec{0}$

$\Longleftrightarrow \boxed{\nabla h = \lambda \nabla g \text{ for some } \lambda \in \mathbb{R} \text{ s.t. } g = c}$

Solve: $\langle -\frac{1}{2}x, -\frac{1}{2}y \rangle = \lambda \langle x, 1 \rangle$

$y + \frac{1}{2}x^2 = 3$

Solution:

(x,y)	h
$(0,3)$	$7/4$
$(-2,1)$	$11/4$
$(2,1)$	$11/4$

Algebra for solution:

$$\begin{cases} -\frac{1}{2}x = \lambda x & (1) \\ -\frac{1}{2}y = \lambda & (2) \\ y + \frac{1}{2}x^2 = 3 & (3) \end{cases}$$

From (1), we have

$$0 = \lambda x + \frac{1}{2}x$$

$$= x\left(\lambda + \frac{1}{2}\right)$$

So either $x = 0$ or $\lambda = -\frac{1}{2}$

If $x = 0$ then we get

(3): $y + 0 = 3$

$y = 3$

and (2): $\lambda = -\frac{3}{2}$

so $(0,3)$ is a possible solution point

If $\lambda = -\frac{1}{2}$ then we get

(2): $-\frac{1}{2}y = -\frac{1}{2}$

$y = 1$

and (3): $1 + \frac{1}{2}x^2 = 3$

$\frac{1}{2}x^2 = 2$

$x^2 = 4$

$x = \pm 2$

so $(-2,1)$, $(2,1)$ are possible solution points.

Day 12 Lecture

Daily Announcements & Reminders: linearization, dir. deriv, gradient, tangent planes

- HW 14.6 due tonight
- Quiz 5 tomorrow, 14.5 & 14.6
  L.O. D2 & D3
- Quiz 4 revision assignment will open after grading is done, due W 2/26
- Do 2 warmup polls

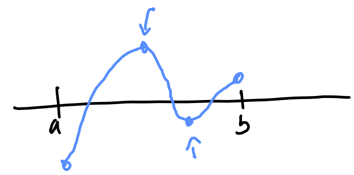
Goals for Today: Sections 14.7, 14.8

- Find global extreme values of continuous functions of two variables on closed & bounded domains
- Apply the method of Lagrange multipliers to find extreme values of functions of two or more variables subject to one or more constraints

- Closed: Contains entire boundary
  Bounded: Fits in a big enough disk

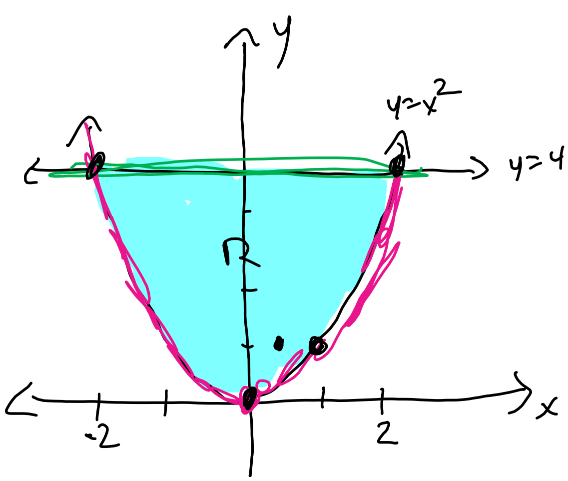
14.7 - Applying Extreme Value Theorem

Strategy for finding global min/max of continuous $f(x, y)$ on a closed & bounded domain R



1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 79. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R consisting of those points with $x^2 \leq y \leq 4$.



$x^2 \leq y$ $\quad y \leq 4$

plot $x^2 = y$ $\quad y = 4$

1) Find crit pts inside R

$Df = [8x-4y \quad -4x+2] = [0 \quad 0]$

$8x - 4y = 0$

$-4x + 2 = 0$ $\rightarrow x = \frac{1}{2}$

so $4 - 4y = 0$

$y = 1$

crit pt $(\frac{1}{2}, 1)$ check in R: $(\frac{1}{2})^2 \leq 1 \leq 4$ ✓

2) Find crit pts of f restricted to boundary of R

On $y=4$: $(x, 4)$ for $-2 \leq x \leq 2$

$f(x,y)$ become $f(x,4) = 4x^2 - 4x(4) + 2(4)$

$g(x) = 4x^2 - 16x + 8$, $-2 \leq x \leq 2$

set $g'(x) = 0$: $8x - 16 = 0 \rightsquigarrow x = 2$ get crit pt on boundary $(2,4)$

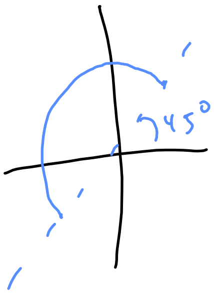
Also include boundary endpoints: $(-2,4)$ & $(2,4)$

On $y=x^2$: Parameterize boundary: $\Leftarrow$ if hard to substitute directly

$\vec{r}(t)=\langle t, t^2\rangle \quad -2\le t\le 2$

Then we get $f(\vec{r}(t)) = 4(t)^2 - 4(t)(t^2)+2(t^2)$

$h(t)= 6t^2-4t^3 \quad , \quad -2\le t\le 2$



$h'(t)=12t-12t^2=0$

$12t(1-t)=0 \quad t=0 \quad t=1$

$r(0)=\langle 0,0\rangle \quad \vec{r}(1)=\langle 1,1\rangle$

so crit pts on boundary $(0,0)$, $(1,1)$

Also include boundary endpoints $t=-2$, $t=2$

$(-2,4)$ & $(2,4)$

3) Evaluate

Test pts	$(\frac{1}{2},1)$	$(-2,4)$	$(2,4)$	$(0,0)$	$(1,1)$
f	1	56	-8	0	2

4) Conclusion:

Global min of f on R is -8 attained $(2,4)$

Global max of f on R is 56 attained $(-2,4)$

14.8 - Lagrange Multipliers

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

λ Lagrange Multipliers

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

Example 80. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

Objective: dist to $(0,0,0)$: $\sqrt{x^2+y^2+z^2}$ $\Leftrightarrow$ $f = x^2+y^2+z^2$

min/max occur at same places as

also works for $\log(\ )$ and $a^{(\ )}$

Constraint: $\underbrace{z^2 - xy}_{g} = 4$

Solve: $\nabla f = \langle 2x, 2y, 2z\rangle$

$\nabla g = \langle -y, -x, 2z\rangle$

$\Rightarrow \begin{cases} \nabla f = \lambda \nabla g \\ g = 4 \end{cases} = \begin{cases} 2x = \lambda(-y) \\ 2y = \lambda(-x) \\ 2z = \lambda(2z) \quad (3) \\ z^2 - xy = 4 \end{cases}$

- Look for simple equations

(3): $2z(1-\lambda) = 0$

cases: $z = 0$ $\quad \lambda = 1$

- Be very careful not to divide by 0

		dist		dist
Test pts:	$(-2,2,0)$	$\sqrt{8}$	$(0,0,2)$	2
	$(2,-2,0)$	$\sqrt{8}$	$(0,0,-2)$	2

Post class: One approach to solving this system.

$$\begin{cases} 2x = \lambda(-y) & (1) \\ 2y = \lambda(-x) & (2) \\ 2z = \lambda(2z) & (3) \\ z^2 - xy = 4 & (4) \end{cases}$$

Start w/ eqn (3): $2z(1-\lambda) = 0$

Case 1: $z = 0$. Now our system is

$$\begin{cases} 2x = -\lambda y & (1) \\ 2y = -\lambda x & (2) \\ -xy = 4 & (4') \end{cases}$$

From (4') we see both x and y are not 0 and that $y = -\frac{4}{x}$, so

(1) and (2) become

(1') $2x = \frac{4\lambda}{x} \rightarrow x^2 = 2\lambda \rightarrow \lambda = \frac{x^2}{2}$

(2') $\frac{-8}{x} = -\lambda x \qquad 8 = \lambda x^2 \rightarrow 16 = x^4$

so $x = 2$ or $x = -2$

$y = -\frac{4}{2} = -2$ or $y = \frac{-4}{-2} = 2$

So we get points $(2,-2,0)$ & $(-2,2,0)$.

Case 2: $\lambda = 1$ Now our system is

$$\begin{cases} 2x = -y & (5) \\ 2y = -x & (6) \\ z^2 - xy = 4 & (4) \end{cases}$$

From (5) into (6) we get $-4x = -x$, so $x = 0$ and then from (5) $y = 0$

and (4) becomes $z^2 = 4$ so $z = \pm 2$

So we get points $(0,0,2)$, $(0,0,-2)$.

Example 81. Set up the system of equations from the method of Lagrange multipliers to find the extreme values of the function $f(x, y, z) = x^2 + y^2 + z^2$ on the curve of intersection of the surfaces $x^2 + y^2 - z = 3$ and $x + 2y - 2z = 2$.

Objective: $f(x,y,z) = x^2+y^2+z^2$

Constraints: $g(x,y,z) = x^2+y^2-z = 3$

$h(x,y,z) = x+2y-2z = 2$

- $\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = 3 \\ h = 2 \end{cases}$

$\nabla f = \langle 2x, 2y, 2z \rangle$

$\nabla g = \langle 2x, 2y, -1 \rangle$

$\nabla h = \langle 1, 2, -2 \rangle$

$$\begin{cases} 2x = 2\lambda x + \mu & (1) \\ 2y = 2\lambda y + 2\mu & (2) \\ 2z = -\lambda - 2\mu & (3) \\ x^2+y^2-z=3 & (4) \\ x+2y-2z=2 & (5) \end{cases}$$

- No equations easily factorable after moving all terms to one side
- Use substitution;

$\mu = \dfrac{2z+\lambda}{-2} \rightarrow$ plug into (1) & (2)

Repeat with λ

$$\begin{cases} (2y + 2z = \lambda yz)\,x \\ (2x + 2z = \lambda xz)\,y \\ (2x + 2y = \lambda xy)\,z \\ xyz = 28 \end{cases}$$

$\cancel{2xy} + 2xz = \cancel{2xy} + 2yz$

$xz = yz$

$x = y$

Day 13 - Double & Iterated Integrals

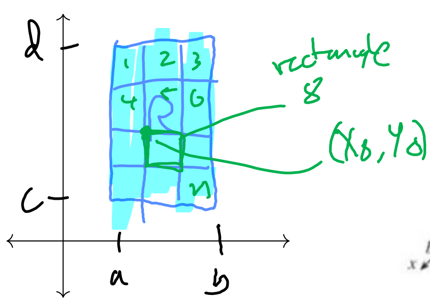
Pre-Lecture

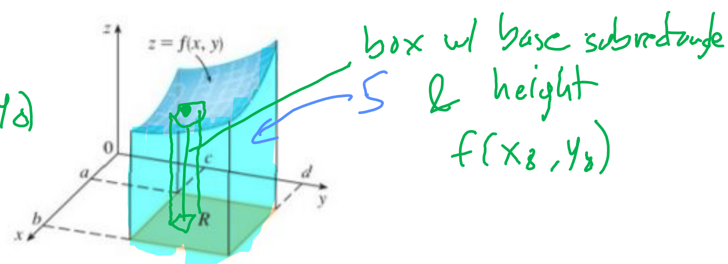
Section 15.1: Introduction to Double Integrals

Volumes and Double integrals. Let R be the closed rectangle defined below:

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, c \le y \le d\}$$

Let $f(x,y)$ be a function defined on R such that $f(x,y) \ge 0$. Let S be the solid that lies above R and under the graph f.





Question: How can we estimate the volume of S?

$$\text{volume}(S) = \sum_{k=1}^{n} f(x_k, y_k)\Delta x_k \Delta y_k$$

Definition 82. The <u>double integral</u> of $f(x,y)$ over a rectangle R is

integrand

$$\iint_R f(x,y)\, dA = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k,y_k)\Delta x_k\, \Delta y_k$$

region of integration $|P|$ = biggest area of a subrectangles

if this limit exists. ← if it exists, f is called <u>integrable</u> on R

- If f is cts on R, then f is integrable R
  - some discts f are integrable
- $\iint_R f(x,y)\, dA$ = <u>signed</u> volume between $z = f(x,y)$ & xy-plane above R.

Day 13 Lecture

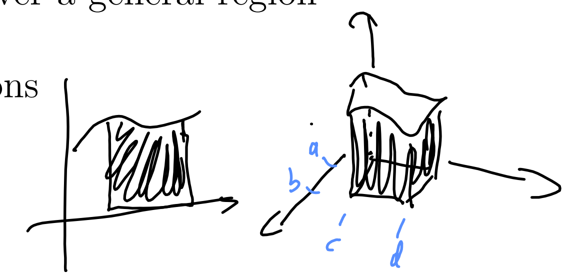
Daily Announcements & Reminders:

- HW 14.7 & 14.8 due tonight
- Quiz 4 Revision Assignment posted, due 11:59 pm 2/26
- Exam 2 is two weeks from today
- Do warmup problem on Ed →

Goals for Today:

Sections 15.1, 15.2

- Introduce double and iterated integrals for functions of two variables on rectangles
- Use Fubini's Theorem to change the order of integration of a iterated integral
- Be able to set up & evaluate a double integral over a general region
- Change the order of integration for general regions

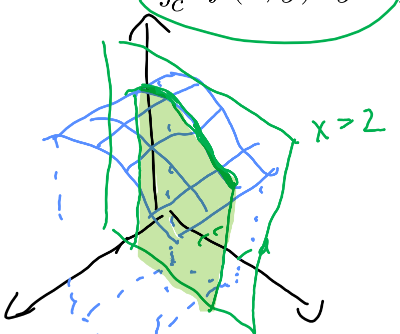


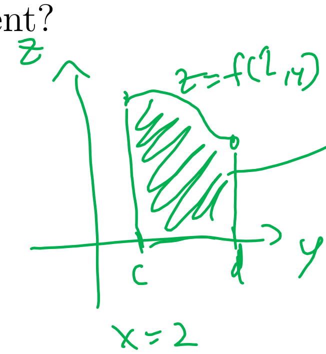
Question: How can we compute a double integral?

Answer: Iterated Integrals

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

What does $\int_c^d f(2, y)dy$ represent?





$\int_c^d f(2,y)dy$ is the cross-sectional area of the solid x=2

What about $\underbrace{\int_c^d f(x,y)dy}$?

treat x as a constant $\rightarrow$ $A(x)$ = cross-sectional area of this solid for each value x

Let $A(x) = \int_c^d f(x,y)dy$. Then,

$$\text{volume} = \int_a^b A(x)dx = \int_a^b \left(\int_c^d f(x,y)dy \right) dx \stackrel{?}{=} \int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

write Differentials

This is called an iterated integral.

Example 83. Evaluate $\int_1^2 \left(\int_3^4 6x^2y\, dy \right) dx$.

$$= \int_1^2 \frac{6x^2y^2}{2} \Big|_{y=3}^{y=4} dx$$

$$= \int_1^2 48x^2 - 27x^2\, dx$$

$$= \int_1^2 \underbrace{21x^2}_{A(x)} dx$$

$$= 7x^3\Big|_1^2 = 7(8-1) = \boxed{49}$$

$$\int_3^4 \underbrace{\int_1^2 6x^2y\, dx}\, dy$$

$$= \int_3^4 2yx^3 \Big|_{x=1}^{x=2} dy$$

$$= \int_3^4 2y(8-1)dy$$

$$= \int_3^4 14y\, dy$$

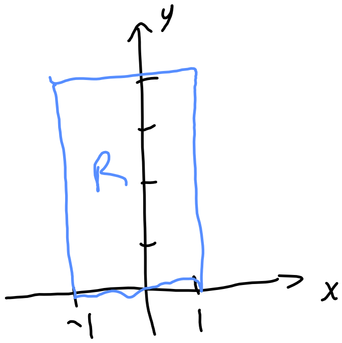
$$= 7y^2\Big|_3^4 = 7(16-9) = 49$$

Theorem 84 (Fubini's Theorem)**.** *If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then*

$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

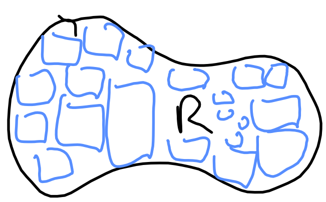
More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

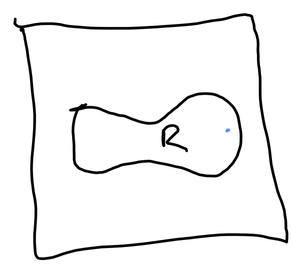
Example 85. Compute $\iint_R x e^{e^{e^y}}\, dA$, where R is the rectangle $[-1,1]\times[0,4]$.



$$V = \int_{-1}^{1}\int_0^4 x e^{e^{e^y}}\, dy\, dx$$
$$= \int_0^4 \int_{-1}^{1} x e^{e^{e^y}}\, dx\, dy$$
$$= \int_0^4 \frac{x^2}{2} e^{e^{e^y}} \Big|_{-1}^{1}\, dy$$
$$= \int_0^4 0\, dy$$
$$= 0$$

Question: What if the region R we wish to integrate over is not a rectangle?

