MATH 2551 D - Dr. Hunter Lehmann

- Dr. Lehmann, Dr. H, Dr. Hunter, as you prefer

Daily Announcements & Reminders:

- Introduce yourself to your neighbors
- Complete syllabus quiz in Canvas modules
- Be able to open Ed Discussion & Webwork
- Blank lecture notes will be posted each week ahead of class on Canvas -> Modules -> Lecture Notes

Goals for Today: Sections 12.1, 12.3, 12.4

- Set classroom norms
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Class Values/Norms:

- Mistakes are a learning opportunity
- Mathematics is collaborative
- Make sure everyone is included
- Criticize ideas, not people
- Be respectful of everyone
- Uphold academic honesty
- Open to new perspectives

Big Idea: Extend differential & integral calculus.

What are some key ideas from these two courses?

Differential Calculus		Integral Calculus	
Limits / continuity	✓	Area under a curve (volumes)	✓
Computing derivatives	✓	Convergence / divergence for a series	X
Optimization	✓	Taylor approx.	X
Unit circle / polar coordinates	✓	Riemann sums	~
Fund. Thm of Calculus	✓	Integration techniques	
		Improper integrals	X

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

- $f(t) = \langle t^2, t^3 \rangle = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix} = (t^2)\vec{i} + (t^3)\vec{j} = (t^2)\vec{e}_1 + (t^3)\vec{e}_2$

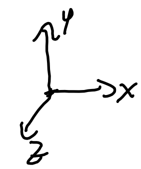
  $\vec{i} = \vec{e}_1 = \langle 1, 0 \rangle \quad \vec{j} = \vec{e}_2 = \langle 0, 1 \rangle$

- $g(x,y) = x^2 e^y$

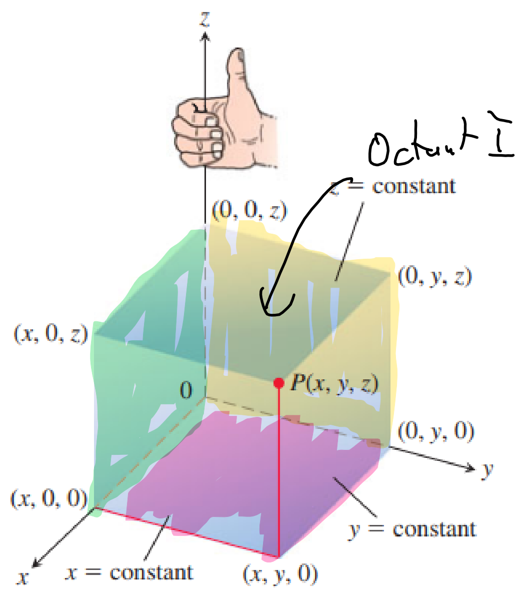
- $h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y^2 \end{bmatrix}$

Section 12.1: Three-Dimensional Coordinate Systems

Phys:



• $\mathbb{R}^3$: (x, y, z)



• Right-handed

• coordinate planes

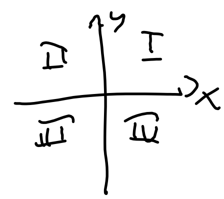
$x=0$ (yz-plane)

$y=0$ (xz-plane)

$z=0$ (xy-plane)

• Eight octants

In $\mathbb{R}^2$

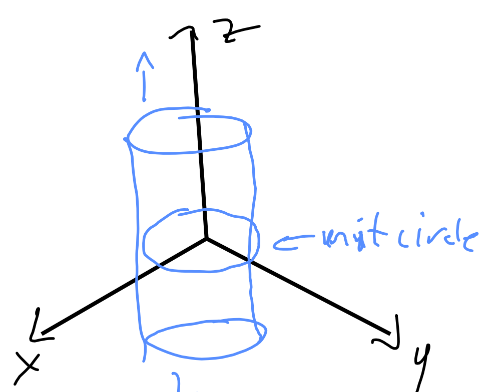


Question: What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation $x^2+y^2 = 1$?

• cylinder
→ no z-component, means circle can be anywhere

• circle
→ no z-component means no thickness

Use 2D intuition to guide 3D

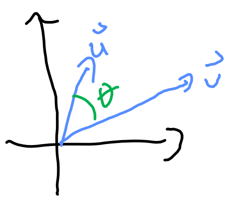


Section 12.3/4: Dot & Cross Products

Definition 1. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

This product tells us about angle between two vectors.



$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos(\theta)$$

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

In particular, two vectors are **orthogonal** if and only if their dot product is 0.

Example 2. Are $\mathbf{u} = \langle 1, 1, 4 \rangle$ and $\mathbf{v} = \langle -3, -1, 1 \rangle$ orthogonal?

$$\vec{u} \cdot \vec{v} = (1)(-3) + (1)(-1) + 4(1)$$

$$= -3 - 1 + 4 = 0$$

So $\vec{u}, \vec{v}$ are orthogonal.

$\times$

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.

1. Distributive: $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$

2. Respect Size: $c(\vec{u}\times\vec{v}) = (c\vec{v})\times\vec{v} = \vec{u}\times(c\vec{v})$

Definition 3. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ in $\mathbb{R}^3$ is

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & v_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Daily Announcements & Reminders:

- HW 12.2, 12.3 due tonight at 10 pm
- Office hours will be:
  9–10, M, virtual
  11–12 R, Skiles 218C
  3–4 F, virtual (starting next week)
  or by appointment
- Do the warm up poll question

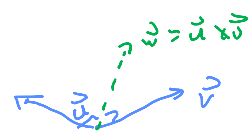
Goals for Today: Sections 12.5-12.6

- Apply the cross product to solve problems
- Learn the equations that describe lines, planes, and quadric surface in $\mathbb{R}^3$
- Solve problems involving the equations of lines and planes
- Sketch quadric surfaces in $\mathbb{R}^3$

Last time, rephrased

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.

1. Right-handed:



- $\vec{u} \times \vec{v}$ is antisymmetric
  $(\vec{u} \times \vec{v}) = -(\vec{v} \times \vec{u})$

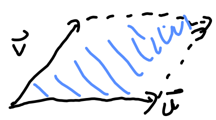
2. Algebraically a product: if $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$:
   - $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
   - $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$

Definition 5. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ in $\mathbb{R}^3$ is the symbolic determinant

$\vec{i} = \langle 1,0,0\rangle$
$\vec{j} = \langle 0,1,0\rangle$
$\vec{k} = \langle 0,0,1\rangle$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\sin\theta$ = area of parallelogram spanned by $\vec{u},\vec{v}$



area $= \|\vec{u}\times\vec{v}\|$

- From Lin. Alg.: volume of parallelipiped w/ sides $\vec{u},\vec{v},\vec{w}\in\mathbb{R}^3$

$$V = \left|\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3\end{pmatrix}\right| = \left|(\vec{u}\times\vec{v})\cdot\vec{w}\right|$$

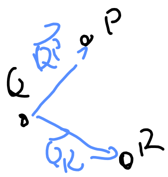
Example 6. Find $\underset{\vec{u}}{\langle 1, 2, 0\rangle} \times \underset{\vec{v}}{\langle 3, -1, 0\rangle}$.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 3 & -1 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} 2 & 0 \\ -1 & 0\end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 3 & 0\end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 3 & -1\end{vmatrix}$$

$$= \vec{i}\,(2(0)-(-1)(0)) - \vec{j}\,((1)(0)-(3)(0)) + \vec{k}\,((1)(-1)-(2)(3))$$

$$= 0\vec{i} + 0\vec{j} - 7\vec{k}$$

$$= \langle 0, 0, -7\rangle$$



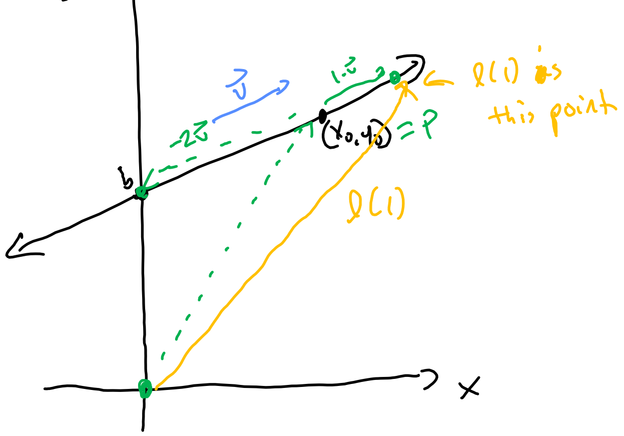
$$\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix}$$

↑ does not produce $\overrightarrow{QP}\times\overrightarrow{QR}$

Section 12.5 Lines & Planes

Lines in $\mathbb{R}^2$, a new perspective:

$y = mx + b$ $\quad$ $y - y_0 = m(x - x_0)$



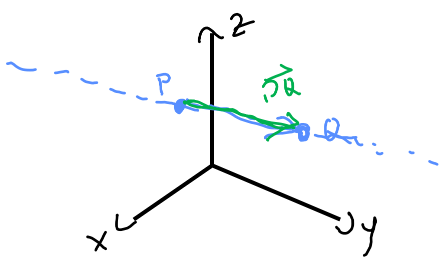
line is all points

$\ell(t) = \overrightarrow{OP} + (t)\vec{v}$ ← $\vec{v}$ is direction vector of the line

↑ P is a point on the line $\quad$ for $t \in \mathbb{R}$

↑ vector equation

Example 7. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.



Need: • point on line; P or Q

• direction: take $\overrightarrow{PQ}$ or $\overrightarrow{QP}$

$\overrightarrow{PQ} = \langle -2-1, 1-0, 1-2 \rangle = \langle -3, 1, -1 \rangle$

$$\ell(t) = \overrightarrow{OP} + t \cdot \overrightarrow{PQ} = \langle 1, 0, 2 \rangle + t\langle -3, 1, -1 \rangle$$
$$= \langle -3t + 1, t, 2 - t \rangle$$

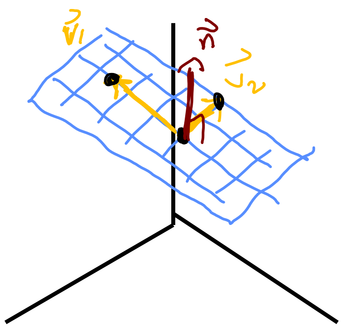
for $t \in \mathbb{R}$

Parametric equations:

$$\begin{cases} x(t) = -3t + 1 \\ y(t) = t \\ z(t) = 2 - t \end{cases}, \quad t \in \mathbb{R}$$

Planes in $\mathbb{R}^3$

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction $\mathbf{n}$, called the *normal vector.*



Need; a) 2 vectors (in the plane) and 1 point on plane

b) 3 points on plane

c) 1 point on plane and 1 vector $\vec{n}$ normal to plane

Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

$$ax + by + cz = d \qquad \text{(2 free variables)}$$

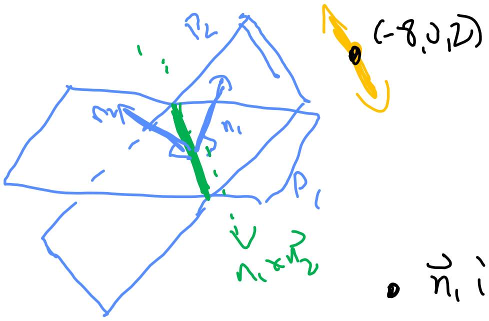
If $P = (x_0, y_0, z_0)$ is in plane and $\vec{n} = \langle a, b, c\rangle$ and $Q = (x, y, z)$ in plane:

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

$$ax + by + cz = d = ax_0 + by_0 + cz_0 = \vec{OP} \cdot \vec{n}$$

Example 8. Consider the planes $y - z = -2$ and $x - y = 0$. Show that the planes intersect and find an equation for the line passing through the point $P = (-8, 0, 2)$ which is parallel to the line of intersection of the planes.



Plane 1 $\quad y - z = -2 \qquad \vec{n}_1 = \langle 0, 1, -1 \rangle$

Plane 2: $\quad x - y = 0 \qquad \vec{n}_2 = \langle 1, -1, 0 \rangle$

- $\vec{n}_1$ is not parallel (not a scalar mult) of $\vec{n}_2$ so planes meet

$2x + 7y - 5z = 0$

$\vec{n} = \langle 2, 7, -5 \rangle$

Goal 2: Find line through $(-8,0,2)$ parallel to line of intersection

So direction vector $\vec{v} = \vec{n}_1 \times \vec{n}_2$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \langle (1)(0) - (-1)(-1), -((0)(0) - (1)(-1)), (0(-1) - (1)(1)) \rangle$$

$$= \langle -1, -1, -1 \rangle$$

So $\boxed{\ell(t) = \langle -1, -1, -1 \rangle t + \langle -8, 0, 2 \rangle}$

Daily Announcements & Reminders:

- Monday office hour → 8-9 am on Zoom
- HW for 12.1, 12.4 due tonight
- Studio attendance credit began on Monday
- Quiz 1 on W; 12.1, 12.4, 12.5
- Do warmup poll →
- PLUS Sessions: Sunday through Wednesday

Goals for Today: Sections 12.6, 13.1

- Sketch quadric surfaces in $\mathbb{R}^3$
- Introduce vector-valued functions
- Plot vector-valued functions and construct them from a graph
- Compute limits, derivatives, and tangent lines for vector-valued functions

Section 12.6 Quadric Surfaces

Definition 11. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

You know several examples already:

spheres: $(x-a)^2+(y-b)^2+(z-c)^2=r^2$; center (a,b,c), radius r

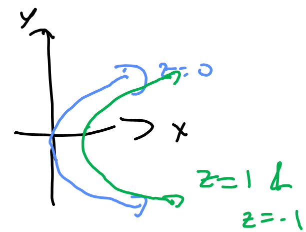
(circular) cylinders: $x^2+y^2=1$

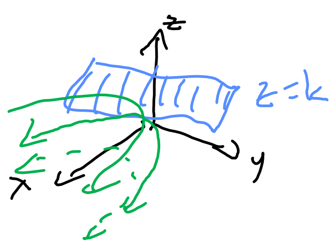
- We will stick to surfaces oriented along coordinate axes; no xy, yz, xz terms.