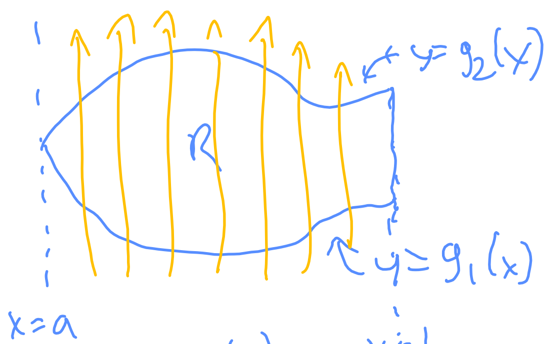




$$g = \begin{cases} f & \text{on } R \\ 0 & \text{on not } R \end{cases}$$

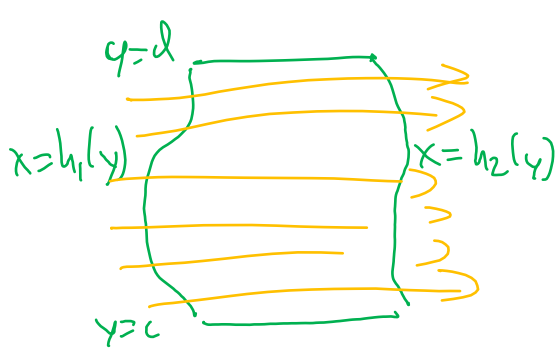
Answer: Repeat same procedure - it will work if the boundary of R is smooth and f is continuous.

Vertically Simple



$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

Horizontally Simple



$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Example 86. Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the xy-plane and whose top is $z = 2 - x - y$.

Vertically simple: Pick up here on Tuesday

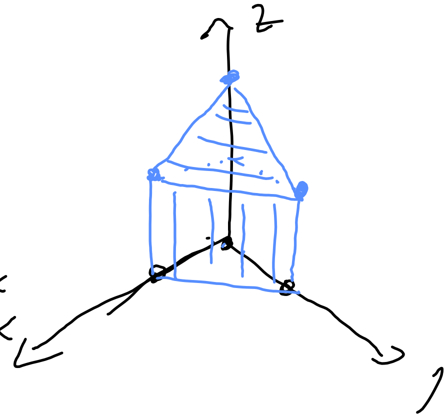
Horizontally simple:

Day 14 - Double Integrals, Area, Average Value

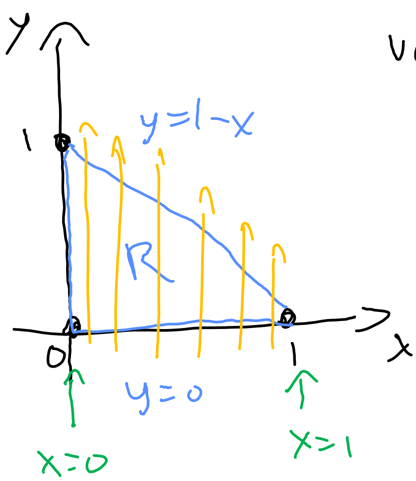
Pre-Lecture

Section 15.2: Double Integrals on General Regions

Example 86. Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the xy-plane and whose top is $z = 2 - x - y$.



Vertically simple:



$$\text{volume} = \iint_R 2-x-y \, dA$$

$$= \int_0^1 \int_0^{1-x} 2-x-y \, dy \, dx$$

$$= \int_0^1 2y - xy - \tfrac{1}{2}y^2 \Big|_0^{1-x} dx$$

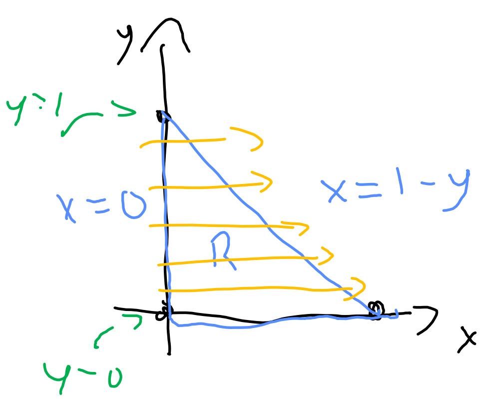
$$= \int_0^1 2(1-x) - x + x^2 - \tfrac{1}{2}(1-x)^2 \, dx$$

$$= -(1-x)^2 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{6}(1-x)^3 \Big|_0^1$$

$$= \left(0 - \tfrac{1}{2} + \tfrac{1}{3} + 0\right) - \left(-1 - 0 + 0 + \tfrac{1}{6}\right)$$

$$= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{6} = \boxed{\tfrac{2}{3}}$$

Horizontally simple:



$$\text{volume} = \iint_R 2-x-y \, dA$$

$$= \int_0^1 \int_0^{1-y} 2-x-y \, dx \, dy$$

$$= \boxed{\tfrac{2}{3}}$$

must be constants!

may have variables other than the one being integrated

Day 14 Lecture

Daily Announcements & Reminders:

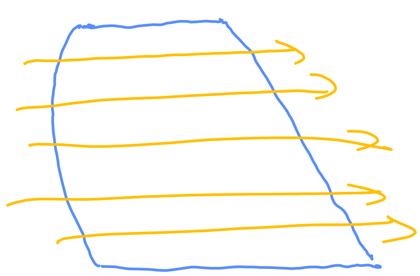
- HW 15.1 due tonight
- Quiz 6 tomorrow in studio, 14.7/8 & 15.1
  - L.O. D4, I1, I2
  - one optimization, one double integral q
- Quiz 4 Revision due tomorrow night
- Do $\underline{\underline{2}}$ warmup polls on Ed
- TA office hour → F 2-3 pm in Stiles 218C

Goals for Today:

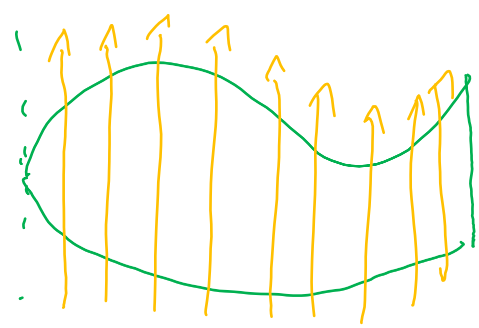
Sections 15.2, 15.3

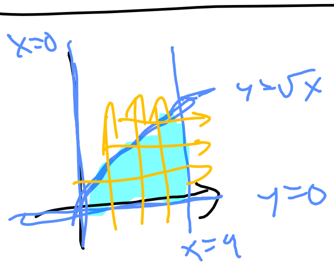
- Be able to set up & evaluate a double integral over a general region
- Change the order of integration for general regions
- Compute areas of general regions in the plane
- Compute the average value of a function of two variables

Horizontally simple: $dx\,dy$

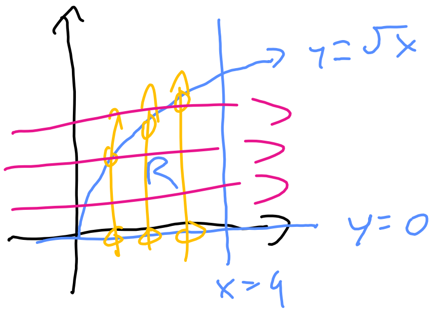


Vertically simple $dy\,dx$





Example 87. Write the two iterated integrals for $\iint_R 1\ dA$ for the region R which is bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.



$$\int_{y=0}^{y=3}\int_{x=y^2}^{x=9} 1\ dx\ dy$$

$$= \int_{x=0}^{x=9}\int_{y=0}^{y=\sqrt{x}} 1\ dy\ dx$$

$$= \iint_R 1\ dA$$

$0 \le x \le y^2$

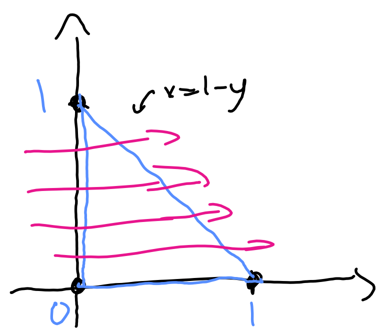
$y^2 \le x \le 9$

$0 \le x \le 9$

Example 88. [Poll] Set up an iterated integral for

$$\iint_R f(x,y)\ dA,$$

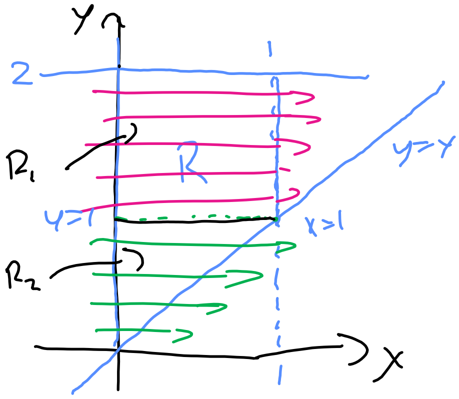
where R is the triangle with vertices $(0,0), (1,0)$, and $(0,1)$, treating the region as horizontally simple.



$$\int_0^1\int_0^{1-y} f(x,y)\ dx\ dy$$

$\int_0^1\int_0^{1-x} f(x,y)\ dy\ dx$ ← vertical slices, same value

Example 89. Set up an iterated integral to evaluate the double integral $\iint_R 6x^2y\,dA$, where R is the region bounded by $x = 0, x = 1, y = 2$, and $y = x$.



R is vertically simple, not horiz. simple

Vertically simple

$$\iint_R 6x^2y\,dA = \int_0^1 \int_x^2 6x^2y\,dy\,dx$$

Horizontal?

$$\iint_R 6x^2y\,dA = \iint_{R_1} 6x^2y\,dA + \iint_{R_2} 6x^2y\,dA$$

$$= \int_1^2 \int_0^1 6x^2y\,dx\,dy + \int_0^1 \int_0^y 6x^2y\,dx\,dy$$

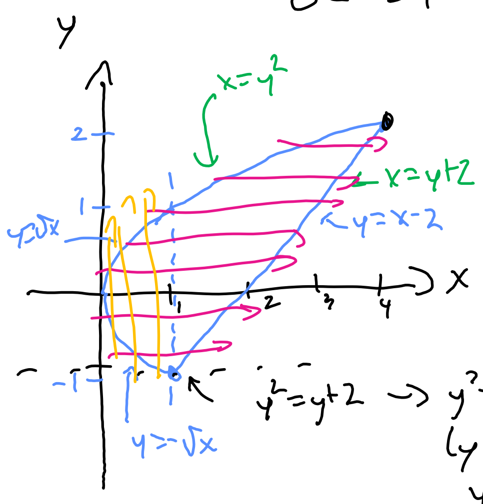
Example 90. [Exam] Sketch the region of integration for the integral expression

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\;dy\;dx + \int_1^4 \int_{x-2}^{\sqrt{x}} f(x,y)\;dy\;dx.$$

Then write an equivalent iterated integral expression in the order $dx\;dy$.

$-\sqrt{x} \leq y \leq \sqrt{x}$
$0 \leq x \leq 1$

$x-2 \leq y \leq \sqrt{x}$
$1 \leq x \leq 4$



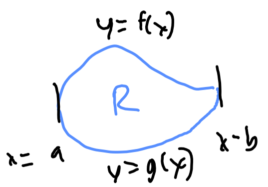
$$= \int_{-1}^{2} \int_{y^2}^{y+2} f(x,y)\,dx\,dy$$

$y^2 = y+2 \rightarrow y^2 - y - 2 = 0$
$(y-2)(y+1) = 0$
$y=2 \quad y=-1$

15.3 - Area and Average Value

Two other applications of double integrals are computing the area of a region in the plane and finding the average value of a function over some domain.

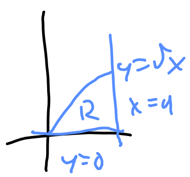
Area: If R is a region bounded by smooth curves, then



$$\text{Area}(R) = \underline{\iint_R 1 \, dA}$$

$$\int_a^b \int_{g(x)}^{f(x)} 1 \, dy \, dx$$
$$= \int_a^b f(x) - g(x) \, dx$$

Example 91. [Poll] Find the area of the region R bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.



$$\text{Area} = \iint_R 1 \, dA$$
$$= \int_0^9 \int_0^{\sqrt{x}} 1 \, dy \, dx \quad \text{or} \quad \int_0^3 \int_{y^2}^9 1 \, dx \, dy$$
$$= 18$$

$$= \int_0^9 \sqrt{x} \, dx$$
$$= \frac{2}{3} x^{3/2} \Big|_0^9 = \frac{2}{3} \cdot 9^{3/2}$$
$$= \frac{2}{3} \cdot 27 = 18$$

Average Value: The average value of $f(x, y)$ on a region R contained in $\mathbb{R}^2$ is

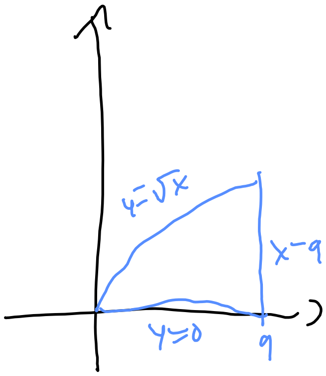
$f(x)$ on $[a, b]$

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x) \, dx$$

$$f_{avg} = \underline{\frac{1}{\text{Area}(R)} \cdot \iint_R f(x,y) \, dA}$$

Example 92. Find the average temperature on the region R in the previous example if the temperature at each point is given by $T(x, y) = 4xy^2$. °C

$R: \ 0 \le y \le \sqrt{x}, \ 0 \le x \le 9$



$$\text{Area}(R) = 18$$

$$T_{avg} = \frac{1}{\text{Area}(R)} \cdot \iint_R T(x,y)\,dA$$

$$= \frac{1}{18} \cdot \int_0^9 \int_0^{\sqrt{x}} 4xy^2 \, dy \, dx$$

$$= \frac{1}{18} \int_0^9 \frac{4}{3} x y^3 \Big|_{y=0}^{y=\sqrt{x}} dx$$

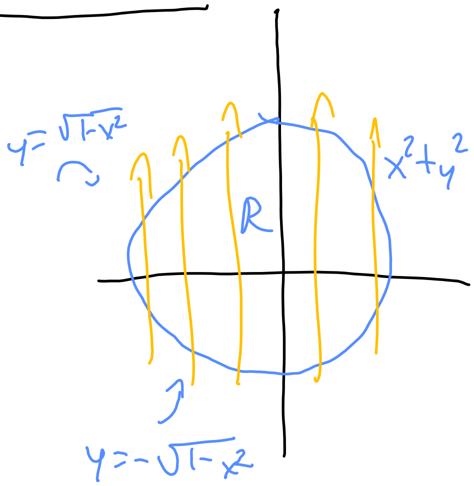
$$= \frac{1}{18} \int_0^9 \frac{4}{3} x^{5/2} \, dx$$

$$= \frac{1}{18} \cdot \frac{4}{3} \cdot \frac{2}{7} \cdot x^{7/2} \Big|_0^9$$

$$= \boxed{\frac{1}{18} \cdot \frac{4}{3} \cdot \frac{2}{7} \cdot 3^7}$$

$$= \boxed{\frac{324}{7}} \approx 46.3^\circ C$$

Next time:



Area(R)

$$= \iint_R 1\, dA$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dy\, dx$$

$$= \int_{-1}^{1} 2\sqrt{1-x^2}\, dx$$

Trig sub!

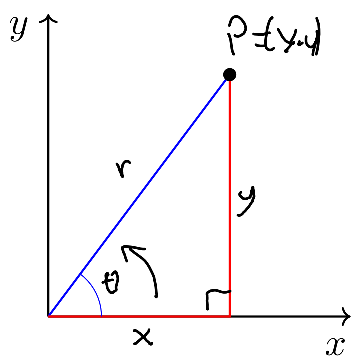
Next time, learn how to avoid!

Day 15 - Polar Integration & Triple Integrals

Pre-Lecture

Polar Coordinates

Polar Coordinates:



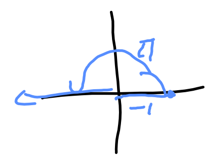
Cartesian coordinates: Give the distances in $\hat{i}$ and $\hat{j}$ directions from $(0,0)$

Polar coordinates: (r,θ)

- r = distance from $(0,0)$ to $P=(x,y)$
- θ = angle between the ray $\overrightarrow{OP}$ and the positive x-axis, measured CCW

Polar to Cartesian: $(r,\theta) \longrightarrow (x,y)$

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$



Cartesian to Polar: $(x,y) \to$ many possible (r,θ) ; $(x,y)=(1,0)$

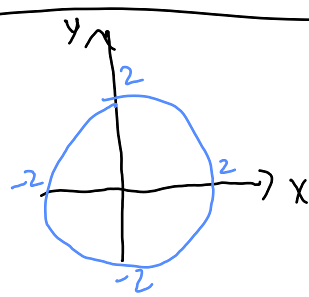
$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$

$\hookrightarrow (r,\theta) = (1,0)$
or
$(r,\theta) = (1, 2\pi)$, $(1, 4\pi)$
or $(r,\theta) = (-1, \pi)$ X

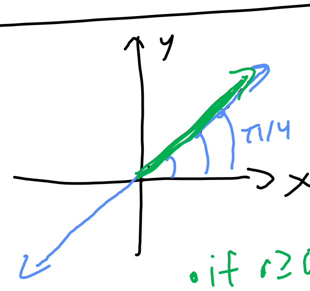
In problems: $r \geq 0$, $0 \leq \theta \leq 2\pi$
for other interval of length 2π $(-\pi \leq \theta \leq \pi)$

Example 92. Find a set of polar coordinates for the point $(x,y) = (-1,1)$. Graph the set of points (x,y) that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the** xy**-plane.**

$(x,y) = (-1,1)$: $r^2 = (-1)^2 + 1^2 = 2$, so $r = \sqrt{2}$
$\tan\theta = \frac{1}{-1} = -1$, $\theta = \frac{3\pi}{4}$



$r = 2$
$\downarrow$
$r^2 = 4$
$\downarrow$
$x^2 + y^2 = 4$



$\theta = \pi/4$
$\tan\theta = 1$
$\frac{y}{x} = 1$
$y = x$

• if $r \geq 0$, only get RH ray

Day 15 Lecture

Daily Announcements & Reminders:

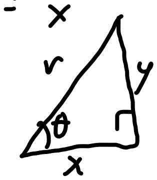
Goals for Today:

Sections 15.4, 15.5

- Introduce the polar coordinate system
- Convert double integrals to iterated polar integrals
- Compute iterated polar integrals
- Define triple integrals and compute basic triple integrals

$x = r\cos\theta$
$y = r\sin\theta$

$x^2+y^2 = r^2$
$\tan\theta = \frac{y}{x}$



Example 94.

a) Write the function $f(x, y) = \sqrt{x^2 + y^2}$ in polar coordinates.

$$\sqrt{x^2+y^2} = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$
$$= \sqrt{r^2(\cancel{\cos^2\theta + \sin^2\theta})}$$
$$= r \quad \text{(b/c we take } r \geq 0)$$

$\sqrt{x^2+y^2} = \sqrt{r^2} = r$

$f(r,\theta) = r$

b) [Poll] Write a Cartesian equation describing the points that satisfy $r = 2\sin(\theta)$.

$2\sin^2\left(\arctan\left(\frac{y}{x}\right)\right) = y$ • circle plus x-axis

$r \cdot \quad r = 2\sin(\theta) \cdot r$

$r^2 = 2r\sin(\theta)$

$x^2 + y^2 = 2y$

$x^2 + y^2 - 2y + 1 = 0 + 1$

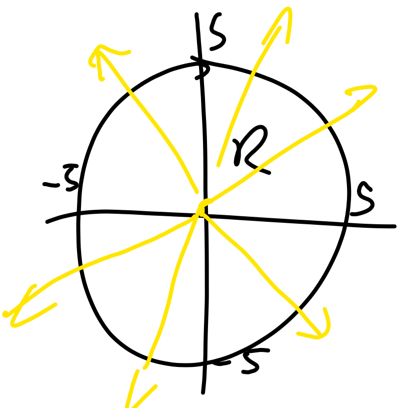
take half of -2 and square and add

$x^2 + (y-1)^2 = 1$

15.4: Double Integrals in Polar Coordinates

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r, \theta)$ on R, compute $\iint_R f(r, \theta)\, dA$.

Example 95. Compute the area of the disk of radius 5 centered at $(0, 0)$. 25π



$0 \le r \le 5$

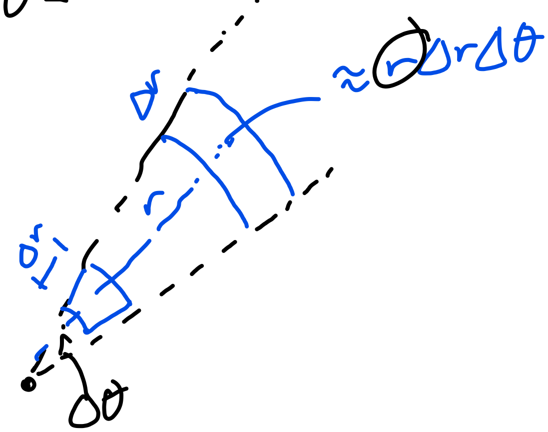
$0 \le \theta \le 2\pi$

$$A = \iint_R 1\, dA$$

$$= \int_0^{2\pi}\int_0^5 1\, dr\, d\theta$$

$$= \int_0^{2\pi} 5\, d\theta$$

$$= 10\pi$$ wrong



instead

$$A = \int_0^{2\pi}\int_0^5 1 \cdot \underbrace{r\, dr\, d\theta}_{dA}$$

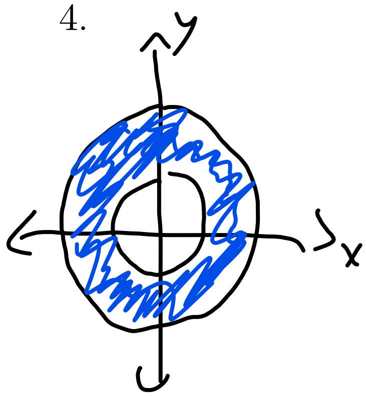
$$= \int_0^{2\pi} \frac{25}{2}\, d\theta$$

$$= 25\pi$$

Remember: In polar coordinates, the area form $dA =$ $r\, dr\, d\theta$

Example 96. Compute $\iint_D e^{-(x^2+y^2)}\,dA$ on the washer-shaped region $1 \le x^2+y^2 \le 4$.

D



$1 \le r^2 \le 4$

$1 \le r \le 2$

$0 \le \theta \le 2\pi$

$$\iint_D e^{-(x^2+y^2)}\,dA$$

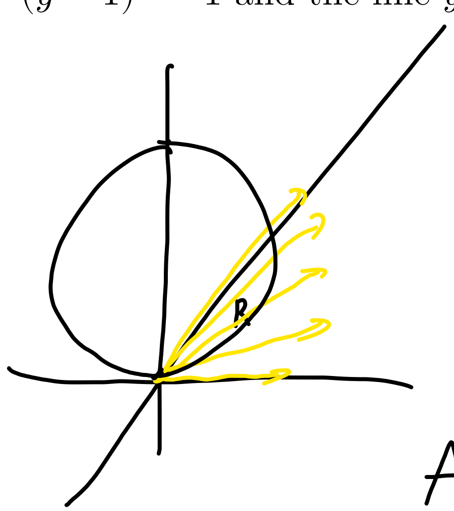
$$= \int_0^{2\pi}\int_1^2 e^{-r^2}\cdot r\,dr\,d\theta \qquad u=-r^2$$

$$= \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_1^2\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}e^{-4}+\frac{1}{2}e^{-1}\,d\theta$$

$$= \boxed{\pi\left(e^{-1}-e^{-4}\right)}$$

Example 97. Compute the area of the smaller region bounded by the circle $x^2+(y-1)^2=1$ and the line $y=x$.



$x^2+(y-1)^2=1 \to r=2\sin\theta$

$y=x \to \theta=\pi/4$

$r\sin\theta = r\cos\theta$

$\tan\theta=1$

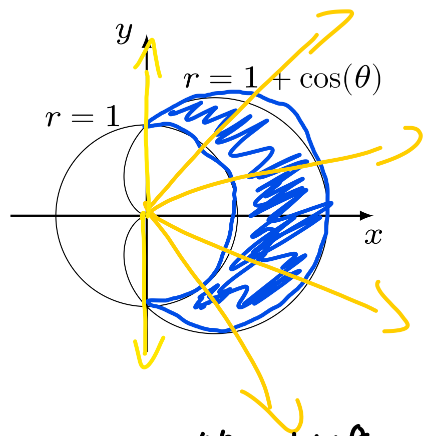
$$A=\int_0^{\pi/4}\int_0^{2\sin\theta} 1\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/4} 2\sin^2\theta\,d\theta$$

$\sin^2\theta=\frac{1}{2}(1-\cos(2\theta))$

$\cos^2\theta=\frac{1}{2}(1+\cos(2\theta))$

Example 98 (Poll)**.** Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



$$\iint_R x \, dA = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} r\cos\theta \cdot r \, dr \, d\theta$$

$$= \int_{3\pi/2}^{5\pi/2} \int_1^{1+\cos\theta} r^2 \cos\theta \, dr \, d\theta$$

15.5 & 15.6 Triple Integrals & Applications

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k.

Taking the limit gives a

$$\rule{6cm}{0.4pt} : \iiint_D f(x, y, z)\ dV$$

Important special case:

$$\iiint_D 1\ dV = \rule{5cm}{0.4pt}$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

Example 99. Compute $\displaystyle\int_0^1 \int_0^2 \int_0^3\ dz\ dy\ dx$ and interpret your answer.

Other important spatial applications:

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta \, dV$ $\quad \delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x \, \delta \, dV, \qquad M_{xz} = \iiint_D y \, \delta \, dV, \qquad M_{xy} = \iiint_D z \, \delta \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta \, dA$ $\quad \delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint_R x \, \delta \, dA, \qquad M_x = \iint_R y \, \delta \, dA$

Center of mass: $\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$

Day 16 - Triple Integrals & More Applications

Pre-Lecture

Section 15.5/6: Triple Integrals and Spatial Applications

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k.

Taking the limit as the volume of the prisms goes to zero gives a

triple integral : $\iiint_D \underbrace{f(x, y, z)}_{\text{integrand}}\ \underbrace{dV}_{\text{volume element}}$

D: region of integration (in $\mathbb{R}^3$)

Important special case:

$$\iiint_D 1\ dV = \text{volume of } D$$

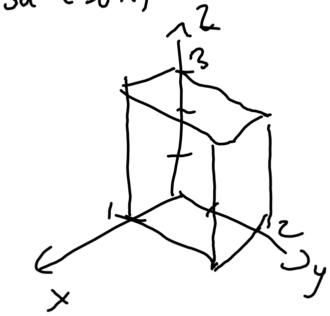
Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals. If $f(x,y,z)$ is cts, then

$$\iiint_D f\ dV = \int_a^b \int_c^d \int_e^f f\ dx\, dy\, dz = \int_a^b \int_e^f \int_c^d f\ dy\, dx\, dz$$
$$= \int_c^d \int_e^f \int_a^b f\ dz\, dx\, dy = \dots$$

Example 99. Compute $\int_0^1 \int_0^2 \int_0^3 1 \, dz\, dy\, dx$ and interpret your answer.

$$= \int_0^1 \int_0^2 z\Big|_0^3 \, dy\, dx$$
$$= \int_0^1 \int_0^2 3 \, dy\, dx$$
$$= \int_0^1 3y\Big|_0^2 \, dx$$
$$= \int_0^1 6 \, dx = 6x\Big|_0^1 = \boxed{6}$$

volume of the region (box)
$0 \leq z \leq 3$
$0 \leq y \leq 2$
$0 \leq x \leq 1$
is 6



Other important spatial applications:

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta \, dV$ $\quad \delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes: • tendency to rotate about the reference

$$M_{yz} = \iiint_D x\,\delta \, dV, \quad M_{xz} = \iiint_D y\,\delta \, dV, \quad M_{xy} = \iiint_D z\,\delta \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta \, dA$ $\quad \delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint_R x\,\delta \, dA, \quad M_x = \iint_R y\,\delta \, dA$

Center of mass: $\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$

Day 16 Lecture

Daily Announcements & Reminders:

- HW 15.4 due tonight, Deriv Review R
- Exam 2 on R, see Canvas
- Do warmup poll on Ed

Goals for Today:

Sections 15.5, 15.6

- Learn how to write triple integrals as iterated integrals.
- Compute triple iterated integrals
- Change the order of integration in a triple iterated integral.
- Apply our work to find the mass and center of mass of objects in $\mathbb{R}^2$ and $\mathbb{R}^3$

Warmup: mass $= \iiint_D \delta(x,y,z)\, dV$

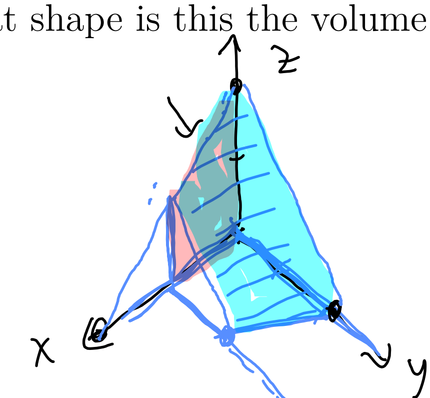
Example 100. Let's practice with triple integrals.

1. **Mechanics:** Compute $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} 1 \, dz\, dy\, dx$.

$$= \int_0^1 \int_0^{2-x} z \Big|_0^{2-x-y} dy\, dx$$
$$= \int_0^1 \int_0^{2-x} 2-x-y \, dy\, dx$$
$$= \int_0^1 (2-x)y - \tfrac{1}{2}y^2 \Big|_0^{2-x} dx$$
$$= \int_0^1 (2-x)^2 - \tfrac{1}{2}(2-x)^2 \, dx$$
$$= \int_0^1 \tfrac{1}{2}(2-x)^2 \, dx$$
$$= \tfrac{1}{6}(2-x)^3 \cdot (-1) \Big|_0^1$$
$$= -\tfrac{1}{6}(1)^3 + \tfrac{1}{6}(2)^3$$
$$= \boxed{\tfrac{7}{6}}$$

2. **Interpretation:** What shape is this the volume of?

$0 \le z \le 2-x-y$
$0 \le y \le 2-x$
$0 \le x \le 1$

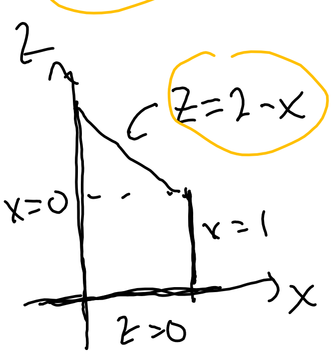


$z = 2-x-y$
$x+y+z=2$
$y = 2-x-z$

3. **Rearrange:** Write an equivalent iterated integral in the order $dy\, dz\, dx$.

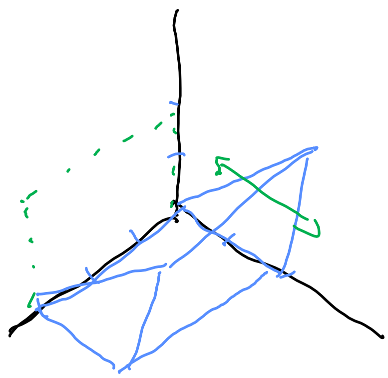
$0 \le y \le 2-x-z$
$0 \le z \le 2-x$
$0 \le x \le 1$

$$\int_0^1 \int_0^{2-x} \int_0^{2-x-z} 1 \, dy\, dz\, dx$$



where these meet

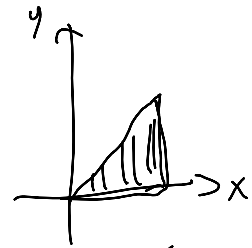
$$\int_0^1 \int_0^1 \int_0^{2-x-z} dy\, dx\, dz$$
$$+ \int_1^2 \int_0^{2-z} \int_0^{2-x-z} dy\, dx\, dz$$



$z \le y \le 2$

$0 \le z \le 2$

$0 \le x \le 3$



$0 \le y \le x$

$0 \le x \le 2$

$\Longleftrightarrow \; y \le x \le 2$

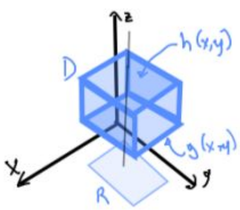
$0 \le y \le 2$

We will think about converting triple integrals to iterated integrals in terms of the shadow of D on one of the coordinate planes.

Case 1: z**-simple**) region. If R is the shadow of D on the xy-plane and D is bounded above and below by the surfaces $z = h(x, y)$ and $z = g(x, y)$, then

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\, dz \right) dy\, dx$$

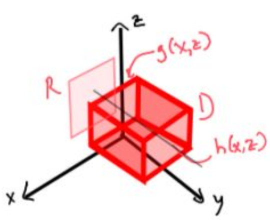
or $dx\,dy$



Case 2: y**-simple**) region. If R is the shadow of D on the xz-plane and D is bounded right and left by the surfaces $y = h(x, z)$ and $y = g(x, z)$, then

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(x,z)}^{h(x,z)} f(x,y,z)\, dy \right) dz\, dx$$

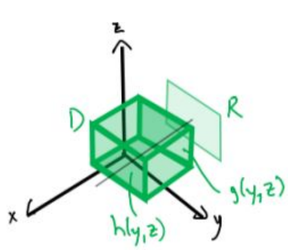
or $dx\,dz$



Case 3: x**-simple**) region. If R is the shadow of D on the yz-plane and D is bounded front and back by the surfaces $x = h(y, z)$ and $x = g(y, z)$, then

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(y,z)}^{h(y,z)} f(x,y,z)\, dx \right) dz\, dy$$

or $dy\,dz$



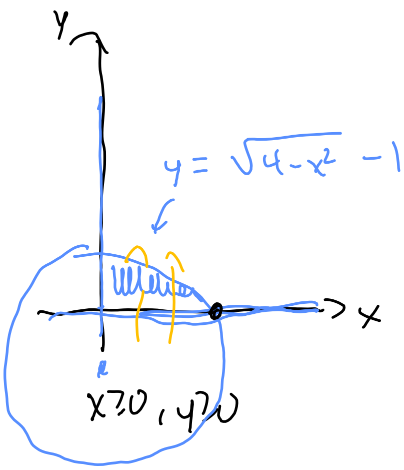
$x \ge 0,\ y \ge 0,\ z \ge 0$

Example 101. Write an integral for the mass of the solid D in the first octant with $2y \le z \le 3 - x^2 - y^2$ with density $\delta(x, y, z) = x^2y + 0.1$ by treating the solid as a) z-simple and b) x-simple. Is the solid also y-simple?

a) z-simple:

$$\text{mass} = \iiint_D \delta \, dV = \int_0^{\sqrt{3}} \int_0^{\sqrt{4-x^2}-1} \int_{2y}^{3-x^2-y^2} x^2y + 0.1 \, dz \, dy \, dx$$

Draw shadow R in xy-plane



eliminate z from inequality:

$$2y \le 3 - x^2 - y^2$$
$$x^2 + y^2 + 2y \le 3$$
$$x^2 + y^2 + 2y + 1 \le 4$$
$$x^2 + (y+1)^2 \le 4$$

$$x^2 + 1^2 = 4$$
$$x = \sqrt{3}$$

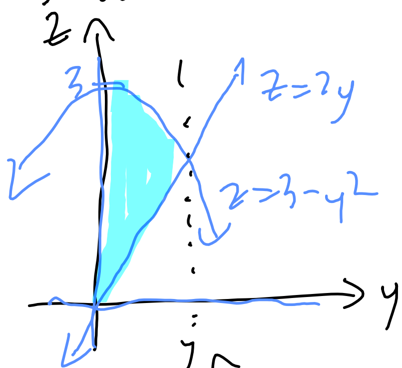
b) x-simple:

$$\text{mass} = \int_0^1 \int_{2y}^{3-y^2} \int_0^{\sqrt{3-y^2-z}} x^2y + 0.1 \, dx \, dz \, dy$$

top x:

$$z = 3 - x^2 - y^2$$
$$x^2 = 3 - y^2 - z$$
$$x = \sqrt{3 - y^2 - z}$$

shadow in yz



$y \ge 0 \quad z \ge 2y$

$z \ge 0$

$$0 \le \sqrt{3 - y^2 - z}$$
$$3 - y^2 - z \ge 0$$
$$z \le 3 - y^2$$

$$3 - y^2 = 2y \to y^2 + 2y - 3 = 0$$
$$(y+3)(y-1) = 0$$

Example 101 (cont.)