The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections.

Example 12. Use cross-sections to sketch and identify the quadric surface $x = z^2 + y^2$.

Horizontal cross sections: Fix $z=0$. $x=0^2+y^2 \Rightarrow x=y^2$ parabola



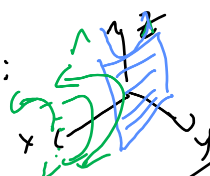


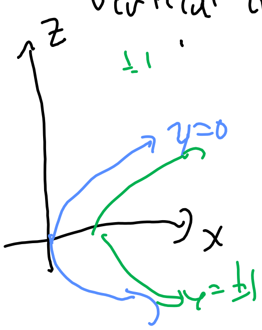
Fix $z=1$. $x=1+y^2$ parabola

Fix $z=-1$. $x=1+y^2$

In general: $z=k$ $x=k^2+y^2$; parabola opening to right in plane $z=k$

Vertical cross sections: Fix values of y:





$y=0$: $x=z^2$

$y=\pm 1$: $x=z^2+1$

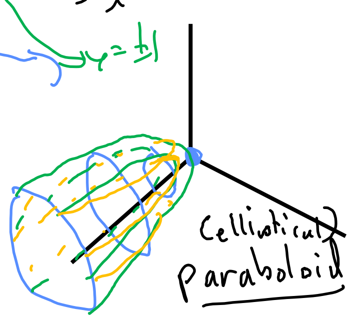
Fix values of x:

$x=0$: $0=z^2+y^2$ point

$x=1$: $1=z^2+y^2$ circle

$x=-1$: $-1=z^2+y^2$ no soln

$x=k>0$: $k=z^2+y^2$ circle of radius $\sqrt{k}$



(elliptical) paraboloid

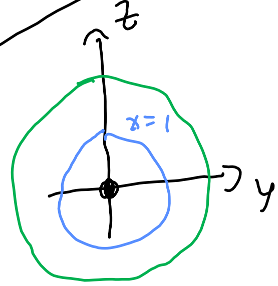
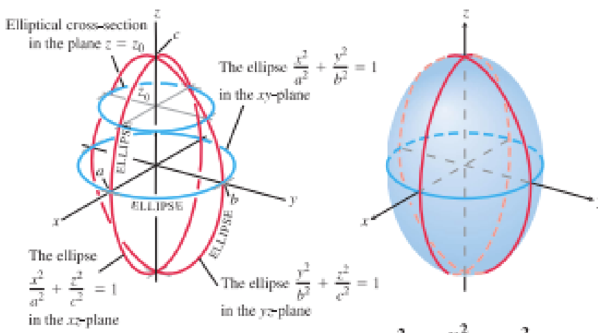
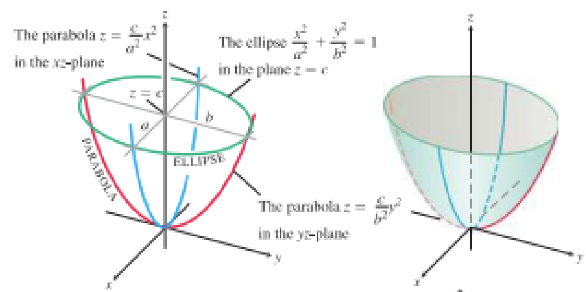


TABLE 12.1 Graphs of Quadric Surfaces



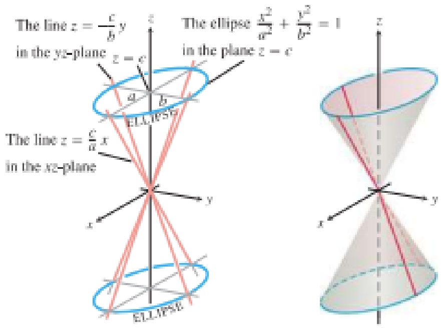
ELLIPSOID $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

• x^2, y^2, z^2 coeffs all same sign



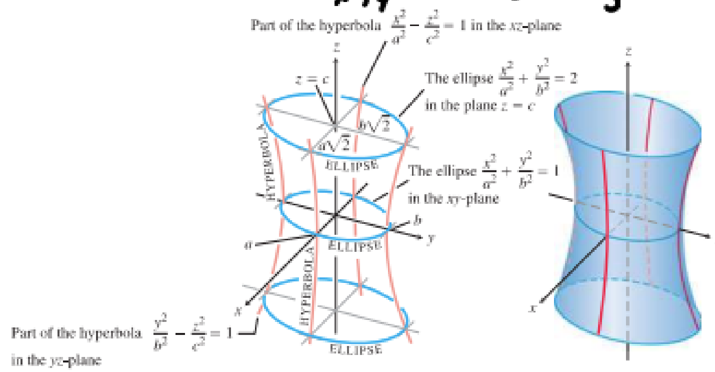
ELLIPTICAL PARABOLOID $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$

• z but no z^2 term, x^2, y^2 coeffs same sign



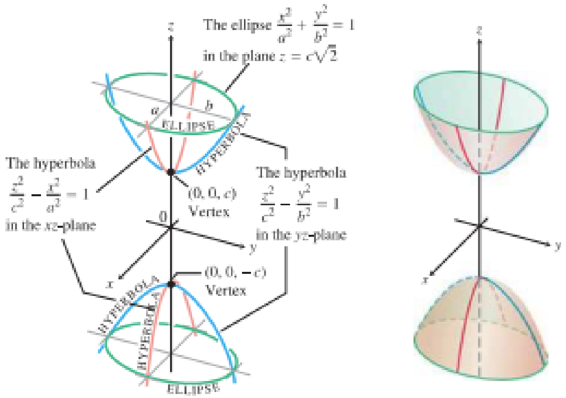
ELLIPTICAL CONE $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$

• no constant term, x^2, y^2 coeffs diff. sign than z^2



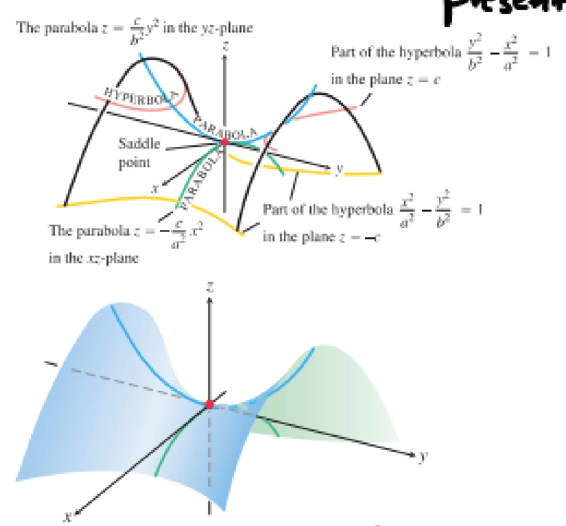
HYPERBOLOID OF ONE SHEET $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

• x^2, y^2 coeffs pos, z^2 neg., constant term present



HYPERBOLOID OF TWO SHEETS $\quad \dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

• x^2, y^2 coeffs neg, z^2 pos, constant term present



HYPERBOLIC PARABOLOID $\quad \dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = \dfrac{z}{c}, c > 0$

• z but no z^2 term, x^2, y^2 coeffs opp. sign

Section 13.1 Curves in Space & Their Tangents

The description we gave of a line last week generalizes to produce other one-dimensional graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$ as well. We said that a function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ with $\mathbf{r}(t) = \mathbf{v}t + \mathbf{r}_0$ produces a straight line when graphed.

constant

This is an example of a **vector-valued function**: its input is a real number t and its output is a vector. We graph a vector-valued function by plotting all of the terminal points of its output vectors, placing their initial points at the origin.

You have seen several examples already:

- lines in $\mathbb{R}^3$
- circles: $x^2+y^2=1$ in $\mathbb{R}^2$

  $\vec{r}(\theta) = \langle \cos(\theta), \sin(\theta) \rangle \quad 0 \le \theta \le 2\pi$
- other curves: $y = x^2 \longleftrightarrow \vec{r}(t) = \langle t, t^2 \rangle \quad t \in \mathbb{R}$

Given a fixed curve C in space, producing a vector-valued function $\mathbf{r}$ whose graph is C is called parameterizing the curve C, and $\mathbf{r}$ is called a parameterization of C.

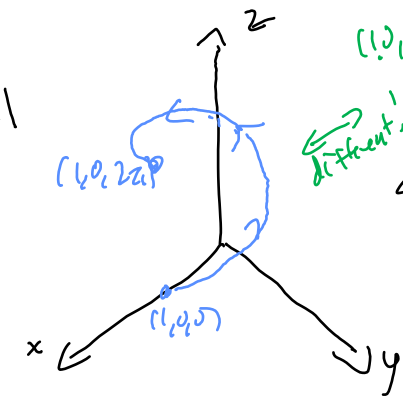
Example 13. Consider $\mathbf{r}_1(t) = \langle \cos(t), \sin(t), t\rangle$ and $\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t\rangle$, each with domain $[0, 2\pi]$. What do you think the graph of each looks like? How are they similar and how are they different?

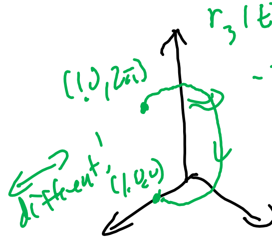
- cylinder
- springs / helix / spiral
- football

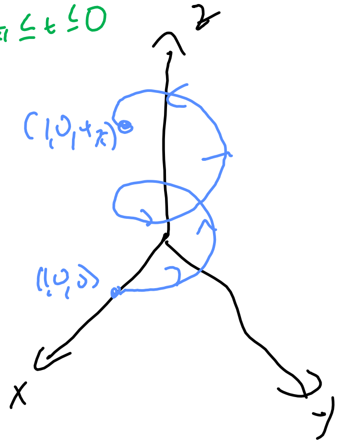
2D, not 1D

$r_3(t) = \langle \cos(-t), \sin(-t), -t\rangle$

$-2\pi \le t \le 0$







- $\vec{r}_1$ & $\vec{r}_2$ parametrize different curves in space
  - domain really matters
  - with domain $\mathbb{R}$, we get same curve, traversed differently

Check your intuition

Calculus of vector-valued functions

Unifying theme: Do what you already know, componentwise.

This works with limits:

Example 14. Compute $\lim_{t\to e}\langle t^2, 2, \ln(t)\rangle$.

$$= \left\langle \lim_{t\to e} t^2, \lim_{t\to e} 2, \lim_{t\to e} \ln(t) \right\rangle$$

$$= \langle e^2, 2, 1 \rangle$$

And with continuity:

Example 15. Determine where the function $\mathbf{r}(t) = t\mathbf{i} - \frac{1}{t^2-4}\mathbf{j} + \sin(t)\mathbf{k}$ is continuous.

- $x(t)$, $y(t)$, $z(t)$ are all cts on their domains.

so $\vec{r}(t)$ is cts on $(\mathbb{R}) \cap \left(\underbrace{(-\infty, -2) \cup (-2, 2) \cup (2, \infty)}_{=} \right) \cap (\mathbb{R})$

And with derivatives:

Example 16. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1 \rangle$, find $\mathbf{r}'(t)$.

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

- $\mathbf{r}'(t)$ gives ______________________________
- $|\mathbf{r}'(t)|$ gives ______________________________
- $\mathbf{r}''(t)$ gives ______________________________

Let's see this graphically

Example 17. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1 \rangle$ at time $t = 2$.

Daily Announcements & Reminders:

- HW 12.5 due tonight
- Do warmup problem

Goals for Today: Sections 13.1-13.3

- Compute limits, derivatives, and tangent lines for vector-valued functions
- Compute integrals of vector-valued functions and solve initial value problems
- Compute arc lengths of curves using parameterizations
- Introduce the idea of an arc-length parameterization

Continuing from last time with <u>derivatives</u>:

Example 16. If $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$, find $\mathbf{r}'(t)$.

$$\vec{r}'(t) = \langle \frac{d}{dt}(2t - \tfrac{1}{2}t^2 + 1), \frac{d}{dt}(t-1) \rangle$$

$$= \langle 2 - t, 1 \rangle$$

$$\vec{r}''(t) = \langle -1, 0 \rangle$$

Interpretation: If $\mathbf{r}(t)$ gives the position of an object at time t, then

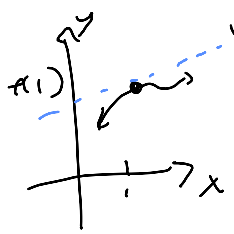
- $\mathbf{r}'(t)$ gives velocity at time t
- $||\mathbf{r}'(t)||$ gives speed at time t
- $\mathbf{r}''(t)$ gives acceleration at time t

Let's see this graphically

Example 17. Find an equation of the tangent line to $\mathbf{r}(t) = \langle 2t - \frac{1}{2}t^2 + 1, t - 1\rangle$ at time $t = 2$.

If $y = f(x)$: to get tangent line at $x=1$

$y = f'(1)(x-1) + f(1)$



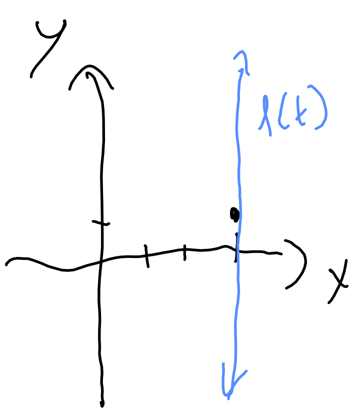
$\vec{r}'(t) = \langle 2-t, 1\rangle$

$\vec{r}'(2) = \langle 0, 1\rangle$

$\vec{r}(2) = \langle 4-2+1, 2-1\rangle$
$= \langle 3, 1\rangle$

For $\vec{r}(t)$: tangent line at $t=2$ is

$$\ell(t) = \vec{r}'(2)(t-2) + \vec{r}(2)$$
$$= \langle 0, 1\rangle(t-2) + \langle 3, 1\rangle$$
$$= \langle 3, t-1\rangle$$



And with integrals:

Example 18. Find $\int_0^1 \underbrace{\langle t, e^{2t}, \sec^2(t)\rangle}_{\vec{r}\,'(t)} \, dt$.