Rules for Triple Integrals for the Sketching Impaired (credit to Wm. Douglas Withers)

Basic

Rule 1: Choose a variable appearing exactly twice for the next integral.

Rule 2: After setting up an integral, cross out any constraints involving the variable just used.

Rule 3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

Intermediate

Rule 4: A square variable counts twice.

Inside of a $\sqrt{\ }$ must be nonnegative.

Rule 5: ~~The region of integration of the next step must lie within the domain of any function used in previous limits.~~

Rule 6: If you do not know which is the upper limit and which is the lower, take a guess - but be prepared to backtrack.

Advanced

Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Example 102. Set up an integral for the volume of the region D defined by

$$\cancel{x + y^2 \leq 8}, \quad \cancel{y^2 + 2z^2 \leq x}, \quad y \geq 0$$

5y

Rule 4

Rule 1: x : 2 times y : 5 times z : 2 times

x first:

$y^2 + 2z^2 \leq x \leq 8 - y^2$

$$\boxed{\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{y^2+2z^2}^{8-y^2} 1 \, dx \, dz \, dy}$$

Rule 2: Cross out used constraints

Rule 3: $\cancel{y^2+2z^2 \le 8-y^2}$ $\quad y\ge 0$

$2y^2+2z^2 \le 8$

$y^2+z^2 \le 4, \quad y \ge 0$

Rule 1: Pick z: $\quad z^2 \le 4-y^2$

$-\sqrt{4-y^2} \le z \le \sqrt{4-y^2}$

Rule 2/3: $0 \le \sqrt{4-y^2}, \quad y \ge 0$

Rule 5: $4-y^2 \ge 0 \quad 4 \ge y^2$

$2 \ge y$

Example 103. Set up a triple iterated integral for the triple integral of $f(x,y,z) = x^3y$ over the region D bounded by

$$x^2+y^2=1, \quad \cancel{z=0}, \quad \cancel{x+y+z=2}.$$

Post class: Rule 1/4: x - 3 times, y - 3 times, z - 2 times

Choose z first

$z=0$ & $z=2-x-y$

Rule 6: guess $0 \le z \le 2-x-y$

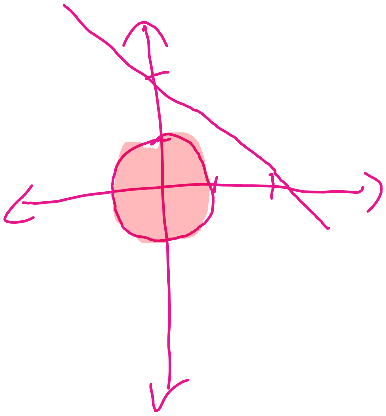
Rule 2: Cross out used constraints

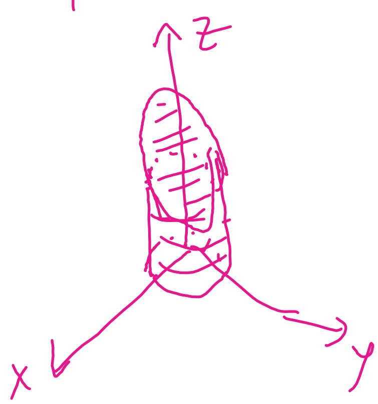
Rule 3: $0 \le 2-x-y$

$y \le 2-x$

& $x^2+y^2=1$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{2-x-y} x^3y \, dz\, dy\, dx$$



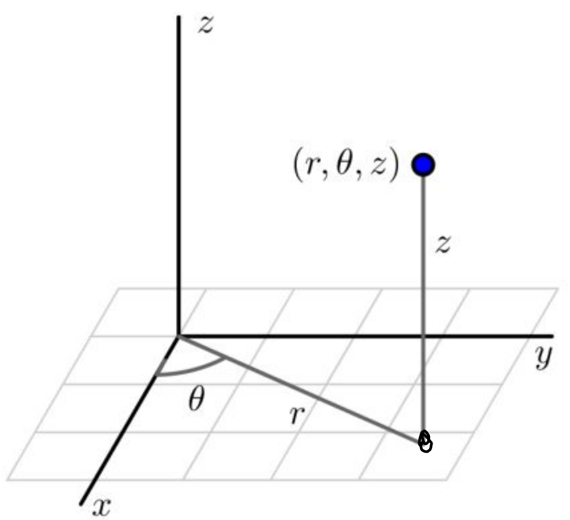


Day 17 - Triple Integrals in Cylindrical & Spherical Coordinates

Pre-Lecture

Section 15.7: Cylindrical Coordinates

Cylindrical Coordinate System



Cylindrical to Cartesian:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

Cartesian to Cylindrical:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$$

For uniqueness:

$r \geq 0$, θ in an interval of length 2π
e.g. $[0, 2\pi]$, $[-\pi, \pi]$

Example 104.

a) Find cylindrical coordinates for the point with Cartesian coordinates $(-1, \sqrt{3}, 3)$.

$x \quad y \quad z$

$r^2 = (-1)^2 + (\sqrt{3})^2 = 4$ $\qquad \tan\theta = \frac{\sqrt{3}}{-1}$

$\boxed{r = 2 \qquad \theta = \frac{4\pi}{3} \qquad z = 3}$

b) Find Cartesian coordinates for the point with cylindrical coordinates $(2, 5\pi/4, 1)$.

r, θ, z

$x = 2\cos(5\pi/4) = -\frac{2}{\sqrt{2}} = -\sqrt{2}$

$y = 2\sin(5\pi/4) = -2/\sqrt{2} = -\sqrt{2}$

$z = 1$

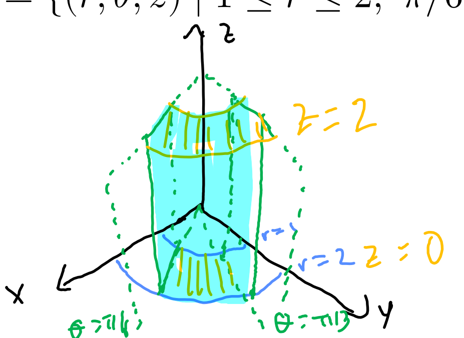
Example 105. In xyz-space sketch the *cylindrical box*

$$B = \{(r, \theta, z) \mid 1 \leq r \leq 2,\ \pi/6 \leq \theta \leq \pi/3,\ 0 \leq z \leq 2\}.$$

• $r = c$ is a circular cylinder around z-axis with radius c

• $\theta = c$ → half vertical plane through z-axis

• $z = c$ horizontal planes



Day 17 Lecture

Daily Announcements & Reminders:

- HW 15.5 due tonight
- Quiz 7 in studio tomorrow: 15.5, 6, 7
  L.O. I1, I3, I4
- Do warmup on Ed $\longrightarrow$

Goals for Today:

Section 15.7

- Be able to convert between Cartesian, cylindrical, and spherical coordinate systems in $\mathbb{R}^3$
- Compute triple integrals expressed in cylindrical coordinates
- Compute triple integrals expressed in spherical coordinates

Triple Integrals in Cylindrical Coordinates

$(r, \theta, z) \to (x, y, z)$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

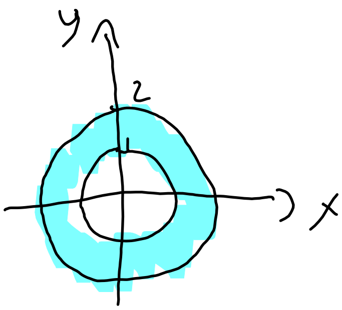
We have $dV =$ $r\,dz\,dr\,d\theta$

or $r\,dr\,dz\,d\theta$

Example 106. Set up a iterated integral in cylindrical coordinates for the volume of the region D lying below $z = x+2$, above the xy-plane, and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$V = \iiint_D 1\, dV = \int_0^{2\pi} \int_1^2 \int_0^{r\cos\theta + 2} 1 \; r\, dz\, dr\, d\theta$$

Shadow:

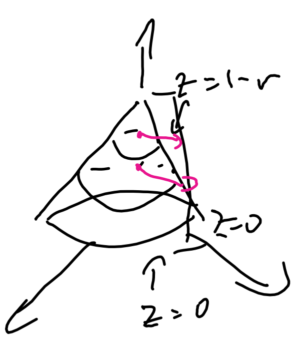


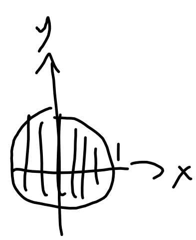
Example 107 (Poll)**.** Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.

$$\text{mass} = \iiint_D z\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{1-r} z r\, dz\, dr\, d\theta$$

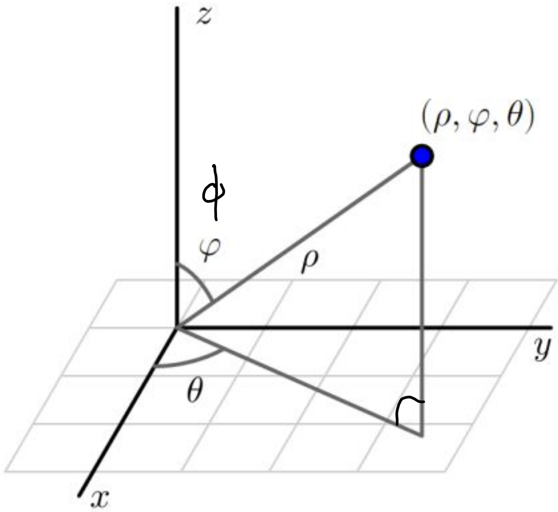
$$= \int_0^{2\pi} \int_0^1 \int_0^{1-z} z r\, dr\, dz\, d\theta$$





$0 = 1 - r \Rightarrow r = 1$

Spherical Coordinate System



For uniqueness:

$\rho \geq 0$, $0 \leq \varphi \leq \pi$, θ in interval of length 2π

$[0, 2\pi]$ or $[-\pi, \pi]$

Spherical to Cartesian:

$$x = \overbrace{\rho \sin(\varphi)}^{r} \cos(\theta)$$
$$y = \rho \sin(\varphi) \sin(\theta)$$
$$z = \rho \cos(\varphi)$$

Cartesian to Spherical:

$$\rho^2 = x^2 + y^2 + z^2$$
$$\tan(\theta) = \frac{y}{x}$$
$$\tan(\varphi) = \frac{\sqrt{x^2 + y^2}}{z}$$

Example 108.

a)Find spherical coordinates for the point with Cartesian coordinates $(-2, 2, \sqrt{8})$.

$\rho^2 = 4+4+8=16 \rightarrow \rho = 4$

$\tan \varphi = \frac{\sqrt{4+4}}{\sqrt{8}} = 1 \rightarrow \varphi = \pi/4$

$\tan \theta = \frac{2}{-2} = -1 \rightarrow \theta = 3\pi/4$

b)Find Cartesian coordinates for the point with spherical coordinates $(2, \pi/2, \pi/3)$.

$\rho \quad \varphi \quad \theta$

$x = 2 \sin \frac{\pi}{2} \cos \frac{\pi}{3} = 1$

$y = 2 \sin \frac{\pi}{2} \sin \frac{\pi}{3} = \sqrt{3}$

$z = 2 \cos \frac{\pi}{2} = 0$

Example 109. In xyz-space sketch the *spherical box*

$$B = \{(\rho, \varphi, \theta) \mid \underbrace{1 \leq \rho \leq 2}, \; 0 \leq \varphi \leq \pi/4, \; \pi/6 \leq \theta \leq \pi/3\}.$$

sphere of radius 1

sphere of radius 2

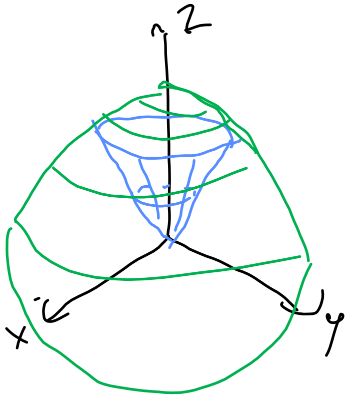
vertical half-planes through z-axis

Triple Integrals in Spherical Coordinates

We have $dV = \rho^2 \sin\varphi \; d\rho \, d\varphi \, d\theta$

Example 110. Write an iterated integral for the volume of the "ice cream cone" D bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the cone $z = \sqrt{3}\sqrt{x^2 + y^2}$.

$x^2+y^2+z^2=1 \Rightarrow \rho^2=1 \Rightarrow \boxed{\rho=1}$



$z=\sqrt{3}\sqrt{x^2+y^2}$ Note: $x^2+y^2=r^2=\rho^2\sin^2\varphi$

$\cancel{\rho}\cos\varphi=\sqrt{3}\cdot\cancel{\rho}\sin\varphi$

$\frac{1}{\sqrt{3}}=\tan\varphi$

$\boxed{\varphi=\pi/6}$

$$V=\int_0^{2\pi}\int_0^{\pi/6}\int_0^1 \rho^2\sin\varphi \, d\rho \, d\varphi \, d\theta$$

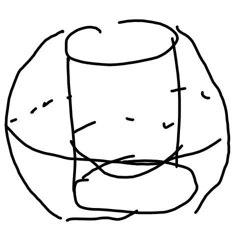
Example 111 (Poll)**.** Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

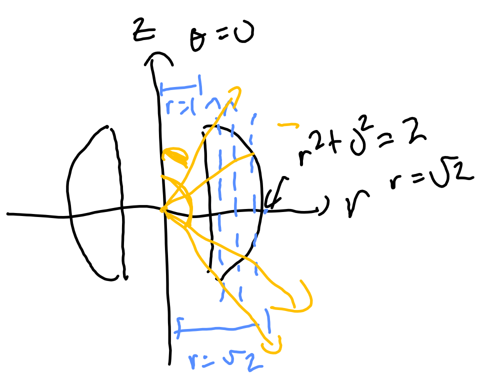
In cylindrical: $r^2+z^2=2$, $r^2=1 \to r=1$

$$V=\int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta$$

$z^2=2-r^2$

$z=\pm\sqrt{2-r^2}$





In spherical: $\rho^2 = 2 \Rightarrow \rho = \sqrt{2}$ $\qquad$ $\rho^2 \sin^2 \varphi = 1 \Rightarrow \rho = \csc\varphi$

$$V = \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_{\csc\varphi}^{\sqrt{2}} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

intersection of sphere & cylinder: $\sqrt{2} = \csc\varphi$

$\sin\varphi = \frac{1}{\sqrt{2}} \Rightarrow \varphi = \pi/4, 3\pi/4$

Question: When should we use these coordinate systems?

- Cylindrical coordinates:
  - cylinders, spheres, paraboloids, horiz/vert planes, cones
- Spherical coordinates:
  - spheres, vert planes, cones
  - cylinders are ok
  - paraboloids are hard

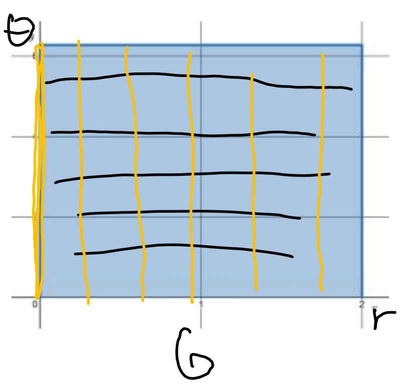
Day 18 - Change of Variables in Multiple Integrals

Pre-Lecture

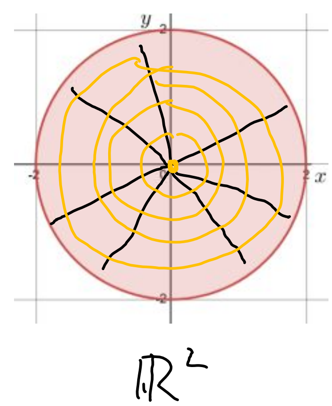
Section 15.8: Coordinate Transformations

We have seen three examples of changing variables in multiple integrals so far: the polar, cylindrical, and spherical coordinate systems.

For example, the change of coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ transforms the rectangle $0 \le r \le 2, 0 \le \theta \le 2\pi$ to the disk $x^2 + y^2 \le 2$:



$$\vec{T}(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$



Goal: Generalize this idea.

Definition 112. A **coordinate transformation** of a region $G \subseteq \mathbb{R}^n$ is a map $\mathbf{T} : G \to \mathbb{R}^n$ which is invertible and differentiable in the interior of G.