$\int_a^b \vec{r}\,'(t)\,dt$ = displacement from $t=a$ to $t=b$
$\vec{r}(b) - \vec{r}(a)$

$$= \langle \int_0^1 t\,dt, \int_0^1 e^{2t}\,dt, \int_0^1 \sec^2(t)\,dt \rangle$$
$$= \langle \tfrac{1}{2}t^2\Big|_0^1, \tfrac{1}{2}e^{2t}\Big|_0^1, \tan(t)\Big|_0^1 \rangle$$
$$= \langle \tfrac{1}{2}-0, \tfrac{1}{2}e^2-\tfrac{1}{2}e^0, \tan(1)-\tan(0) \rangle$$
$$= \langle \tfrac{1}{2}, \tfrac{1}{2}(e^2-1), \tan(1) \rangle$$

At this point we can solve initial-value problems like those we did in single-variable calculus:

Example 19. Wallace is testing a rocket to fly to the moon, but he forgot to include instruments to record his position during the flight. He knows that his velocity during the flight was given by

$$\mathbf{v}(t) = \langle -200\sin(2t), 200\cos(t), 400 - \frac{400}{1+t} \rangle \ m/s.$$

If he also knows that he started at the point $\mathbf{r}(0) = \langle 0,0,0 \rangle$, use calculus to reconstruct his flight path.

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle 100\cos(2t), 200\sin(t), 400t - 400\ln|1+t| \rangle + \vec{C}$$
$$= \langle 100\cos(2t) + C_1, 200\sin(t) + C_2, 400(t - \ln|1+t|) + C_3 \rangle$$

Set $\vec{r}(0) = \langle 0,0,0 \rangle$ and solve

$$\langle 0,0,0 \rangle = \vec{r}(0) = \langle 100 + C_1, C_2, C_3 \rangle$$
$$C_1 = -100, \ C_2 = 0, \ C_3 = 0 \quad \text{OR} \quad \vec{C} = \langle -100, 0, 0 \rangle$$

$$\boxed{\vec{r}(t) = \langle 100(\cos(2t)-1), 200\sin(t), 400(t - \ln|1+t|) \rangle \quad t \geq 0}$$

13.3 Arc length of curves

We have discussed motion in space using by equations like $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

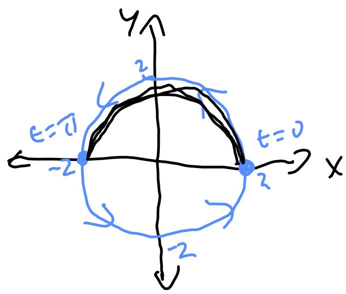
Our next goal is to be able to measure distance traveled or arc length.

Motivating problem: Suppose the position of a fly at time t is

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle,$$

where $0 \le t \le 2\pi$.

a) Sketch the graph of $\mathbf{r}(t)$. What shape is this?



b) How far does the fly travel between $t = 0$ and $t = \pi$?

length of arc of angle θ = $r\theta$

$\pi \cdot 2$

c) What is the speed $\|\mathbf{v}(t)\|$ of the fly at time t?

$\vec{v}(t) = \vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t)\rangle$

$\|\vec{v}(t)\| = \sqrt{4\sin^2(t) + 4\cos^2(t)} = \sqrt{4(\sin^2(t)+\cos^2(t))} = \sqrt{4} = 2$

d) Compute the integral $\int_0^\pi \|\mathbf{v}(t)\|\,dt$. What do you notice?

$= \int_0^\pi 2\,dt = 2t\Big|_0^\pi = 2\pi$

Definition 20. We say that the **arc length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ from $t=a$ to $t=b$ that is traced out exactly once is

$$L = \int_a^b \|\vec{r}\,'(t)\|\,dt = \int_a^b \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$$

Example 21. Set up an integral for the arc length of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from the point $(1,1,1)$ to the point $(2,4,8)$.

$t=1$ $t=2$

Endpoints: $(1,1,1) = \vec{r}(t)$: $1=t$ (x), $1=t^2$ (y), $1=t^3$ (z) | $(2,4,8) = \vec{r}(t)$: $2=t$, $4=t^2$, $8=t^3$

Speed: $\vec{r}\,'(t) = \hat{i} + (2t)\hat{j} + (3t^2)\hat{k}$

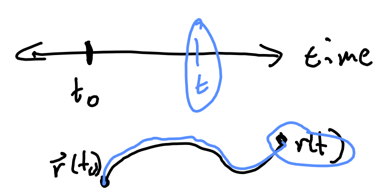
$\|\vec{r}\,'(t)\| = \sqrt{1+4t^2+9t^4}$

Substitute: $\boxed{L = \int_1^2 \sqrt{1+4t^2+9t^4}\,dt}$

Sometimes, we care about the distance traveled from a fixed starting time t_0 to an arbitrary time t, which is given by the **arc length function**.

$$s(t) = \int_{t_0}^{t} \|\vec{v}(T)\|\,dT$$

constant



We can use this function to produce parameterizations of curves where the parameter s measures distance along the curve: the points where $s=0$ and $s=1$ would be exactly 1 unit of distance apart.

arc length parameterization $\Longleftrightarrow$ unit speed parameterization

Example 22. Find an arc length parameterization of the circle of radius 4 about the origin in $\mathbb{R}^2$, $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle$, $0 \le t \le 2\pi$.

not an arc length parameterization: $\|\vec{r}'(t)\| \neq 1$

1) Compute arc length function

• might be hard to integrate

$$s(t) = \int_{t_0}^{t} \|\vec{r}'(\tau)\|\, d\tau$$

• take $t_0 = 0$

$$\vec{r}'(\tau) = \langle -4\sin(\tau), 4\cos(\tau)\rangle$$

$$\|\vec{r}'(\tau)\| = 4$$

$$s(t) = \int_0^t 4\, d\tau = 4\tau\Big|_0^t = 4t$$

2) Invert & solve for t

• always possible, often hard

$$s = 4t \iff t = \frac{s}{4} = f(s)$$

3) Substitute into $\vec{r}$:

$$\vec{r}_2(s) = \vec{r}(f(s)) = \vec{r}\left(\frac{s}{4}\right) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle$$

$$0 \le \frac{s}{4} \le 2\pi \iff 0 \le s \le 8\pi$$

Daily Announcements & Reminders:

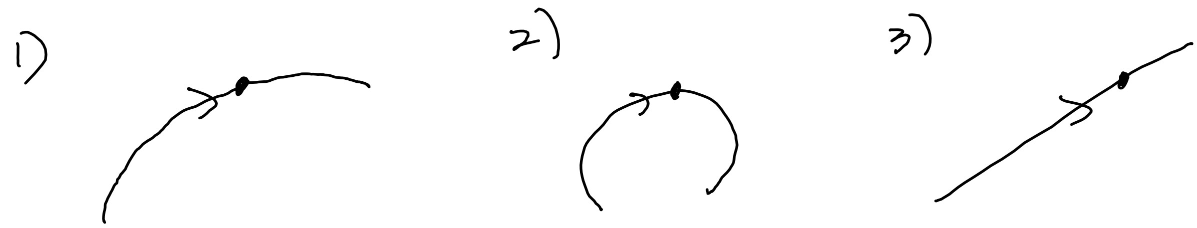
- HW 12.6, 13.1 due tonight ; 13.2 due Th
- Quiz 2 in studio tomorrow on 12.6, 13.1, 13.2 | LO: 63, 66
- Learning Outcomes page published in Canvas Modules
- Do Day 5 Warmup Poll on Ed →

Goals for Today: Sections 13.3-13.4

- Define, interpret, and compute the curvature of a curve
- Compute the unit tangent and principal unit normal vectors of a curve
- Give examples of functions of multiple variables
- Find the domain of functions of two variables
- Graph functions of two variables

13.4 - Curvature, Tangents, Normals

The next idea we are going to explore is the <u>curvature</u> of a curve in space along with two vectors that orient the curve.



Rank curvature of the curves above at the marked points

2 > 1 > 3

- Scale 2nd deriv. appropriately || • curvature measures how fast direction of motion changes

First, we need the **unit tangent vector**, denoted **T**:

- In terms of an arc-length parameter s: $\vec{r}\,'(s)$
- In terms of any parameter t: $\vec{r}\,'(t) / \|\vec{r}\,'(t)\|$

This lets us define the **curvature**, $\kappa(s) =$ $\|\vec{T}\,'(s)\|$

Example 23. Last class we found an arc length parameterization of the circle of radius 4 centered at $(0,0)$ in $\mathbb{R}^2$:

$$\mathbf{r}(s) = \left\langle 4\cos\left(\frac{s}{4}\right), 4\sin\left(\frac{s}{4}\right)\right\rangle, \qquad 0 \le s \le 8\pi.$$

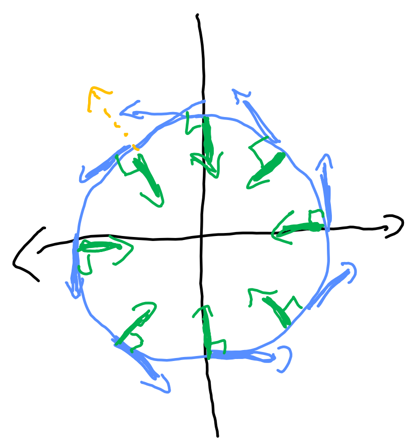
Use this to find $\mathbf{T}(s)$ and $\kappa(s)$.

- $\vec{T}(s) = \vec{r}\,'(s) = \langle -4\sin(\frac{s}{4})\cdot\frac{1}{4}, 4\cos(\frac{s}{4})\cdot\frac{1}{4}\rangle$
$= \langle -\sin(\frac{s}{4}), \cos(\frac{s}{4})\rangle$ • unit vector!

- $\kappa(s) = \|\vec{T}\,'(s)\|$
$= \| \langle -\frac{1}{4}\cos(\frac{s}{4}), -\frac{1}{4}\sin(\frac{s}{4})\rangle \|$
$= \| -\frac{1}{4}\langle \cos(\frac{s}{4}), \sin(\frac{s}{4})\rangle \|$
$= |-\frac{1}{4}| \underbrace{\sqrt{\cos^2(\frac{s}{4}) + \sin^2(\frac{s}{4})}}_{1}$
$= \frac{1}{4}$

• independent of position for this circle

• $\kappa = \frac{1}{r}$ for any circle



$\vec{N}(s) = \langle -\cos(\frac{s}{4}), -\sin(\frac{s}{4})\rangle$

Question: In which direction is **T** changing? $\vec{T}\,'(s)$

This is the direction of the $\underbrace{\textbf{principal}}_{\text{pointing in direction of motion}}$ $\overbrace{\textbf{unit}}^{\text{unit vector}}$ $\underbrace{\textbf{normal}}_{\perp \text{ to curve}/\vec{T}(s)}$, $\mathbf{N}(s) =$ $\vec{T}\,'(s)/\|\vec{T}\,'(s)\|$

We said last time that it is often hard to find arc length parameterizations, so what do we do if we have a generic parameterization $\mathbf{r}(t)$?

- $\mathbf{T}(t) =$ $\vec{r}\,'(t) / \|\vec{r}\,'(t)\|$
- $\mathbf{N}(t) =$ $\vec{T}\,'(t) / \|\vec{T}\,'(t)\|$
- $\kappa(t) =$ $\|\vec{T}\,'(t)\| / \|\vec{r}\,'(t)\|$ or $\dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$

Example 24. Find $\mathbf{T}, \mathbf{N}, \kappa$ for the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t-1\rangle$.

Need $\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t), 1\rangle$

$\|\vec{r}\,'(t)\| = \sqrt{4\sin^2(t) + 4\cos^2(t) + 1} = \sqrt{4+1} = \sqrt{5}$

So $\boxed{\vec{T}(t) = \frac{1}{\sqrt{5}}\langle -2\sin(t), 2\cos(t), 1\rangle}$

Next, $\vec{N}(t) = \vec{T}\,'(t) / \|\vec{T}\,'(t)\|$

$\vec{T}\,'(t) = \frac{1}{\sqrt{5}}\langle -2\cos(t), -2\sin(t), 0\rangle$

$\|\vec{T}\,'(t)\| = \frac{1}{\sqrt{5}}\sqrt{4\cos^2(t) + 4\sin^2(t) + 0} = 2/\sqrt{5}$

$\frac{1}{\sqrt{5}} / (2/\sqrt{5}) = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{2} = \frac{1}{2}$

So $\vec{N}(t) = \frac{1}{2}\langle -2\cos(t), -2\sin(t), 0\rangle$

$= \boxed{\langle -\cos(t), -\sin(t), 0\rangle}$

$\kappa(t) = \dfrac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \dfrac{2/\sqrt{5}}{\sqrt{5}} = \boxed{\dfrac{2}{5}}$

14.1 Functions of Multiple Variables

Definition 25. A function of two variables is a rule that assigns to each pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$.

$$f : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^2$$

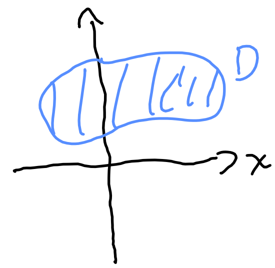
name

domain

codomain

(set of possible pairs (x,y) where f is well-defined)

(range is those real #s that are actually outputs)



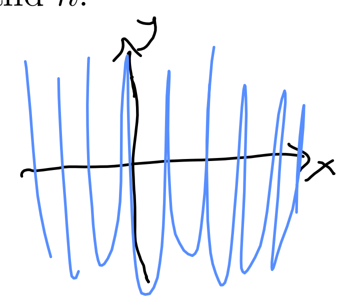
Example 26. Three examples are

$$f(x, y) = x^2 + y^2, \qquad g(x, y) = \ln(x + y), \qquad h(x, y) = \frac{1}{\sqrt{x + y}}.$$

Example 27. Find the largest possible domains of f, g, and h.

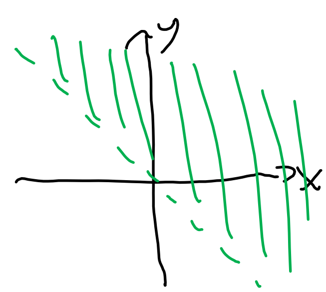
$f(x,y) = x^2+y^2$

Domain is all of $\mathbb{R}^2 = \{ (x,y) \mid x,y \in \mathbb{R} \}$



$g(x,y) = \ln(x+y)$

Domain is $\{ (x,y) \in \mathbb{R}^2 \mid x+y > 0 \} = \{ (x,y) \mid y > -x \}$



$h(x,y) = \dfrac{1}{\sqrt{x+y}}$

from $\sqrt{\ }$

Domain is $\{ (x,y) \mid x+y \geq 0,\ \sqrt{x+y} \neq 0 \} =$

from $\frac{1}{u}$

Domain is NEVER an interval!

Definition 28. If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Here are the graphs of the three functions above.

Example 29. Suppose a small hill has height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x, y). How could we draw a picture that represents the hill in 2D?

In 3D, it looks like this.

Definition 30. The ____________ (also called ____________________) of a function f of two variables are the curves with equations ________________, where k is a constant (in the range of f). A plot of ____________ for various values of z is a ____________________(or ____________________________).

Some common examples of these are:

-
-
-

True or False: The vector-valued function

$$\mathbf{r}(t) = \langle 2t^2 + 1, t \rangle, \quad t \in \mathbb{R}$$

is an arc-length (unit speed) parameterization of the parabola $x = 2y^2 + 1$.

True	35%
False	55%
I don't know	10%

80 votes

$t = y$

$\hookrightarrow x = 2t^2 + 1$

So $x = 2y^2 + 1$ & can get all y-values

To check arclength param:

is $\|\vec{r}\,'(t)\| = 1$?

$\vec{r}\,'(t) = \langle 4t, 1 \rangle$

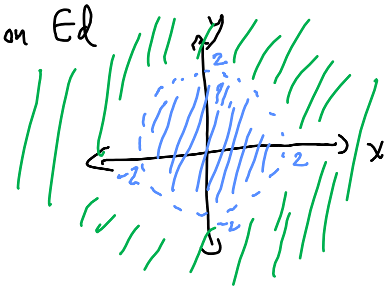
$\|\vec{r}\,'(t)\| = \sqrt{16t^2 + 1}$

$\neq 1$

so not unit-speed / arc-length

Daily Announcements & Reminders:

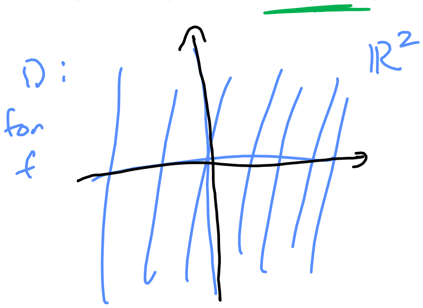
- HW 13.2 due tonight
- Exam 1 is on 9/17
  - look at old exams for practice
  - formula sheet provided will be posted next week
- Do warmup problem →
  - Day 6 Warmup Poll on Ed

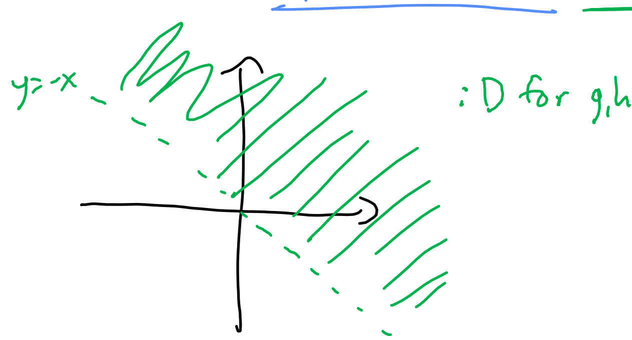


Goals for Today: Section 14.1

- Introduce and sketch traces and contours of functions of two variables
- Find level surfaces of functions of three variables
- Graph functions of two variables

Last time, we discussed the domains of the functions $f(x,y) = x^2 + y^2$, $g(x,y) = \ln(x+y)$, and $h(x,y) = \dfrac{1}{\sqrt{x+y}}$.





Definition 31. If f is a function of two variables with domain D, then the graph of f is the set of all points (x,y,z) in $\mathbb{R}^3$ such that $z = f(x,y)$ and (x,y) is in D.

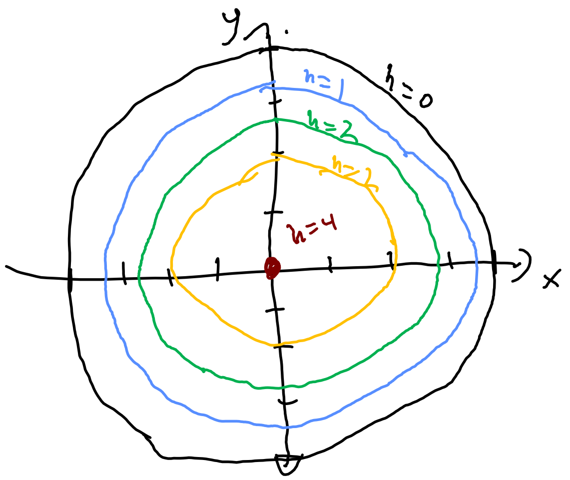
e.g. $(1,3,10)$ is on the graph of f b/c $(1,3)$ is in its domain and $f(1,3)=1+9=10$

Here are the graphs of the three functions above.

Example 32. Suppose a small hill has height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ m at each point (x, y). How could we draw a picture that represents the hill in 2D?

Domain: $\{(x,y) \mid x^2+y^2 \le 16\}$

- Draw curves of all pts in domain with a fixed height.



h = 0 m: $0 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow x^2+y^2 = 16$

h = 1 m: $1 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow x^2+y^2 = 12$

h = 2 m: $2 = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2 \Rightarrow x^2+y^2 = 8$

h = 3 m: $x^2+y^2 = 4$

h = 4: $0 = x^2+y^2$

In 3D, it looks like this.

Definition 33. The contours (also called level curves (or level sets)) of a function f of two variables are the curves with equations $k = f(x,y)$, where k is a constant (in the range of f). A plot of contours for various values of z is a contour plot/map (or level curve map/plot).

Some common examples of these are:

- topographical map
- pole zero plots
- electric field plots
- weather maps
- electron orbital maps

Example 34. Create a contour diagram of $f(x, y) = x^2 - y^2$

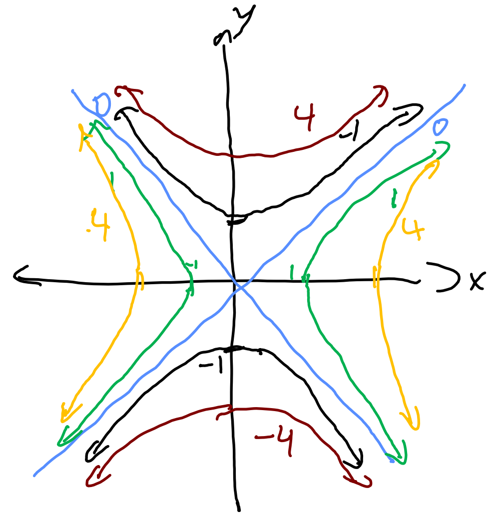
$0 = x^2 - y^2 \Rightarrow x^2 = y^2 \Rightarrow x = y$ or $x = -y$

$1 = x^2 - y^2$ hyperbola $\Leftrightarrow$

$4 = x^2 - y^2$ hyperbola $\Leftrightarrow$

$-1 = x^2 - y^2 \Rightarrow 1 = y^2 - x^2$ hyperbola $\Updownarrow$

$-4 = x^2 - y^2$



Definition 35. The traces of a surface are the curves of intersection of the surface with planes parallel to the xz & yz planes. ($y=k$ or $x=k$)

Example 36. Use the traces and contours of $z = f(x, y) = 4 - 2x - y^2$ to sketch the portion of its graph in the first octant. $x \geq 0, y \geq 0, z \geq 0$

$z = 0$: $0 = 4 - 2x - y^2$

$x = 2 - \frac{1}{2}y^2$

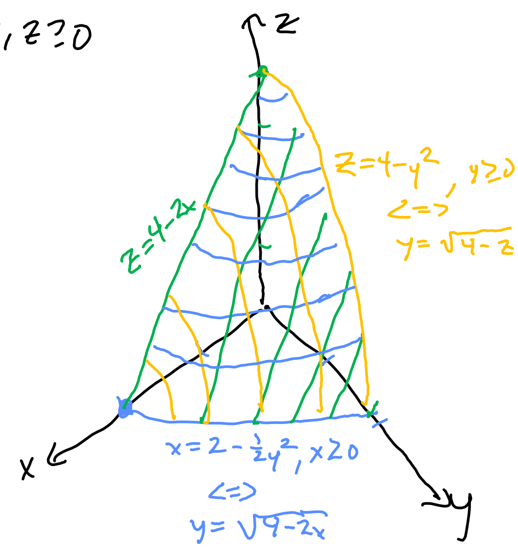
$z = k > 0$: $k = 4 - 2x - y^2$

$x = \frac{4-k}{2} - \frac{1}{2}y^2$

$y = 0$: $z = 4 - 2x$

$y = k > 0$: $z = (4 - k^2) - 2x$

$x = 0$: $z = 4 - y^2$



Let's check our work: https://tinyurl.com/math2551-2var-graph

Daily Announcements & Reminders:

- HW 13.3, 13.4 due tonight
- Quiz 3 tomorrow; Learning Outcomes Q4, Q5, Q6
- Exam 1 in one week. See Canvas announcement.
- Do Day 7 Warmup Poll on Ed ⟶

Goals for Today: Sections 14.2, 14.3

- Evaluate limits of functions of two variables
- Show that a limit does not exist using the two-path test
- Determine the set of points where a function is continuous
- Start to understand how we can measure how a function of two variables is changing

Definition 41. A function of three variables is a rule that assigns to each triple of real numbers (x, y, z) in a set D a unique real number denoted by $f(x, y, z)$. $w = f(x, y, z)$

- The graph of $w = f(x,y,z)$ lies in $\mathbb{R}^4$

$$f : \underbrace{D}_{\text{domain}} \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^3$$

We can still think about the domain and range of these functions. Instead of level curves, we get level surfaces.

Example 42. Describe the domain of the function $f(x,y,z) = \dfrac{1}{4 - x^2 - y^2 - z^2}$.

largest possible

Domain is all (x,y,z) s.t. $4 - x^2 - y^2 - z^2 \neq 0$

$\Longleftrightarrow x^2 + y^2 + z^2 \neq 4$

$\Longleftrightarrow$ all of $\mathbb{R}^3$ <u>except</u> for the sphere of radius 2 centered at $(0,0,0)$.

Example 43. Describe the level surfaces of the function $g(x,y,z) = 2x^2 + y^2 + z^2$.

$0 = 2x^2 + y^2 + z^2 \Rightarrow$ point $(0,0,0)$

$1 = 2x^2 + y^2 + z^2 \Rightarrow$ ellipsoid

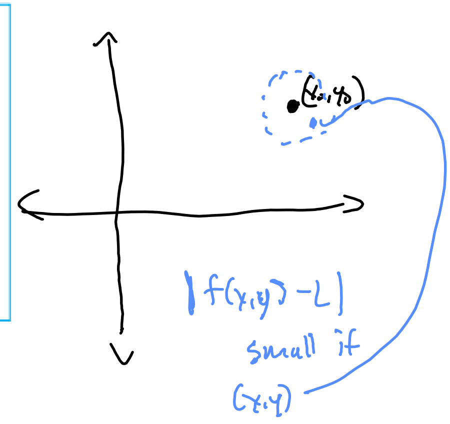
$k > 0$: $k = 2x^2 + y^2 + z^2 \Rightarrow$ ellipsoid

$-1 = 2x^2 + y^2 + z^2 \Rightarrow$ no solution

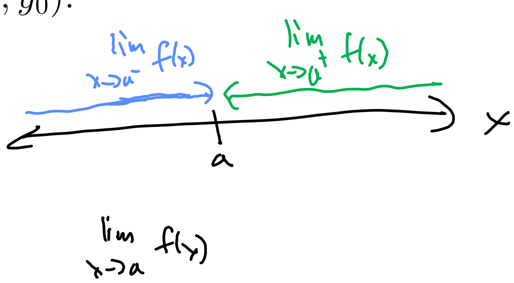
Section 14.2 Limits & Continuity

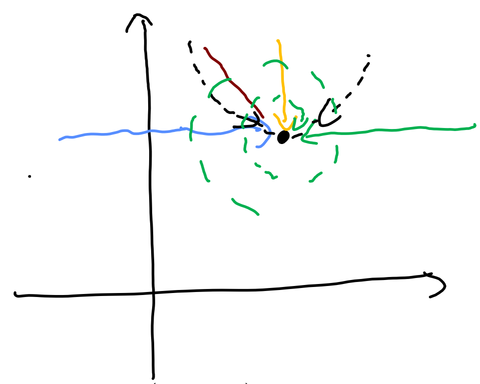
Definition 44. What is a limit of a function of two variables?

> **DEFINITION** We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write
>
> $$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$
>
> if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,
>
> $$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$



We won't use this definition much: the big idea is that $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ if and only if $f(x, y)$ approaches L regardless of how we approach (x_0, y_0).





Definition 45. A function $f(x, y)$ is **continuous** at (x_0, y_0) if

1. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
2. $f(x_0,y_0)$ exists
3. these values are equal

$\Leftrightarrow$ "no holes/jumps"

Key Fact: Adding, subtracting, multiplying, dividing, or composing two continuous functions results in another continuous function.

- $f(x,y)=x$ & $f(x,y)=y$ are cts

  $f(x,y) = x+y$ is cts | $g(x,y) = x^7 + 2x^3y^4 + y^5$ is cts

  $h(x,y) = e^{x^2+y^2}$ is cts : $(x,y) \to x^2y \to e^{x^2y}$

  $t \mapsto e^t$

Example 46. Evaluate $\lim\limits_{(x,y)\to(2,0)} \dfrac{\sqrt{2x-y}-2}{2x-y-4}$, if it exists.