Ex: 1) polar transformation: $\vec{T}(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$

2) Cylindrical / Spherical:

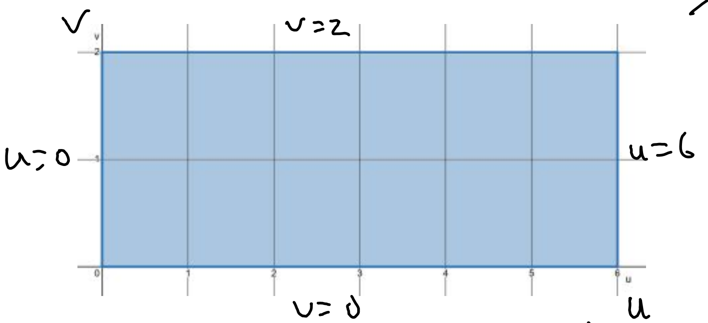
$$\vec{T}_c(r,\theta,z) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix} \qquad \vec{T}_s(\rho,\varphi,\theta) = \begin{bmatrix} \rho\sin\varphi\cos\theta \\ \rho\sin\varphi\sin\theta \\ \rho\cos\varphi \end{bmatrix}$$

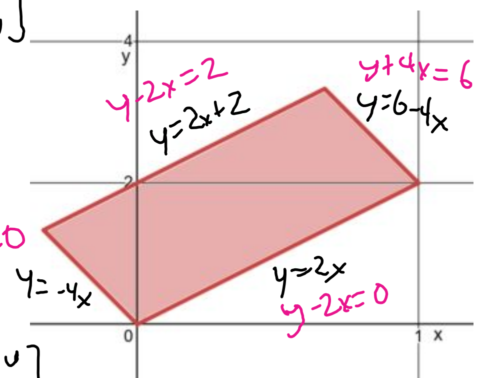
3) (Invertible) Linear transformations

$$\vec{T}(x,y) = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{DT(x,y)} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 113. Find a coordinate transformation $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ that maps the rectangle $[0,6]\times[0,2]$ to the parallelogram bounded by $y = 2x$, $y = 2x+2$, $y = -4x$, and $y = 6 - 4x$.

$$\vec{T}(u,v) = \begin{bmatrix} \frac{1}{6}u - \frac{1}{6}v \\ \frac{1}{3}u - \frac{2}{3}v \end{bmatrix}$$





$$\vec{T}^{-1}(x,y) = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 4x+y \\ -2x+y \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Suggests that: $\begin{cases} u = y + 4x \\ v = y - 2x \end{cases}$

$$T(u,v) = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4+2}\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 & -1/6 \\ 1/3 & 2/3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6}u - \frac{1}{6}v \\ \frac{1}{3}u - \frac{2}{3}v \end{bmatrix}$$

Day 18 Lecture

Daily Announcements & Reminders:

- HW 15.6, 15.7 due tonight
- Office hours 12:30 – 1:30 today
- Exam 2 graded papers & solutions this afternoon
  - regrades through R, 3/27
  - median 82%
- No class next week, get some rest!
- Do warmup on Ed

Goals for Today: Section 15.8

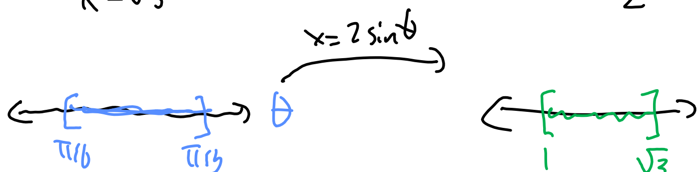
- Change variables in multiple integrals
- Identify choices for changing variables in a given integration problem

Thinking back to single variable calculus: Compute $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

(1) Identify substitution: $x = 2\sin\theta$

(2) Take derivative to relate differentials: $\underline{dx = 2\cos\theta\, d\theta}$

(3) Find a new region of integration:
$x = 1 = 2\sin\theta \to \sin\theta = \frac{1}{2} \to \theta = \pi/6$
$x = \sqrt{3} = 2\sin\theta \to \sin\theta = \frac{\sqrt{3}}{2} \to \theta = \pi/3$



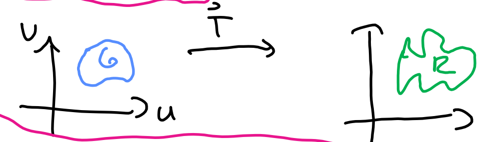
(4) Plug in: $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \int_{\pi/6}^{\pi/3} \frac{1}{\sqrt{\underbrace{4-4\sin^2\theta}_{2\cos\theta}}}\, 2\cos\theta\, d\theta = \int_{\pi/6}^{\pi/3} d\theta = \frac{\pi}{6}$

Theorem 114 (Substitution Theorem)**.** *Suppose* $\mathbf{T}(u, v)$ *is a coordinate transformation that maps the region* G *in the uv-plane to the region* R *in the xy-plane. Then*

like $x = 2\sin\theta$

like $[\pi/6, \pi/3]$

$$\underbrace{\iint_R f(x, y)\, dx\, dy}_{\text{Given: HARD}} = \iint_G f(\mathbf{T}(u, v))\underbrace{|\det(D\mathbf{T}(u, v))|}\, du\, dv. \quad \Big]\ \text{EASIER}$$



Jacobian determinant
- measures area scaling
: if $\det(D\vec{T}(u,v)) = 2$ then $\text{Area}(R) = 2 \cdot \text{Area}(G)$

Example 115. Evaluate $\displaystyle\int_0^4 \int_{y/2}^{y/2+1} \underbrace{\frac{2x - y}{2}}_{u}\, dx\, dy$ via the transformation

$\vec{T}^{-1}(x,y)$: $\Big[\ u = x - y/2, \quad v = y/2.$

1. **Find T:** $\vec{T}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}$

Solve for x & y:

$$\vec{T}^{-1}(x,y) = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T(u,v) = \frac{1}{\frac{1}{2} - 0} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$= \begin{bmatrix} u + v \\ 2v \end{bmatrix}$$

sidebar:

$$\vec{T}(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

$$D\vec{T}(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\det(D\vec{T}(r,\theta)) = r\cos^2\theta + r\sin^2\theta = r$$

2. **Find G and sketch:**

$x = \frac{y}{2}$ $\quad x = \frac{y}{2} + 1$ $\quad y = 0$ $\quad y = 4$

$x - \frac{y}{2} = 0$ $\quad x - \frac{y}{2} = 1$

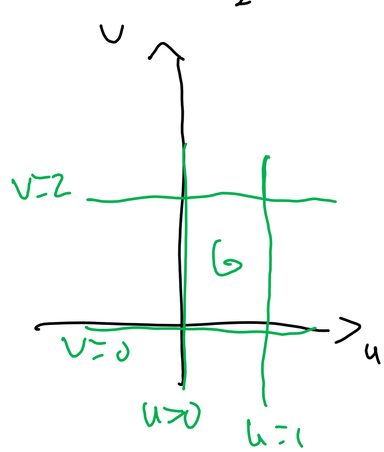
$\downarrow T$:

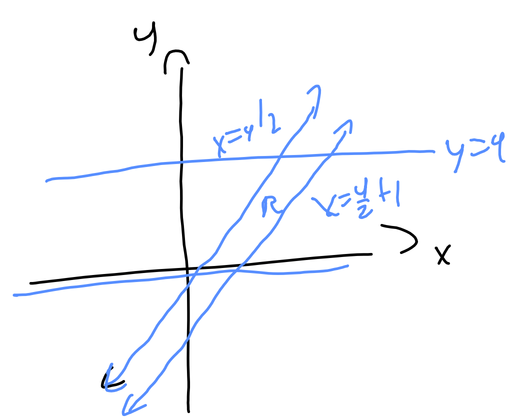
$u + v = \frac{2v}{2} \to u = 0$

$u + v = \frac{2v}{2} + 1 \to u = 1$

$2v = 0 \to v = 0$

$2v = 4 \to v = 2$





$$\vec{T}(u,v) = \begin{bmatrix} u+v \\ 2v \end{bmatrix}$$

3. **Find Jacobian:**

$$D\vec{T}(u,v) = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$|\det(D\vec{T}(u,v))| = |2-0| = 2$$

4. **Convert and use theorem:**

$$\int_0^4 \int_{y/2}^{y/2+1} \underbrace{\frac{2x-y}{2}}\, dx\, dy = \int_0^1 \int_0^2 u \cdot 2 \, dv\, du = 2$$

Example 116.

a) [Poll] Find the Jacobian of the transformation

$$x = u + (1/2)v, \ y = v.$$

$\vec{T}(u,v) = \begin{bmatrix} u + \frac{1}{2}v \\ v \end{bmatrix}$ $\quad D\vec{T} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$

$\det D\vec{T}(u,v) = 1 - 0 = 1$

b) [Poll] Which transformation(s) seem suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2} \, dx \, dy?$$

i. $u = x, v = y$ ← only renames

ii. $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$ ← polar, not helpful

iii. $u = 2x - y, \ v = y^3$

iv. $u = y, \ v = 2x - y$

v. $u = 2x - y, \ v = y$

integrand is nice

$x = \frac{y}{2}$ & $x = \frac{y}{2} + 2$

$2x - y = 0$ $\quad 2x - y = 4$

vi. $u = e^{(2x-y)^2}, \ v = y^3$

Theorem 117 (Derivative of Inverse Coordinate Transformation). *If* $\mathbf{T}(u,v)$ *is a one-to-one differentiable transformation that maps a region* G *in the uv-plane to a region* R *in the xy-plane and* $T(u_0,v_0) = (x_0,y_0)$*, then we have*

$$|\det(D\mathbf{T}(u_0,v_0))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x_0,y_0))|}$$

Example 118. Let's evaluate $\displaystyle\iint_R \frac{y(x+y)}{x^3}$ where R is the region in the xy-plane bounded by $y = x, y = 3x, y = 1 - x$, and $y = 2 - x$. Consider the coordinate transformation $u = x + y, v = y/x$.

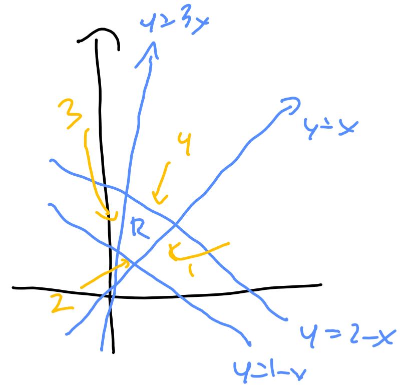
1. **Find the rectangle G in the uv plane that is mapped to R**

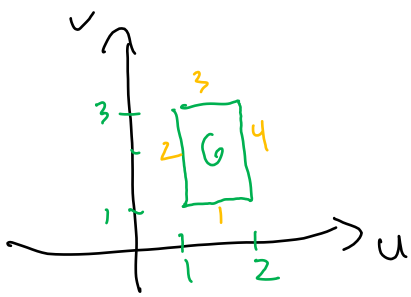
$y = 1 - x \Rightarrow x + y = 1 \Rightarrow u = 1$

$y = 2 - x \Rightarrow x + y = 2 \Rightarrow u = 2$

$y = x \Rightarrow y/x = 1 \Rightarrow v = 1$

$y = 3x \Rightarrow y/x = 3 \Rightarrow v = 3$





2. **Evaluate $f(\mathbf{T}(u,v))|\det(D\mathbf{T}(u,v))|$ in terms of u and v without directly solving for T using the theorem above**

We know: $|\det(DT(u,v))| = \dfrac{1}{|\det(DT^{-1}(x,y))|} = \dfrac{1}{\frac{x}{x}\cdot\frac{1}{x} + \frac{y}{x^2}} = \dfrac{x^2}{x+y}$

$\vec{T}^{-1}(x,y) = \begin{bmatrix} x+y \\ y/x \end{bmatrix} \rightarrow D\vec{T}^{-1}(x,y) = \begin{bmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}$

$f(\vec{T}(u,v))|\det(D\vec{T}(u,v))| = \dfrac{y\cancel{(x+y)}}{x^3} \cdot \dfrac{\cancel{x^2}}{\cancel{x+y}} = \dfrac{y}{x} = v$

3. **Use the Substitution Theorem to compute the integral.**

$$\iint_R \frac{y(x+y)}{x^3}\, dA = \iint_G f(\vec{T}(u,v))\,|\det(\nabla\vec{T}(u,v))|\, dA$$

$$= \int_1^2 \int_1^3 v \, dv\, du$$

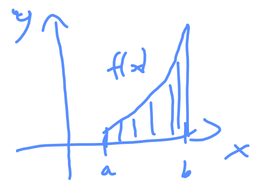
Day 19 - Scalar Line Integrals & Vector Fields

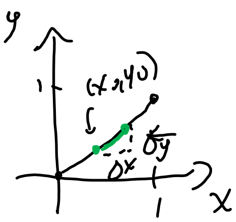
Pre-Lecture

Section 16.1: Scalar Line Integrals

Goal: Extend 1D/2D integrals to 1D/2D objects living in higher-dimensional space

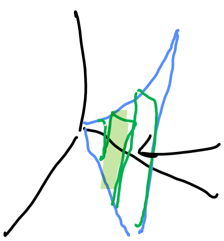
Example 119. Suppose we build a wall whose base is the straight line from $(0,0)$ to $(1,1)$ in the xy-plane and whose height at each point is given by $h(x,y) = 2x + y^2$ meters. What is the area of this wall?

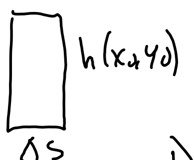




$\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$

$$\text{Area} \approx \sum_{k=1}^{n} h(x_k, y_k)\sqrt{\Delta x^2 + \Delta y^2}$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} h(x_k, y_k)\sqrt{\Delta x^2 + \Delta y^2}$$
$$= \int_C h(x,y)\, ds = \int_a^b h(\vec{r}(t))\, \overbrace{\|\vec{r}\,'(t)\|\, dt}^{ds}$$





1) Parameterize C: $\vec{r}(t) = \langle 1,1\rangle t + \langle 0,0\rangle$; $0 \le t \le 1$

2) Substitute $\text{Area} = \int_0^1 (2t + t^2)\underbrace{\|\langle 1,1\rangle\|}_{=\sqrt{2}}\, dt = \int_0^1 2\sqrt{2}\,t + \sqrt{2}\,t^2\, dt$
$= \sqrt{2}\,t^2 + \frac{\sqrt{2}}{3}t^3\Big|_0^1$
$= \boxed{\frac{4\sqrt{2}}{3}}$

Definition 120. The **line integral** of a scalar function $f(x,y)$ over a curve C in $\mathbb{R}^2$ is

$$\int_C f(x,y)\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\sqrt{\Delta x^2 + \Delta y^2}$$
$$= \int_a^b f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\, dt$$

where $\vec{r}(t)$ parameterizes C with $a \le t \le b$.

What things can we compute with this?

- If $f = 1$: $\int_C 1\, ds = \int_a^b \|\vec{r}\,'(t)\|\, dt$ = arc length of C
- If $f = \delta$ is a density function: $\int_C \delta\, ds$ = mass of a wire lying along C w/ density δ.
- If f is a height: $\int_C f\, ds$ = area of wall w/ base C & height f

Day 19 Lecture

Daily Announcements & Reminders:

- HW 15.8 due tonight
- Quiz 8 on 15.8, 16.1 in studio tomorrow
  - L.O: I3, V1
- Exam 2 regrades through 5pm Th
- Do warmup on Ed

Goals for Today:

Section 16.1, 16.2

- Define a line integral for a scalar function $f(x, y)$ or $f(x, y, z)$
- Compute line integrals using parameterizations
- Define and explore vector fields

Unit 4: Vector Calculus

Goals:

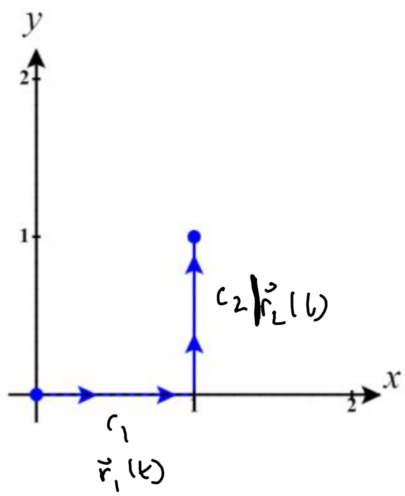
- Extend 1D / 2D integrals to 1D/2D objects living in higher-dimensional space
- Extend the Fund. Thm of Calculus in new ways

We will use tools from everything we have covered so far to do this: parameterizations, derivatives and gradients, and multiple integrals.

Strategy for computing line integrals:

1. Parameterize the curve C with some $\mathbf{r}(t)$ for $a \le t \le b$
2. Compute $ds = \|\mathbf{r}'(t)\|\ dt$
3. Substitute: $\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\ dt$
4. Integrate

Example 121. Compute $\int_C 2x + y^2\ ds$ along the curve C pictured below.



1) Parametrize C: Use $\int_C f\,ds = \int_{c_1} f\,ds + \int_{c_2} f\,ds$

$\vec{r}_1(t) = \langle 1,0\rangle t + \langle 0,0\rangle \quad 0 \le t \le 1$

$\langle 1,0\rangle = \langle 1,0\rangle - \langle 0,0\rangle$

$= \langle t, 0\rangle = \langle x(t), y(t)\rangle$

$\vec{r}_2(t) = \langle 0,1\rangle t + \langle 1,0\rangle \quad 0 \le t \le 1$

$\langle 0,1\rangle = \langle 1,1\rangle - \langle 1,0\rangle$

$= \langle 1, t\rangle$

2) Compute $\|\vec{r}\,'(t)\|$:

$\vec{r}_1{}'(t) = \langle 1,0\rangle \qquad \vec{r}_2{}'(t) = \langle 0,1\rangle$

$\|\vec{r}_1{}'(t)\| = 1 \qquad \|\vec{r}_2{}'(t)\| = 1$

3) Substitute: $\int_C f\,ds = \boxed{\int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt}$

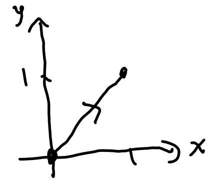
$$\int_C 2x+y^2\,ds = \int_{c_1} 2x+y^2\,ds + \int_{c_2} 2x+y^2\,ds$$

$$= \int_0^1 (2t+0^2)\cdot 1\,dt + \int_0^1 (2(1)+t^2)\cdot 1\,dt$$

$$= t^2\Big|_0^1 + \left(2t + \tfrac{1}{3}t^3\right)\Big|_0^1$$

$$= 1 + 2 + \tfrac{1}{3} = \tfrac{10}{3}$$

Example 122. [Poll] Compute $\int_C 2x + y^2 \; ds$ along the curve C given by $\mathbf{r}(t) = 10t\mathbf{i} + 10t\mathbf{j}$ for $0 \leq t \leq \frac{1}{10}$.



1) Parameterize C : Given!

2) Compute $\|\vec{r}'(t)\|$: $\vec{r}'(t) = \langle 10, 10 \rangle$

$\|\vec{r}'(t)\| = \sqrt{100+100} = \sqrt{200} = 10\sqrt{2}$

3) Substitute: $\int_C f \, ds = \int_a^b f(\vec{r}(t)) \, \|\vec{r}'(t)\| \, dt$

$$\int_C 2x + y^2 \, ds = \int_0^{1/10} (20t + 100t^2) \, 10\sqrt{2} \, dt$$

4) Compute

$$= 10\sqrt{2} \left(10t^2 + \frac{100}{3} t^3 \right) \Big|_0^{1/10}$$

$$= 10\sqrt{2} \left(\frac{1}{10} + \frac{1}{30} \right)$$

$$= \sqrt{2} \left(1 + \frac{1}{3} \right) = \frac{4\sqrt{2}}{3}$$

- In Pre-Lecture: $\int_C 2x + y^2 \, ds$ with C given by $\vec{r}(t) = \langle t, t \rangle$, $0 \leq t \leq 1$

$$= \frac{4\sqrt{2}}{3}$$

- Compare to Ex 122.

- $\int_C f \, ds$ is independent of parameterization & orientation

- $\int_C f \, ds$ is path-dependent (not only endpoints matter)

Example 123 (Poll)**.** Let C be a curve parameterized by $\mathbf{r}(t)$ from $a \le t \le b$. Select all of the true statements below.

a) $\mathbf{r}(t+4)$ for $a \le t \le b$ is also a parameterization of C with the same orientation

False : to fix let $a-4 \le t \le b-4$

b) $\mathbf{r}(2t)$ for $a/2 \le t \le b/2$ is also a parameterization of C with the same orientation

True

c) $\mathbf{r}(-t)$ for $a \le t \le b$ is also a parameterization of C with the opposite orientation

False: goes from $r(-a)$ to $r(-b)$

d) $\mathbf{r}(-t)$ for $-b \le t \le -a$ is also a parameterization of C with the opposite orientation

True.

$\vec{r}(-(t-b))$

e) $\mathbf{r}(b-t)$ for $0 \le t \le b-a$ is also a parameterization of C with the opposite orientation

True.

Parameterizations

- lines / line segments
- segments $y=f(x)$, $x=g(y)$
  ↳ let ind. var. be t
- circles / ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\vec{r}(t) = \langle a\cos(t), b\sin(t) \rangle$

ccw ; $\alpha \le t \le \beta$

Example 124. Find a parameterization of the curve C that consists of the portion of the curve $y = x^2 + 1$ from $(1, 2)$ to $(0, 1)$ and use it to write the integral $\int_C x^2 + y^2\, ds$ as an integral with respect to your parameter.

1) Parameterization: $\vec{r}_1(t) = \langle t, t^2+1 \rangle$ $0 \le t \le 1$ (opposite orientation)

so instead take $\vec{r}(t) = \vec{r}_1(-t)$ $-1 \le t \le 0$

$= \langle -t, t^2+1 \rangle$, $-1 \le t \le 0$

2) Compute $\|\vec{r}\,'(t)\|$: $\vec{r}\,'(t) = \langle -1, 2t \rangle$

so $\|\vec{r}\,'(t)\| = \sqrt{1+4t^2}$

3) Substitute: $\int_C x^2 + y^2\, ds = \int_{-1}^{0} \left(t^2 + (t^2+1)^2\right) \sqrt{1+4t^2}\, dt$

16.2: Vector Fields

Definition 125. A vector field is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ which associates a vector to every point in its domain.

Examples:

- Force fields (gravity, electromagnetic)
- Velocity fields for fluids
- for $f : \mathbb{R}^n \to \mathbb{R}$: ∇f is a vector field
- Tangent vectors to a curve
- Slope fields for differential equations

$$\vec{F}(x,y,z) = \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix} \quad \begin{matrix} M \\ N \\ P \end{matrix}$$

Graphically: For each point (a, b, c) in the domain of $\mathbf{F}$, draw the vector $\mathbf{F}(a, b, c)$ with its base at (a, b, c).

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

$$\vec{F}(x,y) = \langle x, y \rangle$$

$$\vec{F}(x,y) = \langle y, -x \rangle$$

$$\vec{F}(x,y) = \langle \cos(y+x), x+y \rangle$$

Computer Tools:

- CalcPlot3d
- Field Play

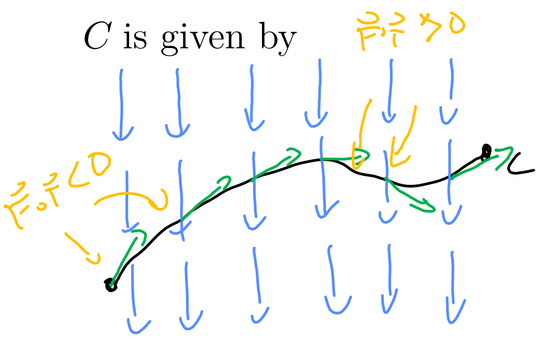
Day 20 - Work, Flow, and Flux Line Integrals

Pre-Lecture

Section 16.2: Work Line Integrals

Idea: In many physical processes, we care about the total sum of the strength of that part of a field that lies either in the direction of a curve or perpendicular to that curve.

1. The work done by a field **F** on an object moving along a curve C is given by

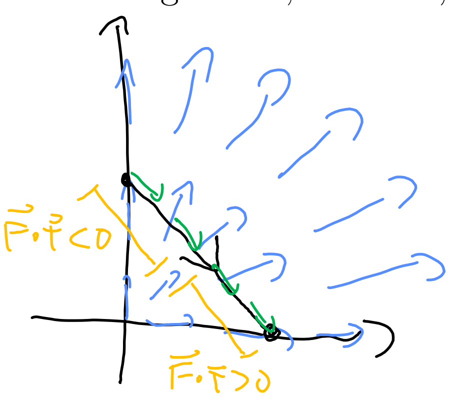


• component of $\vec{F}$ along C : $\vec{F}\cdot\vec{T}$

• work done: $\int_C \vec{F}\cdot\vec{T}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}} \cancel{\|\vec{r}\,'(t)\|}\, dt$

$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

Example 126. Work Done by a Field. Suppose we have a force field $\mathbf{F}(x, y) = \langle x, y \rangle$ N. Find the work done by **F** on a moving object from $(0, 1)$ to $(1, 0)$ in a straight line, where x, y are measured in meters.



1) Parameterize C: $\vec{r}(t) = \langle 1, -1 \rangle t + \langle 0, 1 \rangle \quad 0 \le t \le 1$

2) Find $\vec{r}\,'(t)$: $\vec{r}\,'(t) = \langle 1, -1 \rangle$

3) Substitute:

work done: $\int_C \vec{F}\cdot\vec{T}\, ds = \int_0^1 \langle t, 1-t \rangle \cdot \langle 1, -1 \rangle\, dt$

$= \int_0^1 t - 1 + t\, dt$

$= t^2 - t \Big|_0^1$

$= 0$

Day 20 Lecture

Daily Announcements & Reminders:

- HW 16.1 due tonight
- We will have a sub on 4/13

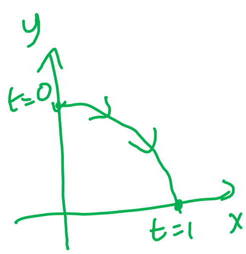
Goals for Today:

Section 16.2

- Define tangential and normal line integrals for vector fields
- Apply vector line integrals to problems involving work, flow, and flux
- Compute vector line integrals using parameterizations

- work done by $\vec{F}$ along C parameterized by $\vec{r}(t)$, $a \le t \le b$

$$= \int_C \underbrace{\vec{F}\cdot\vec{T}}\, ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\, dt$$



take $x^2+y^2=1$, solve for y: $x=\sqrt{1-y^2}$
let $y=t$

Warmup: $\vec{r}(t) = \langle \sqrt{1-t^2}, t\rangle$, $0 \le t \le 1$

is a circle b/c $x(t)^2+y(t)^2=1$

$\vec{r}\,'(t) = \langle \sim, 1\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \langle 0, t\rangle\cdot\langle \sim, 1\rangle = t$

$\int_0^1 t\, dt = \frac{1}{2}$

$\langle t, \sqrt{1-t^2}\rangle$
$0 \le t \le 1$

$\vec{r}\,'(t) = \langle 1, \frac{-t}{\sqrt{1-t^2}}\rangle$

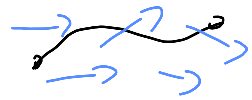
$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = -t$

$\int_0^1 -t\, dt = -\frac{1}{2}$

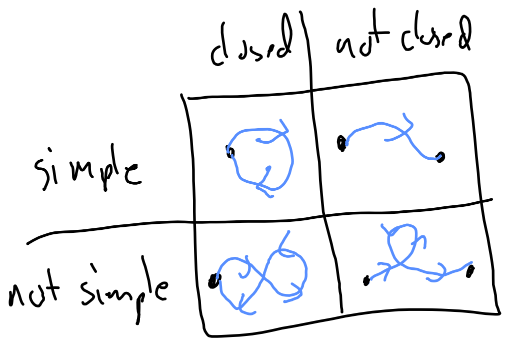
$$\int_C \vec{F}\cdot\vec{T}\, ds = -\int_{-C} \vec{F}\cdot\vec{T}\, ds$$

1. The flow along a curve C of a velocity field **F** for a fluid in motion is given by $\vec{F}$

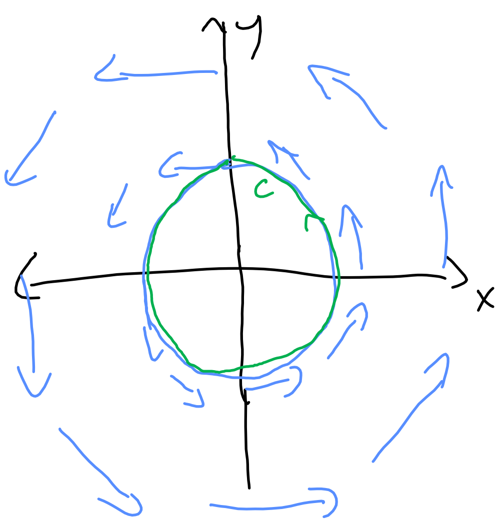
$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$



When C is closed, this is called circulation. C is called simple if it does not intersect itself.



Example 127. Flow of a Velocity Field. Find the circulation of the velocity field $\mathbf{F}(x, y) = \langle -y, x\rangle$ cm/s around the unit circle, parameterized counterclockwise.



1) Parametrize C: $\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$
$0 \le t \le 2\pi$

2) Find $\vec{r}\,'(t)$: $\vec{r}\,'(t) = \langle -\sin(t), \cos(t)\rangle$

3) Substitute:

$\vec{F}(\vec{r}(t)) = \langle -\sin(t), \cos(t)\rangle$

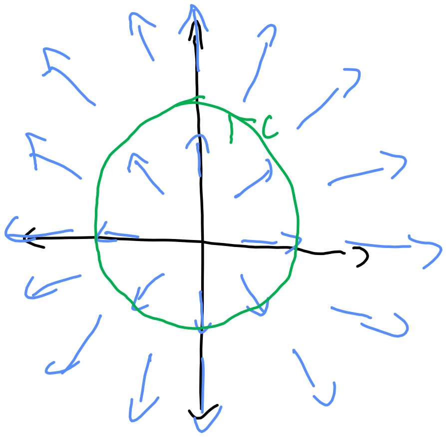
$$\text{Circulation} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \langle -\sin(t), \cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle\,dt$$

$$= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt$$

$$= \int_0^{2\pi} 1\,dt$$

$$= 2\pi \ \text{cm}^2/\text{s}$$

Example 128. [Poll] What is the circulation of $\mathbf{F}(x, y) = \langle x, y\rangle$ around the unit circle, parameterized counterclockwise?



1) $\vec{r}(t) = \langle \cos(t), \sin(t)\rangle \quad 0 \le t \le 2\pi$

2) $\vec{r}\,'(t) = \langle -\sin(t), \cos(t)\rangle$

3) circulation $= \int_C \vec{F} \cdot \vec{T}\, ds$

$$= \int_0^{2\pi} \underbrace{\langle \cos(t), \sin(t)\rangle}_{\vec{F}(\vec{r}(t))} \cdot \langle -\sin(t), \cos(t)\rangle\, dt$$

$$= \int_0^{2\pi} 0\, dt$$

$$= 0$$

∘ positive net flow: more movement in direction of C

Strategy for computing tangential component line integrals

e.g. work, flow, circulation integrals

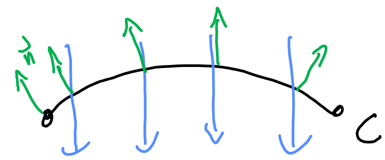
1. Find a parameterization $\mathbf{r}(t)$, $a \le t \le b$ for the curve C.

2. Compute $\mathbf{r}'(t)$.

3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$

4. Integrate

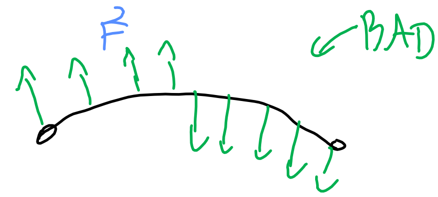
Idea: <u>flux</u> across a plane curve of a 2D-vector field measures the flow of the field across that curve (instead of along it).



We compute this with the integral

$$\int_C \underbrace{\mathbf{F} \cdot \mathbf{n}}\, ds.$$

unit normal



The sign of the flux integral tells us whether the net flow of the field across the curve is in the direction of <u>$\vec{n}$</u> or in the opposite direction.

We can choose **n** to be either of

usual

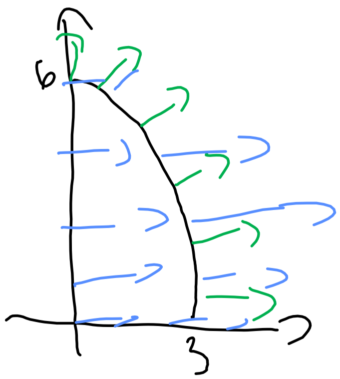
$$\frac{\langle y'(t), -x'(t)\rangle}{\|\vec{r}\,'(t)\|} \quad \text{or} \quad \frac{\langle -y'(t), x'(t)\rangle}{\|\vec{r}\,'(t)\|}$$

Strategy for computing normal component line integrals

e.g. flux integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \le t \le b$ for the curve C.
2. Compute $x'(t)$ and $y'(t)$ and determine which normal to work with.
3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \pm \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t)\rangle\, dt$ (sign based on choice of normal)
4. Integrate

Example 129. Flux of a Velocity Field. Compute the flux of the velocity field $\mathbf{v} = \langle 3 + 2y - y^2/3, 0 \rangle$ cm/s across the quarter of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1$ in the first quadrant, oriented away from the origin.



1) Parameterize: $\vec{r}(t) = \langle 3\cos(t), 6\sin(t) \rangle, \ 0 \le t \le \frac{\pi}{2}$

2) Find $\vec{n}$: $\vec{r}\,'(t) = \langle -3\sin(t), 6\cos(t) \rangle$

$\vec{n}(t) = \langle 6\cos(t), 3\sin(t) \rangle$

to get correct direction

3) Plug in:

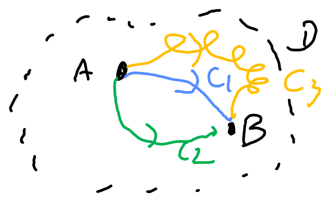
$$\text{flux} = \int_C \vec{F} \cdot \hat{n}\, ds = \int_0^{\pi/2} \langle 3 + 12\sin(t) - 36\sin^2(t) \rangle \cdot \langle 6\cos(t), 3\sin(t) \rangle\, dt$$

Day 21 - Conservative Vector Fields & the FToLI

Pre-Lecture

Section 16.3: Path Independence

Definition 130. A vector field $\mathbf{F}$ is **path independent** on an open region D if $\int_C \vec{F}\cdot\vec{T}\,ds$ are the same for all paths C in the region that have the same endpoints.



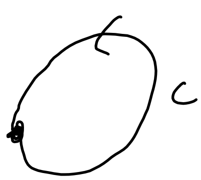
• gravitational force & spring force

When $\mathbf{F}$ is path independent, we can use the simplest path from point A to point B to compute a line integral, and will often denote the line integral with points as bounds, e.g.

$$\int_{(0,1,2)}^{(3,1,1)} \mathbf{F}\cdot\mathbf{T}\,ds \qquad \text{or} \qquad \int_{(a,b)}^{(c,d)} \mathbf{F}\cdot d\mathbf{r}.$$

Example 131. If C is any closed path and $\mathbf{F}$ is path independent on a region containing C, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 0$$



Cor: If $\int_C \vec{F}\cdot d\vec{r} \neq 0$ for some closed curve C, $\vec{F}$ is not path independent.