1) Evaluate. $\lim\limits_{(x,y)\to(2,0)} \dfrac{\sqrt{2x-y}-2}{2x-y-4} = \dfrac{\sqrt{4-0}-2}{4-0-4} = \dfrac{0}{0}$

Warmup

$\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4} = \dfrac{1}{4}$

HARD in 2 vars

• L'Hospital's $= \lim\limits_{x\to 4} \dfrac{\frac{1}{2\sqrt{x}}}{1} = \dfrac{1}{4}$

• $\dfrac{\sqrt{x}-2}{x-4} \cdot \dfrac{\sqrt{x}+2}{\sqrt{x}+2} = \dfrac{\cancel{(x-4)}}{\cancel{(x-4)}(\sqrt{x}+2)}$

so $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4} = \lim\limits_{x\to 4} \dfrac{1}{\sqrt{x}+2} = \dfrac{1}{4}$

Use this

2) Try to Simplify: $\lim\limits_{(x,y)\to(2,0)} \dfrac{\sqrt{2x-y}-2}{2x-y-4} \cdot \dfrac{\sqrt{2x-y}+2}{\sqrt{2x-y}+2}$

$= \lim\limits_{(x,y)\to(2,0)} \dfrac{(2x-y)-4}{((2x-y)-4)(\sqrt{2x-y}+2)}$

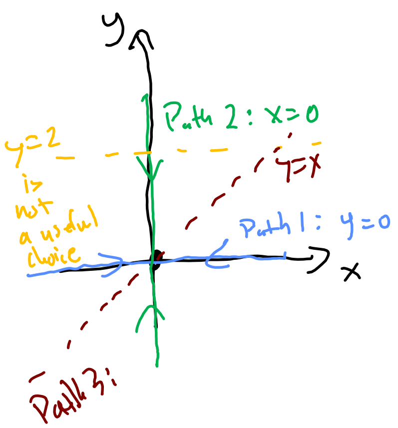
$= \lim\limits_{(x,y)\to(2,0)} \dfrac{1}{\sqrt{2x-y}+2} = \dfrac{1}{\sqrt{4+0}+2} = \dfrac{1}{4}$

Sometimes, life is harder in $\mathbb{R}^2$ and limits can fail to exist in ways that are very different from what we've seen before.

Big Idea: Limits can behave differently along different paths of approach

Example 47. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$, if it exists. Here is its graph.

- Evaluate: $\frac{0}{0+0} = \frac{0}{0}$? Can't simplify.



On $y=0$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+0^2} = \lim_{x\to0} \frac{x^2}{x^2} = \lim_{x\to0} 1 = 1$

On $x=0$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0^2}{0^2+y^2} = \lim_{y\to0} \frac{0}{y} = \lim_{y\to0} 0 = 0$

Since these limits along different paths through $(0,0)$ are different, the overall limit does not exist.

$\displaystyle\lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{x\to0} \frac{1}{2} = \frac{1}{2}$

CAUTION: Converse is false! There can be two paths on which f's limit agrees without the overall limit existing.

This idea is called the **two-path test:**

If we can find <u>two paths of approach</u> to (x_0, y_0) along which <u>the limit of $f(x,y)$</u> takes on two different values, then <u>the limit does not exist</u>.

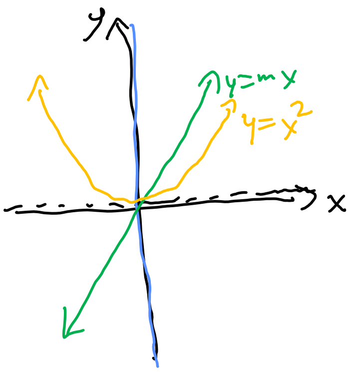
Example 48. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

does not exist.

Along both $y=0$ & $x=0$:

$\lim\limits_{(x,0)\to(0,0)} \frac{0}{x^4+0} = 0$ $\quad \lim\limits_{(0,y)\to(0,0)} \frac{0}{0+y^2} = 0$



Along $y=mx$:

$\lim\limits_{(x,mx)\to(0,0)} \frac{x^2(mx)}{x^4+(mx)^2} = \lim\limits_{x\to 0} \frac{mx}{x^2+m^2} = \frac{0}{0+m^2} = 0$

Along $y=x^2$:

$\lim\limits_{(x,x^2)\to(0,0)} \frac{x^2(x^2)}{x^4+(x^2)^2} = \lim\limits_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$

So by the Two-Path Test, the limit does not exist

Example 49. [Challenge:] Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4y}{x^4+y^2}$$

does exist using the Squeeze Theorem.

Theorem 50 (Squeeze Theorem)**.** *If $f(x,y) = g(x,y)h(x,y)$, where $\lim_{(x,y)\to(a,b)} g(x,y) = 0$ and $|h(x,y)| \le C$ for some constant C near (a,b), then $\lim_{(x,y)\to(a,b)} f(x,y) = 0$.*

$g\to 0$, $|h|\le C$ is bounded

To use: 1) Recognize g, h

2) Bound h

Ex: 1) $\frac{x^4y}{x^4+y^2} = \frac{x^4}{x^4+y^2}\cdot y$ $\quad g(x,y)=y$; $\lim\limits_{(x,y)\to(0,0)} y = 0$

$h(x,y) = \frac{x^4}{x^4+y^2}$

2) Bound h: $y^2\ge 0$, so $x^4+y^2\ge x^4$ so $1 \ge \frac{x^4}{x^4+y^2} \ge 0$

So by the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} \frac{x^4y}{x^4+y^2} = 0$

14.3: Partial Derivatives

Goal: Describe how a function of two (or three, later) variables is changing at a point (a, b).

Example 51. Let's go back to our example of the small hill that has height

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point (x, y). If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive y-direction), at what rate will our height change? What if we walk due east (the positive x-direction)?

Let's investigate graphically.

Daily Announcements & Reminders:

- HW 14.1 due tonight, 14.2 & Vector Function Review on T night
- Exam 1 on Tuesday. See Canvas announcement.
  - Formula sheet updated to add limit theorems
- Quiz 3 grades back by Monday morning
- Do warmup problem on Ed
- Friday office hour is now 4-5pm on Zoom.

Goals for Today: Section 14.3

- Learn how to compute partial derivatives of functions of multiple variables
- Learn how to compute higher-order partial derivatives
- Understand Clairaut's theorem
- Define the total derivative

14.3: Partial Derivatives

Goal: Describe how a function of two (or three, later) variables is changing at a point (a, b).

Example 52. Let's go back to our example of the small hill that has height

$$h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

meters at each point (x, y). If we are standing on the hill at the point with $(2, 1, 11/4)$, and walk due north (the positive y-direction), at what rate will our height change? What if we walk due east (the positive x-direction)?

Let's investigate graphically.

If we walk due north: we have $(x, y) = (2, y)$

so $h(2, y) = 4 - \frac{1}{4}(2)^2 - \frac{1}{4}y^2 = 3 - \frac{1}{4}y^2$

How fast is this changing as y increases?

A: $\left.\frac{d}{dy}\left(h(2, y)\right)\right|_{y=1} = \left.\frac{d}{dy}\left(3 - \frac{1}{4}y^2\right)\right|_{y=1} = \left.-\frac{1}{2}y\right|_{y=1} = -\frac{1}{2}$ m height / m North

- The slope of the hill in the positive y-direction at $(2, 1, \frac{11}{4})$ is $-\frac{1}{2}$

This is the partial derivative of h w.r.t. y

As x changes (partial derivative of h w.r.t. x):

$\left.\frac{d}{dx}\left(h(x, 1)\right)\right|_{x=2} = \left.\frac{d}{dx}\left(\frac{15}{4} - \frac{1}{4}x^2\right)\right|_{x=2} = \left.-\frac{1}{2}x\right|_{x=2} = -1$ m height/m E

Definition 53. If f is a function of two variables x and y, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \qquad f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Notations:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f)$$

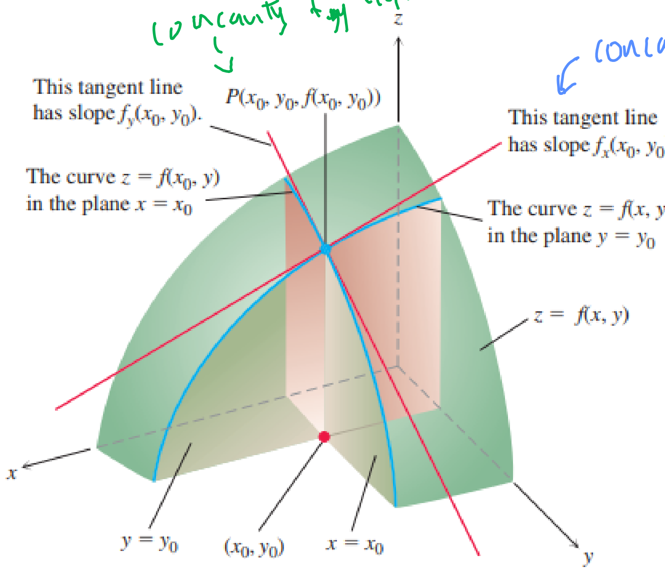
$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(f)$$

Interpretations:

- f_{xy} & f_{yx} measure "twist"

concavity $f_{yy}(x_0,y_0)$

concavity $f_{xx}(x_0,y_0)$



- Computationally:

  - Treat variable(s) that we are not taking derivatives w.r.t. as constants.

Example 54. Find $f_x(1, 2)$ and $f_y(1, 2)$ of the functions below.

a) $f(x, y) = \sqrt{5x - y}$

$$f_x = \frac{\partial}{\partial x}\left(\sqrt{5x-y}\right)$$
$$= \frac{1}{2}(5x-y)^{-1/2} \cdot \frac{\partial}{\partial x}(5x-y) \quad (5-0)$$
$$= \frac{5}{2}(5x-y)^{-1/2} \quad \text{so } f_x(1,2) = \frac{5}{2}(3)^{-1/2} = 5/2\sqrt{3}$$

$$f_y = \frac{\partial}{\partial y}\left(\sqrt{5x-y}\right)$$
$$= \frac{1}{2}(5x-y)^{-1/2} \cdot \frac{\partial}{\partial y}(5x-y) \quad (0-1)$$
$$= -\frac{1}{2}(5x-y)^{-1/2}$$
$$\text{so } f_y(1,2) = -\frac{1}{2\sqrt{3}}$$

b) $f(x, y) = \tan(xy)$

$$f_x(x,y) = \sec^2(xy) \cdot \frac{\partial}{\partial x}(xy) = y\sec^2(xy) \qquad f_x(1,2) = 2\sec^2(2)$$
$$f_y(x,y) = \sec^2(xy) \cdot \frac{\partial}{\partial y}(xy) = x\sec^2(xy) \qquad f_y(1,2) = \sec^2(2)$$

Question: How would you define the second partial derivatives?

Take partial derivatives of the partial derivatives

Notation:

"pure"

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

"mixed"

• inside to outside

Example 55. Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} of the function below.

a) $f(x, y) = \sqrt{5x - y}$

$$f_x = \frac{5}{2}(5x-y)^{-1/2} \qquad f_y = -\frac{1}{2}(5x-y)^{-1/2}$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{5}{2}(5x-y)^{-1/2}\right) = -\frac{5}{4}(5x-y)^{-3/2} \cdot 5$$

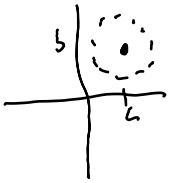
$$f_{yx} = \frac{\partial}{\partial x}\left(-\frac{1}{2}(5x-y)^{-1/2}\right) = \frac{1}{4}(5x-y)^{-3/2} \cdot 5$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{5}{2}(5x-y)^{-1/2}\right) = -\frac{5}{4}(5x-y)^{-3/2} \cdot (-1) = \frac{5}{4}(5x-y)^{-3/2}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(-\frac{1}{2}(5x-y)^{-1/2}\right) = \frac{1}{4}(5x-y)^{-3/2} \cdot (-1)$$

What do you notice about f_{xy} and f_{yx} in the previous example?

Theorem 56 (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions $f, f_x, f_y, f_{xy}, f_{yx}$ are all continuous on D, then*



$$f_{xy} = f_{yx}$$

• If all 1st, 2nd, 3rd order partial derivs are cts on a disk near (a,b)

$$f_{xxy} = f_{xyx} = f_{yxx} \neq f_{yyx} = f_{yxy} = f_{xyy}$$

Example 57. What about functions of three variables? How many partial derivatives should $f(x, y, z) = 2xyz - z^2y$ have? Compute them.

3

$$f_x = \frac{\partial}{\partial x}(2xyz - z^2 y) = 2yz - 0$$

$$f_y = 2xz - z^2$$

$$f_z = 2xy - 2zy$$

Example 58. How many rates of change should the function $f(s,t) = \begin{bmatrix} s^2 + t \\ 2s - t \\ st \end{bmatrix}$ $\begin{array}{l} x(s,t) \\ y(s,t) \\ z(s,t) \end{array}$

have? Compute them. 6 (or 2)

$x_s = 2s$ $\quad x_t = 1$

$y_s = 2$ $\quad y_t = -1$

$z_s = t$ $\quad z_t = s$

$$f_s = \begin{bmatrix} 2s \\ 2 \\ t \end{bmatrix} \qquad f_t = \begin{bmatrix} 1 \\ -1 \\ s \end{bmatrix}$$

1 col for s derivs

1 col for t derivs

the total derivative is $Df(s,t) = \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix}$ $\begin{array}{l} \leftarrow \text{1 row for } x \\ \leftarrow \text{1 row for } y \\ \leftarrow \text{1 row for } z \end{array}$

In the previous example, we computed 6 partial derivatives. How might we **organize** this information?

A matrix!

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$,

we have n inputs, m outputs, and nm partial derivatives, which we can use to form the **total derivative**.

This is a linear map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an matrix, with one column per input and one row per output.

It has the formula $Df_{ij} = \frac{\partial}{x_j}(f_i)$

Examples next time.

Daily Announcements & Reminders:

- HW 14.3 due tonight
- Exam grades back by W, no discussion until then
- Do warmup problem on Ed →

Goals for Today: Sections 14.4-14.6

- Discuss the total derivative and linear approximation
- Learn the Chain Rule for derivatives of functions of multiple variables
- Be able to compute implicit partial derivatives

hold all but one variable constant and find rate of change w.r.t. remaining variable

Last time, we computed partial derivatives. How might we **organize** this information?

Last time: $\vec{f}(s,t) = \begin{bmatrix} s^2+t \\ 2s-6 \\ st \end{bmatrix}$ $n=2$, $m=3$

For any function $f : \mathbb{R}^n \to \mathbb{R}^m$ having the form $f(x_1, \dots, x_n) = \begin{bmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{bmatrix}$,

we have n inputs, m output, and $n \cdot m$ partial derivatives, which we can use to form the **total derivative**.

At each point $\vec{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$,

This is a linear map from $\mathbb{R}^n \to \mathbb{R}^m$, denoted Df, and we can represent it with an matrix, with one column per input and one row per output.

It has the formula $Df_{ij}(\vec{a}) = \dfrac{\partial}{\partial x_j}\left(f_i(x_1,\dots,x_n)\right)\Big|_{\vec{a}}$

Example 59. Find the total derivatives of each function:

a) $f(x) = x^2 + 1$

$Df = \begin{bmatrix} 2x \end{bmatrix}$ 1×1

$f'(x) = 2x$

b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

$D\vec{r}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{bmatrix}$ 3×1

c) $f(x, y) = \sqrt{5x - y}$

$Df(x,y) = \begin{bmatrix} \frac{5}{2\sqrt{5x-y}} & \frac{-1}{2\sqrt{5x-y}} \end{bmatrix}$ 1×2

$f_x = \frac{5}{2\sqrt{5x-y}}$

$f_y = \frac{-1}{2\sqrt{5x-y}}$

d) $f(x, y, z) = 2xyz - z^2 y$

$Df(x,y,z) = \begin{bmatrix} 2yz & 2xz - z^2 & 2xy - 2zy \end{bmatrix}$ 1×3

$f_x = 2yz$

$f_y = 2xz - z^2$

$f_z = 2xy - 2zy$

e) $\mathbf{f}(s, t) = \langle s^2 + t, 2s - t, st \rangle$

$$D\vec{f}(s,t) = \begin{array}{c} \begin{array}{cc} \text{deriv w.r.t. } s \downarrow & \text{deriv w.r.t. } t \downarrow \end{array} \\ \begin{bmatrix} 2s & 1 \\ 2 & -1 \\ t & s \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{deriv of } x \\ \leftarrow \text{deriv of } y \\ \leftarrow \text{deriv of } z \end{array}$$

3×2

$x_s = 2s$ $x_t = 1$

$y_s = 2$ $y_t = -1$

$z_s = t$ $z_t = s$

What does it mean? In differential calculus, you learned that one interpretation of the derivative is as a slope. Another interpretation is that the derivative measures how a function transforms a neighborhood around a given point.

Check it out for yourself. (credit to samuel.gagnon.nepton, who was inspired by 3Blue1Brown.)

In particular, the (total) derivative of **any** function $f : \mathbb{R}^n \to \mathbb{R}^m$, evaluated at $\mathbf{a} = (a_1, \dots, a_n)$, is the linear function that best approximates $f(\mathbf{x}) - f(\mathbf{a})$ at $\mathbf{a}$.

This leads to the familiar linear approximation formula for functions of one variable: $f(x) = f(a) + f'(a)(x - a)$.

Theme: Our ideas from single var. calc (mostly) hold with total derivative replacing single var. deriv.

Definition 60. The **linearization** or **linear approximation** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $\mathbf{a} = (a_1, \dots, a_n)$ is

$$L(\mathbf{x}) = f(\vec{a}) + \overbrace{Df(\vec{a})\,\underbrace{(\vec{x} - \vec{a})}_{\begin{bmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{bmatrix}}}^{\text{matrix-vector multiplication}}$$

Example 61. Find the linearization of the function $f(x, y) = \sqrt{5x - y}$ at the point $(1, 1)$. Use it to approximate $f(1.1, 1.1)$.

$f(1.1, 1.1) = 2.098$

1) Find linearization: need $f(1,1)$ & $Df(1,1)$

$f(1,1) = \sqrt{5-1} = 2$

$Df(x,y) = \begin{bmatrix} \frac{5}{2\sqrt{5x-y}} & \frac{-1}{2\sqrt{5x-y}} \end{bmatrix}$

$Df(1,1) = \begin{bmatrix} \frac{5}{4} & -\frac{1}{4} \end{bmatrix}$

$L(x,y) = f(1,1) + Df(1,1)\begin{bmatrix} x-1 \\ y-1 \end{bmatrix} = \boxed{2 + \begin{bmatrix} \frac{5}{4} & -\frac{1}{4} \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}}$

$= 2 + \frac{5}{4}(x-1) - \frac{1}{4}(y-1)$

rate of change of f wrt y

change in y

2) Approximate $f(1.1, 1.1)$:

$f(1.1, 1.1) \approx L(1.1, 1.1) = 2 + \frac{5}{4}(1.1-1) - \frac{1}{4}(1.1-1) = 2 + \frac{5}{4}(.1) - \frac{1}{4}(.1) = 2.1$

Question: What do you notice about the equation of the linearization?

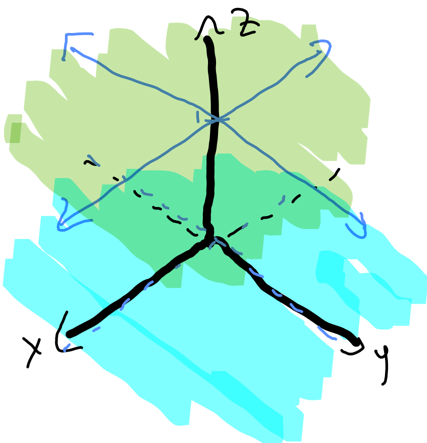
Its graph is a plane. In particular, it is tangent to $z = f(x,y)$ at $(a, b, f(a,b))$.

We say $f : \mathbb{R}^n \to \mathbb{R}$ is **differentiable** at $\mathbf{a}$ if its linearization is a good approximation of f near $\mathbf{a}$.

$$\lim_{(x,y)\to(a,b)} \frac{\left|\overbrace{f(x,y) - L(x,y)}^{f(x,y) - f(a,b) - Df(a,b)\begin{bmatrix} x-a \\ y-b \end{bmatrix}}\right|}{\|(x,y) - (a,b)\|} = 0.$$

In particular, if f is a function $f(x,y)$ of two variables, it is differentiable at (a,b) if ~~it~~ its graph has a unique tangent plane at ~~(a,b)~~ $(a,b,f(a,b))$.

Example 62. Determine if $f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$ is differentiable at $(0,0)$.



- $f = 1$ above x,y-axes and 0 elsewhere
- $f_x = 0$
  $f_y = 0$ $\Rightarrow$ suggests $Df(0,0) = [0 \ \ 0]$

CAUTION: All partial derivatives can exist at a point without being differentiable.

The Chain Rule

Recall the Chain Rule from single variable calculus:

$$\frac{d}{dx}\left(f(g(x))\right) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x)$$

Similarly, the **Chain Rule** for functions of multiple variables says that if $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are both differentiable functions then

$$D(f(g(\mathbf{x}))) = \underbrace{\overbrace{Df(g(\mathbf{x}))}^{(m \times p)}\overbrace{Dg(\mathbf{x})}^{(p \times n)}}_{\text{matrix mult.}}.$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R}^p \arrow[r, "f"] & \mathbb{R}^m \end{tikzcd}$$

$g \circ f$; $\mathbb{R}^p \to \mathbb{R}^m \quad ? \quad \mathbb{R}^n \to \mathbb{R}^p$

Example 63. Suppose we are walking on our hill with height $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ along the curve $\mathbf{r}(t) = \langle t+1, 2-t^2\rangle$ in the plane. How fast is our height changing at time $t = 1$ if the positions are measured in meters and time is measured in minutes?

$Dh(t) = Dh(\vec{r}(t)) \cdot D\vec{r}(t)$

$h(t) = h(\vec{r}(t)) = h(x(t), y(t))$

So compute: $Dh(1) = Dh(\vec{r}(1)) \cdot D\vec{r}(1)$

$Dh(x,y) = \begin{bmatrix} -\frac{1}{2}x & -\frac{1}{2}y \end{bmatrix}$

$$= \begin{bmatrix} -\frac{1}{2}x & -\frac{1}{2}y \end{bmatrix}\Big|_{(x,y)=(2,1)} \begin{bmatrix} 1 \\ -2t \end{bmatrix}_{t=1}$$

$D\vec{r}(t) = \begin{bmatrix} 1 \\ -2t \end{bmatrix}$ $\quad \vec{r}(1) = \langle 2, 1\rangle$

$$= \begin{bmatrix} -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= -1 + 1 = 0$$

Example 64. Suppose that $W(s,t) = F(u(s,t), v(s,t))$, where F, u, v are differentiable functions and we know the following information.

$$u(1,0) = 2 \qquad v(1,0) = 3$$
$$u_s(1,0) = -2 \qquad v_s(1,0) = 5$$
$$u_t(1,0) = 6 \qquad v_t(1,0) = 4$$
$$F_u(2,3) = -1 \qquad F_v(2,3) = 10$$

$W = F(g(s,t))$

$g(s,t) = \begin{bmatrix} u(s,t) \\ v(s,t) \end{bmatrix}$ $\quad g(1,0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Find $W_s(1,0)$ and $W_t(1,0)$.

Same as asking for $DW(s=1, t=0)$

$$DW(s=1,t=0) = DF(g(s,t))\Big|_{(u,v)=g(1,0)} \cdot D\vec{g}(s,t)\Big|_{(s,t)=(1,0)}$$

$$= \begin{bmatrix} F_u(u,v) & F_v(u,v) \end{bmatrix}\Big|_{(u,v)=g(1,0)} \begin{bmatrix} u_s(s,t) & u_t(s,t) \\ v_s(s,t) & v_t(s,t) \end{bmatrix}\Bigg|_{(1,0)}$$

$$= \begin{bmatrix} F_u(2,3) & F_v(2,3) \end{bmatrix} \begin{bmatrix} u_s(1,0) & u_t(1,0) \\ v_s(1,0) & v_t(1,0) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 10 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 52 & 34 \end{bmatrix}$$

Post class

Application to Implicit Differentiation: If $F(x, y, z) = c$ is used to *implicitly* define z as a function of x and y, then the chain rule says:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \& \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Example 65. Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the sphere $x^2 + y^2 + z^2 = 4$.

$F(x, y, z) = x^2 + y^2 + z^2$

so $F_x = 2x$, $F_y = 2y$, $F_z = 2z$

thus $\dfrac{\partial z}{\partial x} = -\dfrac{2x}{2z} = \boxed{-\dfrac{x}{z}}$ & $\dfrac{\partial z}{\partial y} = -\dfrac{2y}{2z} = \boxed{-\dfrac{y}{z}}$

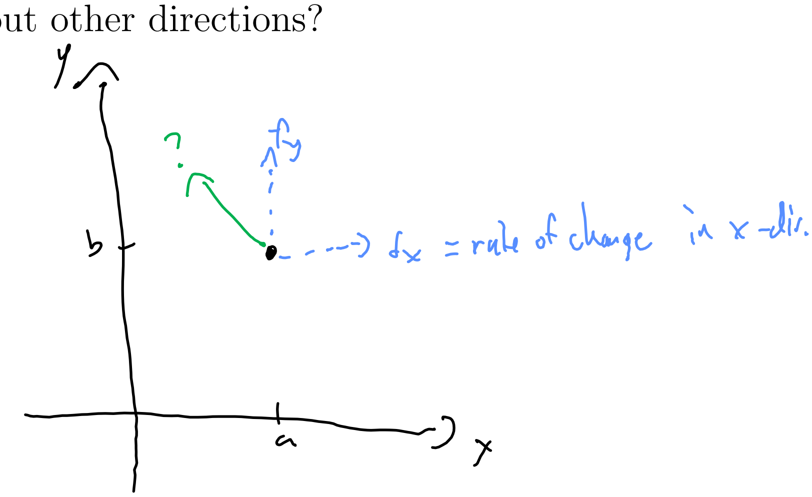
Daily Announcements & Reminders:

- HW 14.4 due tonight
- Quiz 4 in studio tomorrow : 14.3 & 14.4
  - L.O.: D1 & D2
- Exam 1 grades released by Th morning
- Do warmup problem on Ed ⟶

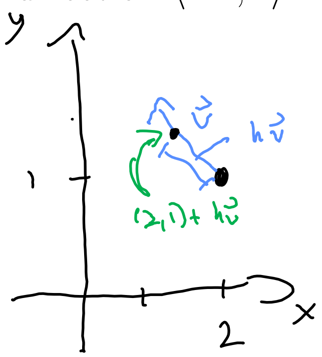
Goals for Today: Sections 14.4-14.6

- Learn to compute the rate of change of a multivariable function in any direction
- Investigate the connection between the gradient vector and level curves/surfaces
- Discuss tangent planes to surfaces, how to find them, and when they exist

Example 66. Recall that if $z = f(x, y)$, then f_x represents the rate of change of z in the x-direction and f_y represents the rate of change of z in the y-direction. What about other directions?



Let's go back to our hill example again, $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$. How could we figure out the rate of change of our height from the point $(2, 1)$ if we move in the direction $\langle -1, 1\rangle$? $h(2,1) = 11/4$ $Dh(2,1) = [-1, -\frac{1}{2}]$



- 1) Normalize direction vector to get geometric answer:

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

2) Construct diff quotient & take limit:

rate of change: $\displaystyle \lim_{h \to 0} \frac{f((2,1)+h\vec{u}) - f(2,1)}{h} = \lim_{h\to 0} \frac{f\left(2 - \frac{h}{\sqrt{2}}, 1 + \frac{h}{\sqrt{2}}\right) - 11/4}{h}$

...

$$= \frac{1}{2\sqrt{2}}$$

- Inst. rate of change of height (slope) of hill in the $\langle -1, 1\rangle$ or $\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$ dir. from $(2,1)$ is $\frac{1}{2\sqrt{2}}$ m height / m traveled

Definition 67. The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\mathbf{p}$ in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(\mathbf{p}) = \lim_{h \to 0} \frac{f(\vec{p} + h\vec{u}) - f(\vec{p})}{h}$$

if this limit exists.