Question: Given $\mathbf{F}$, how do we tell if it is path independent on a particular region?

For example, is $\mathbf{F}(x,y) = \langle x, y\rangle$ a path independent vector field on its domain?

Don't know?

Day 21 Lecture

Daily Announcements & Reminders:

- HW 16.2 tonight
- Quiz 9 tomorrow; 16.2/16.3
  L.O V1, V2, V3
- No OH on Th. See Canvas.
- Warmup on Ed

Goals for Today: Section 16.3

- Define conservative vector fields and recognize examples from physics
- Learn how to check if a field is conservative
- Compute potential functions
- Apply the Fundamental Theorem of Line Integrals to compute line integrals of conservative vector fields

Example 132 (Poll)**.** Last time, we saw that if C is the unit circle about the origin, oriented counterclockwise, then $\int_C \langle -y, x\rangle \cdot d\mathbf{r} = 2\pi$. From this, we can conclude:

$\vec{F} = \langle -y, x \rangle$ is not path independent b/c

C is closed but $\int_C \vec{F} \cdot d\vec{r} = 2\pi \neq 0$

A different idea: Suppose $\mathbf{F}$ is a gradient vector field, i.e. $\mathbf{F} = \nabla f$ for some function of multiple variables f. f is called a potential function for $\mathbf{F}$. In this case we also say that $\mathbf{F}$ is **conservative.**

$\vec{F} = \nabla f$; i.e. there is some f s.t. $\vec{F} = \langle f_x, f_y \rangle$

Ex: Is $\vec{F} = \langle x, y \rangle$ conservative?

Yes! $\vec{F} = \nabla f$ for $f = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C$

Can we find $f(x,y)$ s.t. $f_x = x$ & $\boxed{f_y = y}$?

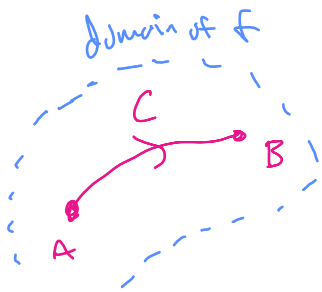
$f = \int f_x\,dx = \int x\,dx = \frac{1}{2}x^2 + \boxed{C(y)}$

OR: $f = \int f_y\,dy = \int y\,dy = \frac{1}{2}y^2 + C_2(x)$

Compare & resolve

$y = f_y = \frac{\partial}{\partial y}\left(\frac{1}{2}x^2 + C(y)\right) = C'(y) \Rightarrow C(y) = \frac{1}{2}y^2 + C$

Theorem 133 (Fundamental Theorem of Line Integrals)**.** *If C is a smooth curve from the point A to the point B in the domain of a function f with continuous gradient on C, then*



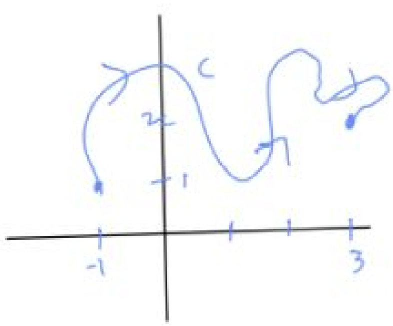
$$\int_C \nabla f \cdot \mathbf{T}\, ds = f(B) - f(A)$$

FTC:
$\int_a^b f'(x)\,dx = f(b) - f(a)$

- all conservative fields are path-independent

Example 134. Compute $\int_C \underbrace{\langle x, y \rangle}_{\text{conservative}} \cdot d\mathbf{r}$ for the curve C shown below from $(-1, 1)$ to $(3, 2)$.

- Before today, impossible to compute!



so use FToLI:

$\int_C \langle x,y \rangle \cdot d\vec{r} = \int_C \nabla\left(\frac{1}{2}x^2 + \frac{1}{2}y^2\right) \cdot \vec{T}\,ds$

$= \frac{1}{2}x^2 + \frac{1}{2}y^2 \Big|_{(-1,1)}^{(3,2)}$

$= \left(\frac{1}{2}\cdot 9 + \frac{1}{2}\cdot 4\right) - \left(\frac{1}{2}(1) + \frac{1}{2}(1)\right)$

$= 11/2$

Example 135. Find a potential function for $\langle -y, x\rangle$, if possible.

$$f = \int -y\,dx = -xy + C_1(y)$$

$$f = \int x\,dy = xy + C_2(x)$$

b/c terms with both x & y are different, f is not conservative.

Example 136. Find a potential function for $\vec{F} = \langle \overset{P}{xy + yz}, \overset{Q}{\tfrac{1}{2}x^2 + xz + e^y}, \overset{R}{xy + z^2}\rangle$, if possible.

① $f = \int P\,dx = \int xy + yz\,dx = \frac{1}{2}x^2y + xyz + C_1(y,z)$

② $f = \int Q\,dy = \int \frac{1}{2}x^2 + xz + e^y\,dy = \frac{1}{2}x^2y + xyz + e^y + C_2(x,z)$

③ $f = \int R\,dz = \int xy + z^2\,dz = xyz + \frac{1}{3}z^3 + C_3(x,y)$

Can we choose C_1, C_2, C_3 so that ① = ② = ③.

$C_1 = e^y + \frac{1}{3}z^3$, $C_2 = \frac{1}{3}z^3$, $C_3 = \frac{1}{2}x^2y + e^y$

$$\boxed{f = xyz + \frac{1}{2}x^2y + e^y + \frac{1}{3}z^3 + C}$$

It follows that **every conservative field is path independent.**

In fact, by carefully constructing a potential function, we can show the converse is also true: path independence ⟹ conservative

$$f(x,y) = \int_{(a,b)}^{(x,y)} \vec{F}\cdot\vec{T}\, ds$$

This leads to a better way to test for path-independence and a way to apply the FToLI.

Curl Test for Conservative Fields: Let $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ be a vector field defined on a **simply-connected** region. If $\operatorname{curl}\mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 0,0,0\rangle$, then $\mathbf{F}$ is conservative.

$\vec{F} = \langle P, Q, 0\rangle$

- If $\mathbf{F}$ is a 2-d vector field, $\operatorname{curl}\mathbf{F} =$ $\langle 0, 0, Q_x - P_y\rangle = \langle 0,0,0\rangle \iff Q_x = P_y$
- This is also called the **mixed-partials test**, because

$P_y = Q_x$

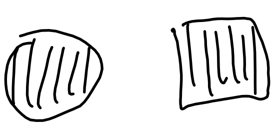
$R_y = Q_z$

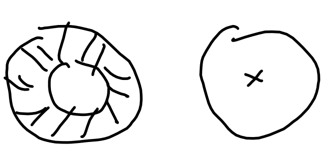
$P_z = R_x$

simply-connected

"no holes"

ex: $\mathbb{R}^2, \mathbb{R}^3$





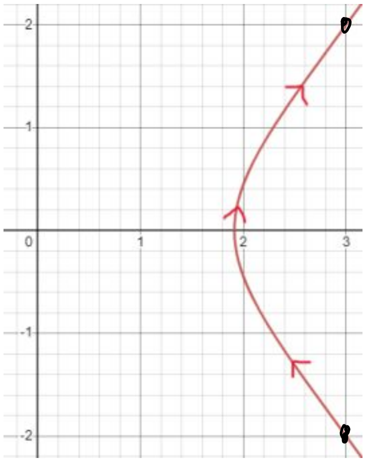
non-ex

if $\vec{F} = \nabla f$: $\langle f_x, f_y\rangle$

so $Q_x = f_{yx}$

$P_y = f_{xy}$

Example 137. Evaluate $\int_C (10x^4 - 2xy^3)\ dx - 3x^2y^2\ dy$ where C is the part of the curve $x^5 - 5x^2y^2 - 7x^2 = 0$ from $(3, -2)$ to $(3, 2)$.



$$\int_C (10x^4 - 2xy^3)\,dx + (-3x^2y^2)\,dy$$

$$= \int_C \vec{F} \cdot \vec{T}\,ds \quad \text{for } \vec{F} = \langle 10x^4 - 2xy^3, -3x^2y^2 \rangle$$

- Is $\vec{F}$ conservative? Use Mixed Partials Test.

  Is $Q_x = P_y$? Yes!

  $Q_x = -6xy^2 \qquad P_y = 0 - 6xy^2 \quad \checkmark$

- Find potential f : $\vec{F} = \nabla f$.

$$f = \int P\,dx = 2x^5 - x^2y^3 + C_1(y)$$

$$f = \int Q\,dy = -x^2y^3 + C_2(x)$$

$$C_1(y) = 0 \qquad C_2(x) = 2x^5$$

so $\boxed{f = -x^2y^3 + 2x^5}$

- Apply FToLI:

$$\int_C \vec{F} \cdot \vec{T}\,ds = f(B) - f(A)$$

$$= -x^2y^3 + 2x^5 \Big|_{(3,-2)}^{(3,2)} = -144$$

Day 22 - Div, Curl, & Green's Theorem

Pre-Lecture

Divergence and Curl

Useful notation: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

↑ instructions

So if $f(x, y, z)$ is a function of three variables, $\nabla f = \left\langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f) \right\rangle$

If $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ is a vector field:

- $\nabla \cdot \mathbf{F} =$ $\frac{\partial}{\partial x}(P) + \frac{\partial}{\partial y}(Q) + \frac{\partial}{\partial z}(R) = P_x + Q_y + R_z$

  divergence or $\operatorname{div} \vec{F}$

  works in dim n, not just $\mathbb{R}^3$

- $\nabla \times \mathbf{F} =$ $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle$ • only in $\mathbb{R}^3$

  curl $\vec{F}$

  <u>not</u> functions, instructions

Example 138. Find the divergence and curl of $\mathbf{F} = \langle \overset{P}{xy}, \overset{Q}{2y^2}, \overset{R}{x + z} \rangle$.

$\nabla \cdot \vec{F} = P_x + Q_y + R_z = y + 4y + 1$

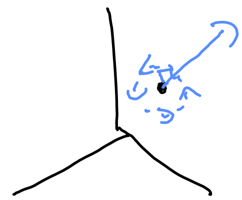
$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2y^2 & x+z \end{vmatrix} = \left\langle \frac{\partial}{\partial y}(x+z) - \frac{\partial}{\partial z}(2y^2), -\left(\frac{\partial}{\partial x}(x+z) - \frac{\partial}{\partial z}(xy)\right), \left(\frac{\partial}{\partial x}(2y^2) - \frac{\partial}{\partial y}(xy)\right) \right\rangle$

$= \langle 0, -(1-0), 0 - x \rangle = \langle 0, -1, -x \rangle$

How do we measure the change of a vector field?

1. **Curl** (in $\mathbb{R}^3$)

   - Tells us circulation density
   - Measures instantaneous/local circulation
   - Is a vector
   - Direction gives RHR axis of local rotation
   - Magnitude gives rate of rotation
   - curl $\mathbf{F} = \nabla \times \vec{F}$ $\qquad (\nabla \times \vec{F}) \cdot \hat{k}$
   - If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$: we use $\nabla \times \mathbf{F} = \nabla \times \langle P, Q, 0\rangle = \langle 0, 0, \underbrace{Q_x - P_y}_{\text{scalar curl}} \rangle$
   - If $\nabla \times \mathbf{F}$ ($\leftarrow$ vector) is zero:
     $\vec{F}$ is conservative; $\vec{F} = \nabla f$
     and $\vec{F}$ is irrotational



2. **Divergence** (in any $\mathbb{R}^n$)

   - Tells us flux density
   - Measures local expansion / compression
   - Is a scalar
   - div $\mathbf{F} = \nabla \cdot \vec{F} = P_x + Q_y + R_z$
   - If $\nabla\mathbf{F}$ ($\leftarrow$ scalar) is zero: $\vec{F}$ is incompressible and $\vec{F} = \nabla \times \vec{G}$ for some vector field $\vec{G}$

Day 22 Lecture

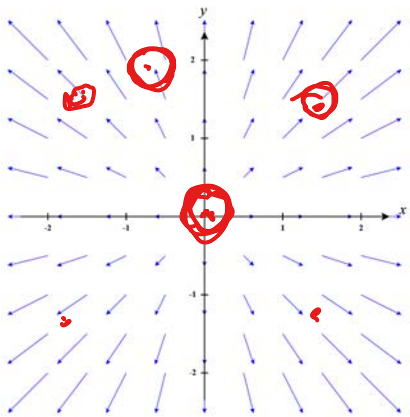
Daily Announcements & Reminders:

Goals for Today:

Section 16.4

- Define the divergence and curl of a vector field
- Interpret divergence and curl geometrically
- Apply Green's Theorem to compute line integrals over the boundary of a simply-connected region

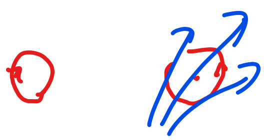
Example 139. Let $\mathbf{F}(x, y) = \langle x, y\rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



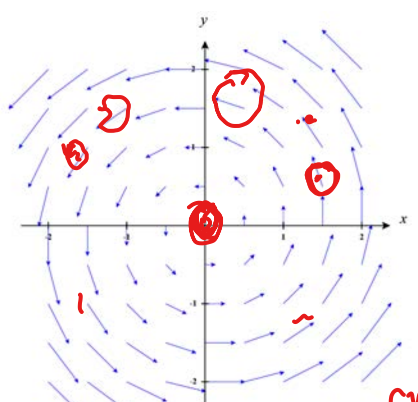
$\operatorname{div} \vec{F} = P_x + Q_y = 1 + 1 = 2$

$\operatorname{curl} \vec{F} \cdot \vec{k} = Q_x - P_y = 0 - 0 = 0$

$\operatorname{div} \vec{F} > 0$ at point P : water flow $\vec{F}$ has secret source at P



Example 140 (Poll)**.** Let $\mathbf{F}(x, y) = \langle -y, x\rangle$. $\sqrt{x^2+y^2}$ Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



$\operatorname{div} \vec{F} = P_x + Q_y = 0$

$\operatorname{curl} \vec{F} \cdot \vec{k} = Q_x - P_y = 1 - (-1)$
$= 2$

$\operatorname{curl} \vec{F} \cdot \vec{k} > 0$ at P : force field rotate a small disk fixed at P counterclockwise

Question: How is this useful?

$\int_a^b f'(x)\,dx = f(b) - f(a)$

Answer: We can relate rates of change of vector field inside a region to the behavior of the vector field on the boundary of the region.

Theorem 141 (Green's Theorem)**.** *Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane with outward oriented unit normal* $\mathbf{n}$. *If $\mathbf{F} = \langle P, Q\rangle$ has continuous partial derivatives around R, then*

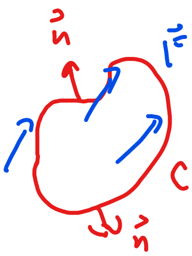
a)Circulation form:

circulation density

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA \qquad \textit{or} \qquad \int_C P\,dx + Q\,dy = \iint_R Q_x - P_y\,dA$$

$= -Q$ $\quad$ $= P$ $\quad$ $= P_x + Q_y$

b)Flux form:



flux density

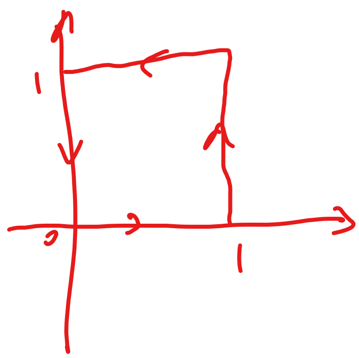
$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R (\nabla\cdot\mathbf{F})\,dA \qquad \textit{or} \qquad \int_C P\,dy - Q\,dx = \iint_R P_x + Q_y\,dA$$

$\iint_S (\text{curl}\,\vec{F})\cdot\vec{k}\,dA \approx$ area of S × curl at P

$=$ circulation around small region

Special case: $\text{curl}\,\vec{F}\cdot\vec{k} = 0$

Example 142. Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$ for the vector field $\mathbf{F} = \langle \underbrace{-y^2}_{P}, \underbrace{xy}_{Q} \rangle$ where C is the boundary of the square bounded by $x = 0, x = 1, y = 0,$ and $y = 1$ oriented counterclockwise.