E.g. for our hill example above we have:

$$D_{\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle} h(2,1) = \frac{1}{2\sqrt{2}}$$

Note that $D_{\mathbf{i}} f = f_x$ $D_{\mathbf{j}} f = f_y$ $D_{\mathbf{k}} f = f_z$

Definition 68. If $f : \mathbb{R}^n \to \mathbb{R}$, then the gradient of f at $\mathbf{p} \in \mathbb{R}^n$ is the vector function ∇f (or grad f) defined by

$$\nabla f(\mathbf{p}) = \langle f_{x_1}(\vec{p}),\ f_{x_2}(\vec{p}), \ldots, f_{x_n}(\vec{p}) \rangle$$
$$= Df(\vec{p})^T$$

$$\nabla h(x,y) = \langle -\tfrac{1}{2}x, -\tfrac{1}{2}y \rangle = \begin{bmatrix} -1/2\, x \\ -1/2\, y \end{bmatrix}$$

Note: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\mathbf{p}$, then f has a directional derivative at $\mathbf{p}$ in the direction of any unit vector $\mathbf{u}$ and

$$D_{\mathbf{u}} f(\mathbf{p}) = Df(\vec{p})\,\vec{u} = \nabla f(\vec{p}) \cdot \vec{u}$$

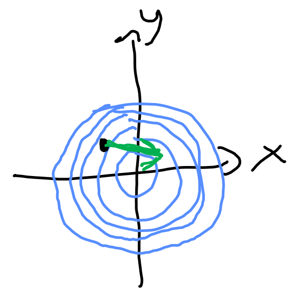
Example 69. Find the gradient vector and the directional derivative of each function at the given point $\mathbf{p}$ in the direction of the given vector $\mathbf{u}$.

a) $f(x,y) = \ln(x^2+y^2), \mathbf{p} = (-1,1), \mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$

$$D_u f(\vec{p}) = Df(\vec{p})\,\vec{u} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} = \frac{-3}{\sqrt{5}}$$

1) Find grad: $\nabla f = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$

2) Evaluate at $\vec{p}$: $\nabla f(-1,1) = \left\langle \frac{-2}{2}, \frac{2}{2} \right\rangle = \langle -1, 1 \rangle = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



3) Check unit vector: $\|\vec{u}\| = \sqrt{\frac{1}{5} + \frac{4}{5}} = 1$ ✓

4) $D_{\vec{u}} f(\vec{p}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} = -\frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}} = \boxed{\frac{-3}{\sqrt{5}}}$

b) $g(x,y,z) = x^2 + 4xy^2 + z^2$, $\mathbf{p} = (1,2,1)$, $\mathbf{u}$ the unit vector in the direction of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

1) $\nabla g = \langle 2x + 4y^2, 8xy, 2z \rangle$

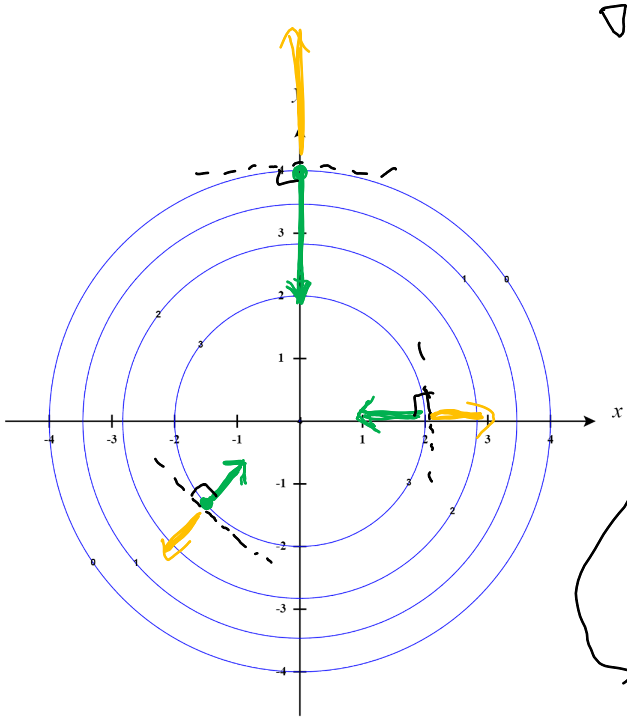
2) Evaluate at $\vec{p}$: $\nabla g(1,2,1) = \langle 2 + 16, 8(1)(2), 2(1) \rangle = \langle 18, 16, 2 \rangle$

3) unit vector? $\vec{u} = \frac{\langle 1,2,-1 \rangle}{\|\langle 1,2,-1 \rangle\|} = \frac{1}{\sqrt{6}} \langle 1,2,-1 \rangle$

4) $D_{\vec{u}}\, g(1,2,1) = \langle 18, 16, 2 \rangle \cdot \frac{1}{\sqrt{6}} \langle 1, 2, -1 \rangle = \frac{1}{\sqrt{6}} (18 + 32 - 2) = \boxed{\frac{48}{\sqrt{6}}}$

Example 70. If $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, the contour map is given below. Find and draw ∇h on the diagram at the points $(2, 0), (0, 4)$, and $(-\sqrt{2}, -\sqrt{2})$. At the point $(2, 0)$, compute $D_{\mathbf{u}}h$ for the vectors $\mathbf{u}_1 = \mathbf{i}, \mathbf{u}_2 = \mathbf{j}, \mathbf{u}_3 = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

$$\nabla h = \langle -\tfrac{1}{2}x, \tfrac{1}{2}y \rangle = \begin{bmatrix} -1/2\, x \\ -\frac{1}{2} y \end{bmatrix}$$



(a,b)	$\nabla h(a,b)$
$(2,0)$	$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$
$(0,4)$	$\begin{bmatrix} 0 \\ -2 \end{bmatrix}$
$(-\sqrt{2},-\sqrt{2})$	$\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$

- All point towards center (highest pt on hill)
- $\|\nabla h\|$ related to dist. from center

- At $\vec{p}$, $\nabla f(\vec{p})$ is $\perp$ to level set containing $\vec{p}$
- $\nabla f(\vec{p})$ points in direction of greatest increase of f
- $\|\nabla f(\vec{p})\|$ is greatest rate of change of f

b/c: $D_{\vec{u}} f(\vec{p}) = \nabla f(\vec{p}) \cdot \vec{u}$
$= \|\nabla f(\vec{p})\| \cdot \cancel{\|\vec{u}\|} \cancel{\cos(\theta)}$ $\theta = 0°$ $\Leftrightarrow$ $\vec{u}$ in dir of ∇f

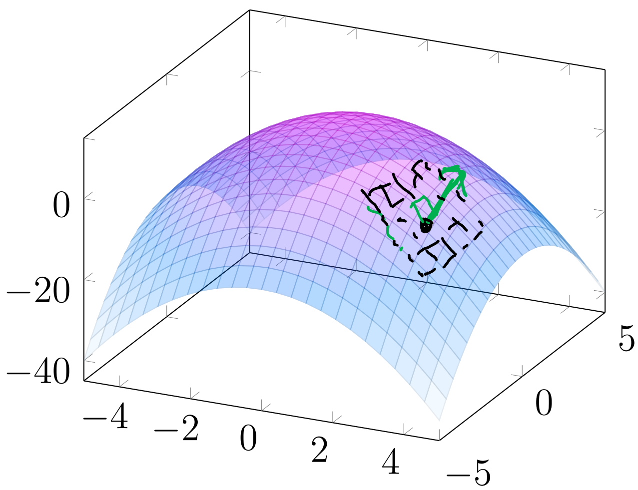
- $-\nabla f(\vec{p})$ points in direction of greatest decrease (least change) of f & this rate is $-\|\nabla f(\vec{p})\|$.

Note that the gradient vector is <u>orthogonal</u> to level curves. of $f(x,y)$

Similarly, for $f(x, y, z)$, $\nabla f(a, b, c)$ is <u>orthogonal to level surfaces</u>

Tangent planes to level surfaces

Suppose S is a surface with equation $F(x, y, z) = k$. How can we find an equation of the tangent plane of S at $P(x_0, y_0, z_0)$?



$x^2 + y^2 + z = 10$, $P = (-1, 3, 0)$

To construct tangent plane:

- need point: given point $\vec{p}$
- need normal $\vec{n}$: ∇F is normal at $\vec{p}$

1) Identify F that defines my surface

$F(x,y,z) = x^2 + y^2 + z$ w/ $F(x,y,z) = 10$ OR $x^2 + y^2 + z - 10 = 0$

$G(x,y,z) = x^2 + y^2 + z - 10$

2) Take point: $(-1, 3, 0)$

3) Find normal: $\nabla \vec{F} = \langle 2x, 2y, 1 \rangle$

$\vec{n} = \nabla \vec{F}(\vec{p}) = \langle -2, 6, 1 \rangle$

4) Give plane equation: $-2(x+1) + 6(y-3) + (z-0) = 0$

Example 71. Find the equation of the tangent plane at the point $(-2, 1, -1)$ to the surface given by

$$z = 4 - x^2 - y = f(x,y)$$

1) Identify $F = c$:

$z + x^2 + y - 4 = 0$ so $F(x,y,z) = z + x^2 + y - 4$

2) Point: $(-2, 1, -1)$

3) Normal: $\nabla F = \langle 2x, 1, 1 \rangle$

$\vec{n} = \nabla F(2, 1, -1) = \langle -4, 1, 1 \rangle$

4) Plane: $-4(x+2) + 1(y-1) + 1(z+1) = 0$

$$z = \underbrace{-1}_{f(-2,1)} + \underbrace{4}_{f_x(-2,1)}(x+2) \underbrace{-}_{f_y(-2,1)}(y-1)$$

Special case: if we have $z = f(x, y)$ and a point $(a, b, f(a, b))$, the equation of the tangent plane is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

This should look familiar: it's the linearization

Daily Announcements & Reminders:

- HW 14.5 due tonight
- Exam 1 grades are out
  - 25th percentile 56.5%, median 69%, 75th 80%, max 100%
  - Regrade requests open until 9 am 10/13 – give a specific item in correct solution that was missed in your solution
  - Take more advantage of office hours / Math Lab / PLUS
- Do warmup poll on Ed
- Megathread on Ed for Q&A today

Goals for Today: Section 14.7

- Define local & global extreme values for functions of two variables
- Learn how to find local extreme values for functions of two variables
- Learn how to classify critical points for functions of two variables
- Learn how to find global extreme values on a closed & bounded domain

$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$

Last time: If $f(x, y)$ is a function of two variables, we said $\nabla f(a, b)$ points in the direction of greatest change of f.

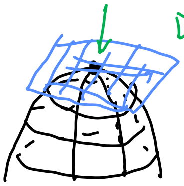
Back to the hill $h(x, y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$! What should we expect to get if we compute $\nabla h(0, 0)$? Why? What does the tangent plane to $z = h(x, y)$ at $(0, 0, 4)$ look like?

$\nabla h = \langle -\frac{1}{2}x, -\frac{1}{2}y \rangle$ so $\nabla h(0,0) = \langle 0, 0 \rangle$

so there is no direction of greatest increase of h at $(0,0)$

$z = 4 + 0(x-0) + 0(y-0) = 4$

local max



$Dh = [0\ 0]$

$\nabla h = \langle 0, 0 \rangle$

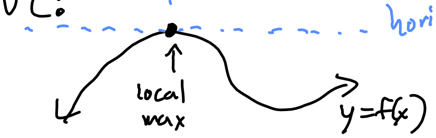
&

horiz. tangent plane

In SVC:

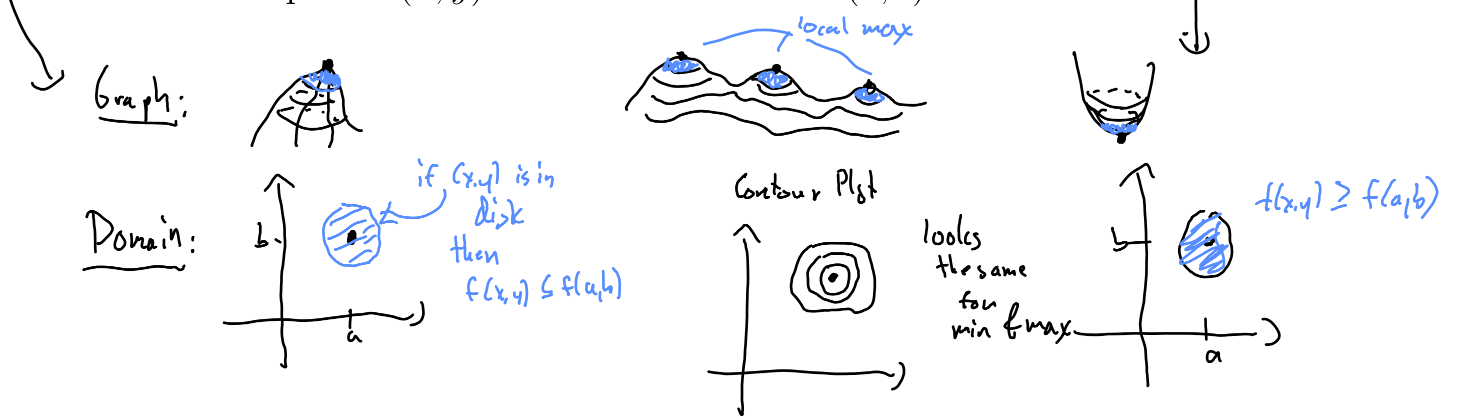
$f'(a) = 0$

horiz. tangent line

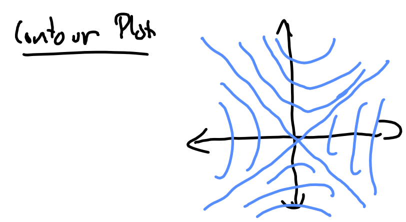


Definition 63. Let $f(x, y)$ be defined on a region containing the point (a, b). We say

- $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)
- $f(a, b)$ is a local minimum value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in a disk centered at (a, b)



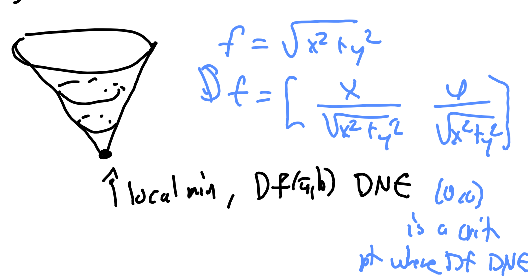
In $\mathbb{R}^3$, another interesting thing can happen. Let's look at $z = x^2 - y^2$ (a hyperbolic paraboloid!) near $(0, 0)$.