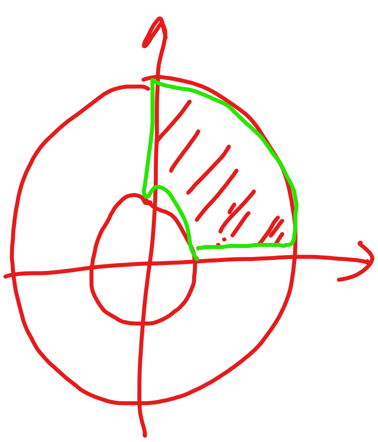
$$\int_C \vec{F}\cdot\vec{T}\,ds \overset{\text{Green}}{=} \iint_R (Q_x - P_y)\,dA$$

$$= \iint_R y-(-2y)\,dA$$

$$= \int_0^1\int_0^1 3y\ dx\,dy$$

$$= \int_0^1 3y\ dy = 3/2$$

Example 143. Compute the flux out of the region R which is the portion of the annulus between the circles of radius 1 and 3 in the first ~~octant~~ quadrant for the vector field $\mathbf{F} = \langle \frac{1}{3}x^3, \frac{1}{3}y^3 \rangle$.



$$\int_C \vec{F}\cdot\vec{n}\,ds \overset{\text{Green}}{=} \iint_R P_x + Q_y\ dA$$

$$= \iint_R x^2+y^2\ dA$$

polar

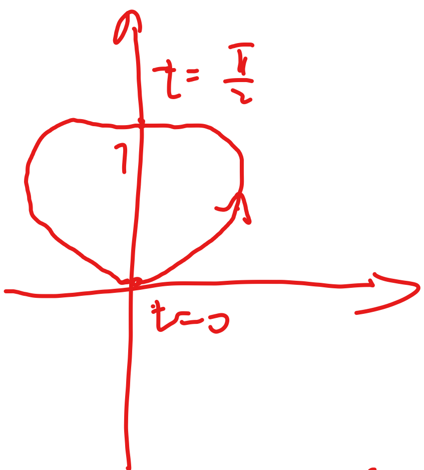
$$= \int_0^{\pi/2}\int_1^3 r^2\ \ r\,dr\,d\theta$$

$$= \dots\dots$$

$$= 10\pi$$

Example 144. Let R be the region bounded by the curve $\mathbf{r}(t) = \langle \sin(2t), \sin(t)\rangle$ for $0 \le t \le \pi$. Find the area of R, using Green's Theorem applied to the vector field $\mathbf{F} = \frac{1}{2}\langle x, y\rangle$.

$$\iint_R \tfrac{1}{2} + \tfrac{1}{2}\, dA = \text{area}$$



$$\int_C \vec{F}\cdot\vec{n}\, ds \overset{\text{Green}}{=} \iint_R \operatorname{div}(\vec{F})\, dA$$

$$= \int_C P\,dy - Q\,dx$$

$x = \sin 2t$
$y = \sin t$

$$\int_0^{\pi} \frac{\sin 2t}{2}\, d\sin t - \frac{\sin t}{2}\, d\sin 2t$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \int_0^{\pi} \sin t\cos^2 t\, dt - \sin t\left(\cos^2 t - \sin^2 t\right) dt$$

$$= \int_0^{\pi} \sin^3 t\, dt$$

$$= \int_0^{\pi} -\sin^2 t\, d\cos t = \int_0^{\pi} (\cos^2 t - 1)\, d\cos t = \frac{\cos^3 t}{3} - \cos t\,\Big|_0^{\pi} = 4/3$$

Note: This is the idea behind the operation of the measuring instrument known as a planimeter.

Example 145. Determine which methods we have learned so far that could be used to compute an answer to the problem below.

[Poll]

Compute the work done by

$$\mathbf{F}(x, y) = \langle ye^{xy}, xe^{xy} + 1 \rangle$$

on a particle moving along $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

- Parameterize the curve and use the formula $\int_C f\, ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\|\, dt$

- Parameterize the curve and use the formula $\int_C \mathbf{F}\, d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$

- Parameterize the curve and use the formula $\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t) \rangle\, dt$

- Find a potential function and use the Fundamental Theorem of Line Integrals

$Q_x = P_y$

- Apply the circulation form of Green's Theorem

- Apply the flux form of Green's Theorem

- None of these are appropriate methods for this problem.

Day 23 - Surfaces & Surface Integrals

Pre-Lecture

Section 16.5 Surfaces

Different ways to think about curves and surfaces:

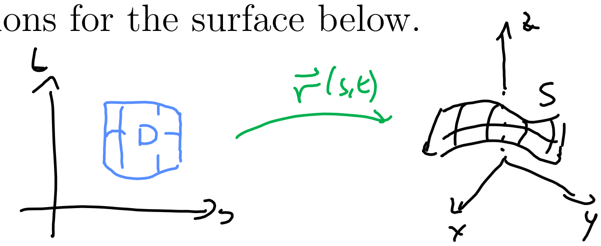
	Curves	Surfaces
Explicit:	$y = f(x)$ $y=\sin(x)$	$z = f(x,y)$ $z=2x+3y$ | $z=x^2+y^2$
Implicit:	$F(x,y) = 0$ $x^2+y^2=4$ $x^2+y^2-4=0$	$F(x,y,z) = 0$ $x^2+y^2+z^2=4$ $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}-1=0$
Parametric Form:	$\mathbf{r}(t) = \langle x(t), y(t)\rangle$ $\vec{r}(t)=\langle 2\cos(t), 2\sin(t)\rangle$ $0\le t\le 2\pi$	$\vec{r}(s,t)=\langle x(s,t), y(s,t), z(s,t)\rangle$ or u,v $\quad (s,t)\in D\subseteq\mathbb{R}^2$

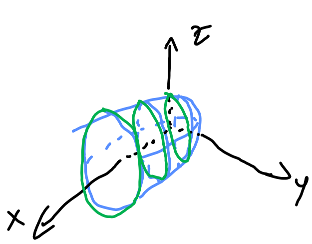
Example 146. Give parameteric representations for the surface below.

a) $x = y^2 + \frac{1}{2}z^2 - 2$

Since $x=f(y,z)$, let $y=s$, $z=t$

$\vec{r}(s,t)=\langle s^2+\frac{1}{2}t^2-2, s, t\rangle$; $(s,t)\in\mathbb{R}^2$





$x=u$; $-2\le u<\infty$

$u=y^2+\frac{1}{2}z^2-2$

$u+2=y^2+\frac{z^2}{2}$

$1=\frac{y^2}{u+2}+\frac{z^2}{2u+4}$

$y=\sqrt{u+2}\cos(v)$

$z=\sqrt{2u+4}\sin(v)$

$0\le v\le 2\pi$

$\vec{r}_2(u,v)=\langle u, \sqrt{u+2}\cos(v), \sqrt{2u+4}\sin(v)\rangle$

$u\ge -2$

$0\le v\le 2\pi$

Day 23 Lecture

Daily Announcements & Reminders:

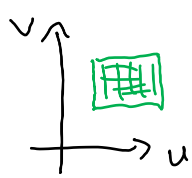
- HW 16.4 due tonight
- Quiz 10 tomorrow in studio, 16.4 & 16.5
  - L.O V3, V4, V5
- Exam 3 on Th 4/17, info on Canvas today/tomorrow
- Final Exam: 5/1 8-10:50, 3 parts, optional
- Do warmup on Ed

Goals for Today:

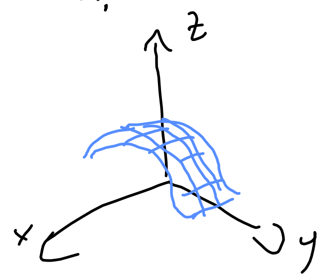
Sections 16.5/16.6

- Describe surfaces in $\mathbb{R}^3$ with a parameterization
- Define and compute surface integrals
- Use surface integrals to compute meaningful quantities: surface areas, masses, flux, etc.
- Interpret the physical significance of flux surface integrals

Goal: Extend double integrals to surfaces.



$$\vec{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$$



$$x = y^2 + \tfrac{1}{2}z^2 - 2$$

$$\vec{r}_1(s,t) = \langle s^2 + \tfrac{1}{2}t^2 - 2, s, t \rangle$$

$$(s,t) \in \mathbb{R}^2$$

$$\vec{r}_2(u,\theta) = \langle u, \sqrt{u+2}\cos(\theta), \sqrt{2u+4}\sin(\theta) \rangle$$

$$u \geq -2$$

$$0 \leq \theta \leq 2\pi$$

Example 147. Give parameteric representations for the surfaces below.

a) The portion of the surface $x = y^2 + \frac{1}{2}z^2 - 2$ which lies behind the yz-plane.

$x \leq 0$

$\vec{r}_1(s,t) = \langle s^2 + \frac{1}{2}t^2 - 2, s, t \rangle$

$s^2 + \frac{1}{2}t^2 \leq 2 \Rightarrow \frac{s^2}{2} + \frac{t^2}{4} \leq 1$

$\vec{r}_2(u,\theta) = \langle u, \sqrt{u+2}\cos(\theta), \sqrt{2u+4}\sin(\theta) \rangle$

$-2 \leq u \leq 0$

$0 \leq \theta \leq 2\pi$

b) $x^2 + y^2 + z^2 = 9$ $\Rightarrow \rho = 3$

In spherical coords:

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

$$\vec{r}(\varphi,\theta) = \begin{bmatrix} 3\sin\varphi\cos\theta \\ 3\sin\varphi\sin\theta \\ 3\cos\varphi \end{bmatrix}$$

$0 \leq \varphi \leq \pi$

$0 \leq \theta \leq 2\pi$

c) $x^2 + y^2 = 25$ $r = 5$

cylindrical:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$\longrightarrow$

$$\vec{r}(\theta,z) = \begin{bmatrix} 5\cos\theta \\ 5\sin\theta \\ z \end{bmatrix}$$

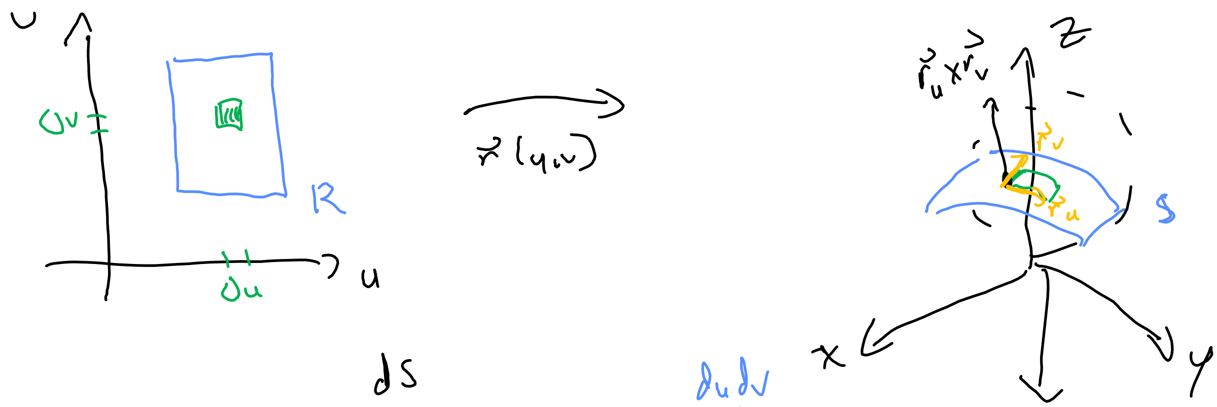
$0 \leq \theta \leq 2\pi$

$z \in \mathbb{R}$

What can we do with this?

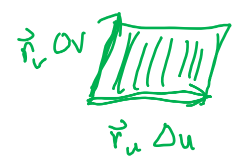
If our parameterization is **smooth** ($\mathbf{r}_u, \mathbf{r}_v$ not parallel in the domain), then:

- $\mathbf{r}_u \times \mathbf{r}_v$ is normal to the surface S
- A rectangle of size $\Delta u \times \Delta v$ in the uv-domain is mapped to a ~~rectangle~~ approximately parallelogram of size $\|\vec{r}_u \times \vec{r}_v\| \, \Delta u \, \Delta v$ on the surface in $\mathbb{R}^3$.



- Thus, $\text{Area}(S) = \iint_S 1 \, d\sigma = \iint_R \|\vec{r}_u \times \vec{r}_v\| \, dA$

  $d\sigma$ or dS (surface integral); $dA = du\,dv$ (double integral)



$\text{Area} = \|\vec{r}_u \Delta u \times \vec{r}_v \Delta v\| = \|\vec{r}_u \times \vec{r}_v\| \Delta u \Delta v$

Example 148 (Poll)**.** Find the area of the portion of the cylinder $x^2 + y^2 = 25$ between $z = 0$ and $z = 1$. Parametrize

1) $\vec{r}(\theta, z) = \langle 5\cos\theta, 5\sin\theta, z \rangle \qquad 0 \le \theta \le 2\pi \quad 0 \le z \le 1$

2) Compute $\|\vec{r}_\theta \times \vec{r}_z\|$

$$\vec{r}_\theta \times \vec{r}_z = \begin{array}{c} \\ \vec{r}_\theta \\ \vec{r}_z \end{array} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5\sin\theta & 5\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 5\cos\theta, 5\sin\theta, 0 \rangle$$

$\|\vec{r}_\theta \times \vec{r}_z\| = 5$

3) Compute: $SA = \int_0^{2\pi} \int_0^1 5 \, dz \, d\theta = 10\pi$

Strategy for computing surface integrals

e.g. surface area, mass, center of mass

1. Find a parameterization $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $(u,v) \in D$ for the surface S.

2. Compute the area scaling factor $\|\mathbf{r}_u \times \mathbf{r}_v\|$

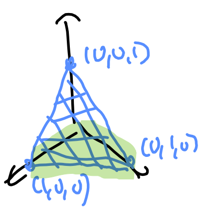
3. Substitute: $\iint_S f(x,y,z)d\sigma = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\, dA$

4. Integrate

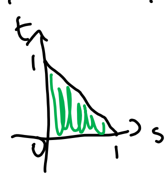
Example 149. Suppose the density of a thin plate S in the shape of the portion of the plane $x + y + z = 1$ in the first octant is $\delta(x,y,z) = 6xy$. Find the mass of the plate.

Goal: Find mass of plate. mass $= \iint_S \delta(x,y,z)\, d\sigma$

1) Parameterize S : $z = 1 - x - y$



$\vec{r}(s,t) = \langle s, t, 1-s-t\rangle$ $\left\{ \begin{array}{l} 0 \le s \le 1 \\ 0 \le t \le 1-s \end{array} \right.$



$s \ge 0$
$t \ge 0$
$1-s-t \ge 0$

$\delta = 6xy$

2) Find $\|\vec{r}_s \times \vec{r}_t\|$: $\vec{r}_s \times \vec{r}_t = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1\rangle$

$\|\vec{r}_s \times \vec{r}_t\| = \sqrt{3}$

3) Substitute: $\iint_S \delta(x,y,z)\, d\sigma = \iint_R \delta(\vec{r}(s,t))\, \|\vec{r}_s \times \vec{r}_t\|\, dA$

$= \int_0^1 \int_0^{1-s} 6st \cdot \sqrt{3}\, dt\, ds = \frac{\sqrt{3}}{4}$ kg

16.6: Flux through Surfaces

Pre Lecture

Goal: If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, find the total flux of $\mathbf{F}$ through a surface S.

Note: If the flux is positive, that means the net movement of the field through S is in the direction of ____________________

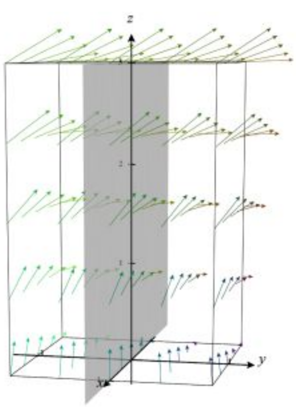
If $\mathbf{r}(u, v)$ is a smooth parameterization of S with domain R, we have

$$\text{flux of } \mathbf{F} \text{ through } S = \iint_S (\mathbf{F} \cdot \mathbf{n})\, d\sigma = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

next time

Example 150 (Poll)**.** Suppose S is a smooth surface in $\mathbb{R}^3$ and $\mathbf{F}$ is a vector field in $\mathbb{R}^3$. **True or False:** If $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma > 0$, then the angle between $\mathbf{F}$ and $\mathbf{n}$ is acute at all points on S.

Example 151 (Poll)**.** Based on the plot of the vector field $\mathbf{F}$ and the surface S below, oriented in the positive y-direction, is the flux integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ positive, negative, or zero?



16.6: Flux through Surfaces

Goal: If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, find the total flux of $\mathbf{F}$ through a surface S.

Note: If the flux is positive, that means the net movement of the field through S is in the direction of the normal vectors to S

If $\mathbf{r}(u,v)$ is a smooth parameterization of S with domain R, we have

unit normal

$$\text{flux of } \mathbf{F} \text{ through } S = \iint_S \underbrace{(\mathbf{F}\cdot\mathbf{n})}_{\substack{\text{normal component}\\ \text{of } \vec{F} \text{ to } S}} \, d\sigma = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot \underbrace{(\mathbf{r}_u \times \mathbf{r}_v)}_{\text{normal to } S} \, dA.$$

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \quad \text{so } \vec{n}\, d\sigma$$

$$= \frac{\vec{r}_u \times \vec{r}_v}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}} \cdot \cancel{\|\vec{r}_u \times \vec{r}_v\|}\, dA$$

Day 24 - Flux Integrals & the Generalized FTC Part I

Pre-Lecture

Section 16.6: Computing Flux Integrals

Example 152. Find the flux of the vector field

$$\mathbf{F}(x, y, z) = \langle 1, y, z \rangle$$

across the portion of the plane $x + 2y - z = 4$ above the rectangle $[0, 4] \times [0, 2]$ in the xy-plane, oriented in the $\langle 1, 2, -1 \rangle$ direction.

$$\text{flux} = \iint_S \vec{F} \cdot \vec{n}\, d\sigma$$

1) Parameterize: $\vec{r}(u,v) = \langle u, v, u+2v-4 \rangle$, $0 \le u \le 4$, $0 \le v \le 2$

2) $\vec{r}_u \times \vec{r}_v$ normal: $\vec{r}_u = \langle 1, 0, 1 \rangle$

$\vec{r}_v = \langle 0, 1, 2 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle -1, -2, 1 \rangle$ • opposite orientation!

use $\langle 1, 2, -1 \rangle$

3) Substitute:

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \int_0^2 \int_0^4 \underbrace{\langle 1, v, u+2v-4 \rangle}_{\vec{F}(\vec{r}(u,v))} \cdot \underbrace{\langle 1, 2, -1 \rangle}_{-(\vec{r}_u \times \vec{r}_v)}\, du\, dv$$

$$= \int_0^2 \int_0^4 1 + 2v - u - 2v + 4 \, du\, dv$$

$$= \int_0^2 \int_0^4 5 - u \, du\, dv$$

$$= 24$$