This is called a saddle point

Notice that in all of these examples, we have a horizontal tangent plane at the point in question, i.e.

$Df(a,b) = [0 \ \ 0]$ $\iff$ $\nabla f(a,b) = \langle 0, 0 \rangle = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ OR $Df(a,b)$ does not exist



Definition 64. If $f(x, y)$ is a function of two variables, a point (a, b) in the domain of f with $Df(a, b) =$ $[0 \ \ 0]$ or where $Df(a, b)$ fails to exist is called a critical points of f.

Example 65. Find the critical points of the function $f(x, y) = x^3 + y^3 - 3xy$.

1) Find Df: $Df(x,y) = \begin{bmatrix} 3x^2 - 3y & 3y^2 - 3x \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

and set equal [0 0]

2) Solve for x & y: $\begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases} \Rightarrow \begin{cases} x^2 - y = 0 & (1) \\ y^2 - x = 0 & (2) \end{cases}$ solve for (1) for y: $y = x^2$

Substitute $y = x^2$ into (2):

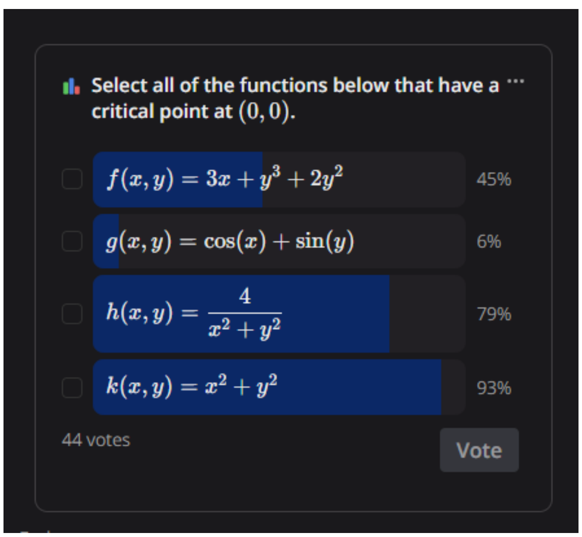
$(x^2)^2 - x = 0 \Rightarrow x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0$

so either $x = 0$, $y = 0^2 = 0$ or $x^3 - 1 = 0$, $x^3 = 1$, $x = \sqrt[3]{1} = 1$, $y = (1)^2 = 1$

Critical points: (0,0) & (1,1)

Q: What about Df does not exist?

Here Df exists everywhere!



- $f(0,0) = 0$ but $Df = [3 \quad 3y^2 + 4y]$ ← 3 not 0 at (0,0)
- $Dg = [-\sin(x) \quad \cos(y)]$
  $Dg(0,0) = [0 \quad 1]$ so (0,0) is not crit pt
- $Dh = \left[\dfrac{-8x}{(x^2+y^2)^2} \quad \dfrac{-8y}{(x^2+y^2)^2} \right]$ DNE at (0,0)
  BUT (0,0) is not in domain of h, so (0,0) is not a crit pt.
- $Dk = [2x \quad 2y] = [0 \quad 0]$ at (0,0)
  so (0,0) is a crit pt.

To classify critical points, we turn to the **second derivative test** and the **Hessian matrix**. The **Hessian matrix** of $f(x, y)$ at (a, b) is

$$Hf(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

$D^2 f(a,b)$

Theorem 66 (2nd Derivative Test). *Suppose (a, b) is a critical point of $f(x, y)$ and f has continuous second partial derivatives. Then we have:*

- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) > 0$, $f(a, b)$ *is a local minimum*
- *If* $\det(Hf(a, b)) > 0$ *and* $f_{xx}(a, b) < 0$, $f(a, b)$ *is a local maximum*
- *If* $\det(Hf(a, b)) < 0$, f *has a saddle point at* (a, b)
- *If* $\det(Hf(a, b)) = 0$, *the test is inconclusive.*

λ is an eigenvalue of A w/ eigen vector $\vec{x} \neq \vec{0}$ if $A\vec{x} = \lambda\vec{x}$

More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ has a critical point at $\mathbf{p}$ then

if $n=2$ this is equivalent to $\det(Hf(a,b))>0$

- If all eigenvalues of $Hf(\mathbf{p})$ are positive, f is concave up in every direction from $\mathbf{p}$ and so has a local minimum at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative, f is concave down in every direction from $\mathbf{p}$ and so has a local maximum at $\mathbf{p}$.
- If some eigenvalues of $Hf(\mathbf{p})$ are positive and some are negative, f is concave up in some directions from $\mathbf{p}$ and concave down in others, so has neither a local minimum or maximum at $\mathbf{p}$. (saddle point)
- If all eigenvalues of $Hf(\mathbf{p})$ are positive or zero, f may have either a local minimum or neither at $\mathbf{p}$.
- If all eigenvalues of $Hf(\mathbf{p})$ are negative or zero, f may have either a local maximum or neither at $\mathbf{p}$.

Example 67. Classify the critical points of $f(x,y) = x^3 + y^3 - 3xy$ from Example 65.

Crit pts: $(0,0)$ & $(1,1)$

$$Df = \begin{bmatrix} 3x^2 - 3y & 3y^2 - 3x \end{bmatrix}$$

1) Find Hf: $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$

At $(0,0)$: $Hf(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$

$\det(Hf(0,0)) = 0 - (-3)(-3) = -9 < 0$

By the 2nd Deriv. Test, f has a saddle point at $(0,0)$

At $(1,1)$: $Hf(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$

$\det(Hf(1,1)) = 36 - 9 = 27 > 0$

and $f_{xx}, f_{yy} > 0$

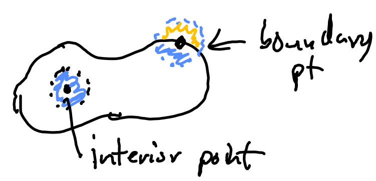
so by the 2nd Deriv. Test, f has a local min at $(1,1)$.

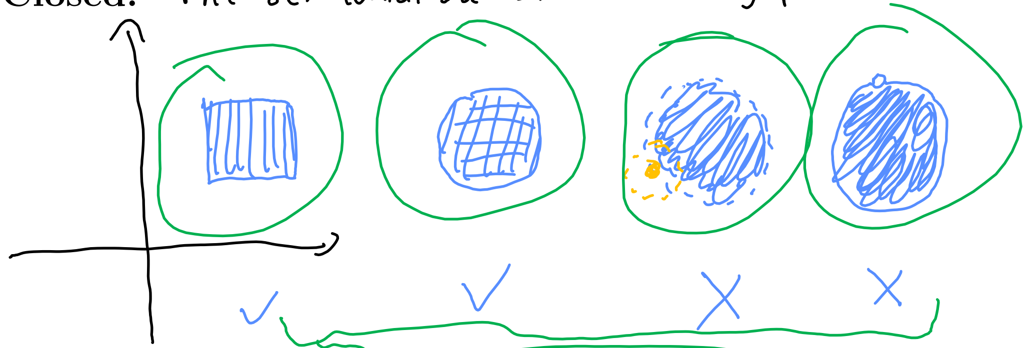
Two Local Maxima, No Local Minimum: The function $g(x, y) = -(x^2-1)^2 - (x^2y - x - 1)^2 + 2$ has two critical points, at $(-1, 0)$ and $(1, 2)$. Both are local maxima, and the function never has a local minimum!

A global maximum of $f(x, y)$ is like a local maximum, except we must have $f(a, b) \geq f(x, y)$ for **all** (x, y) in the domain of f. A global minimum is defined similarly.

Theorem 68. *On a closed & bounded domain, any continuous function $f(x, y)$ attains a global minimum & maximum.* Extreme Value Theorem

Closed: The set contains all of its boundary points.



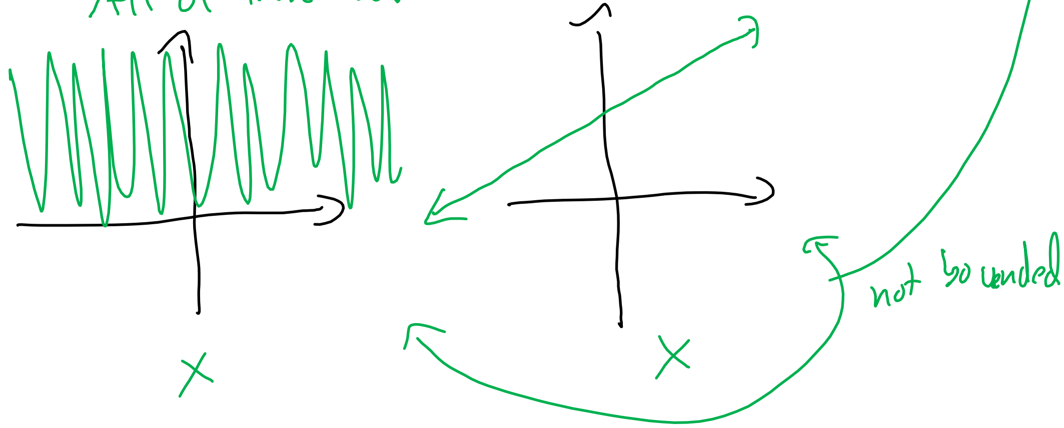


$\mathbb{R}^2$

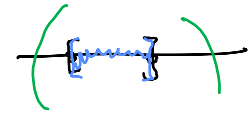
b/c it has no boundary

Bounded: The set fits in a large enough circle

All of these four



In $\mathbb{R}$



Strategy for finding global min/max of $f(x, y)$ on a closed & bounded domain R

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 69. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

Start here Tuesday

Daily Announcements & Reminders:

- HW 14.6 due tonight, 14.7 & 14.8 on Th
- Quiz 5 in studio tomorrow; LO. D3, D4, D5
  - gradient, dir. derivative, tangent planes, linearization, local min/max
- Exam 1 #7b regraded to count using the general arclength formula $\int_a^b \|\vec{r}\,'(t)\|\,dt$ instead of your function from a) as correct
- Do the two warmup polls on Ed

Goals for Today: Sections 14.7, 14.8

- Find global extreme values of continuous functions of two variables on closed & bounded domains
- Apply the method of Lagrange multipliers to find extreme values of functions of two or more variables subject to one or more constraints

- Closed: the set contains all boundary points
- Bounded: the set fits in a big enough circle

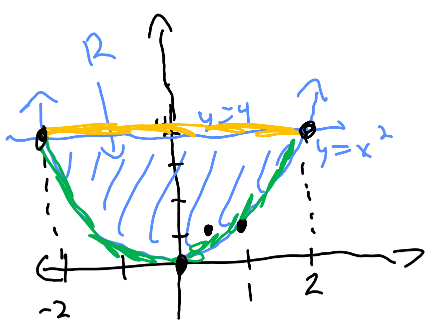
Strategy for finding global min/max of continuous $f(x, y)$ **on a closed & bounded domain** R

• Analogue of Extreme Value Theorem

1. Find all critical points of f inside R.
2. Find all critical points of f on the boundary of R
3. Evaluate f at each critical point as well as at any endpoints on the boundary.
4. The smallest value found is the global minimum; the largest value found is the global maximum.

Example 70. Find the global minimum and maximum of $f(x, y) = 4x^2 - 4xy + 2y$ on the closed region R bounded by $y = x^2$ and $y = 4$.

0: Draw region R



1) Find crit pts of f inside R

$Df = [\,8x-4y \quad -4x+2\,] = [\,0 \quad 0\,]$

$\begin{cases} 8x-4y=0 & ① \\ -4x+2=0 & ② \end{cases} \longrightarrow 4x=2 \to x=\frac{1}{2}$

Plug into ① $8(\frac{1}{2})-4y=0 \to y=1$

crit pt $(\frac{1}{2}, 1)$

2) Find crit pts of f restricted to boundary of R

On $y=x^2$: (x, x^2) for some $x \in [-2, 2]$

$f(x,y)$ becomes

$f(x, x^2) = 4x^2 - 4x(x^2) + 2(x^2)$

$g(x) = 6x^2 - 4x^3 \qquad -2 \le x \le 2$

Set $g'(x)=0$ and solve: $g'(x) = 12x - 12x^2 = 0$

$12x(1-x) = 0$

$x=0$ or $x=1$

plug into $y=x^2$: $y=0$, $y=1$

Endpoints/corners are $(-2,4), (2,4)$

3) Evaluate

Test pts	f
$(\frac{1}{2}, 1)$	1
$(0,0)$	0
$(1,1)$	2
$(-2,4)$	56
$(2,4)$	−8

4) Conclusion

- Global min of f on R is −8 attained at (2,4)
- Global max of f on R is 56 attained at (−2,4)

On $y=4$: $(x,4)$ for $x \in [-2,2]$

$f(x,y)$ becomes $f(x,4) = 4x^2 - 4x(4) + 2(4)$

$h(x) = 4x^2 - 16x + 8$

Set $h'(x) = 8x - 16 = 0$

$x = 2$

Plug into $y=4$ $\quad y=4$

Endpoints are $(-2,4)$, $(2,4)$

Constrained Optimization

Goal: Maximize or minimize $f(x,y)$ or $f(x,y,z)$ subject to a *constraint*, $g(x,y) = c$.
or
$g(x,y,z) = c$

Example 71. A new hiking trail has been constructed on the hill with height $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$, above the points $y = -0.5x^2 + 3$ in the xy-plane. What is the highest point on the hill on this path?

① Identify
Objective function: $h(x,y) = 4 - \frac{1}{4}x^2 - \frac{1}{4}y^2$ (the thing we are maximizing or minimizing)

② **Constraint equation:** • Know ∇g is orthogonal to $g=3$ curve

$$\underbrace{y + \frac{1}{2}x^2}_{g(x,y)} = 3$$

③ Goal: Find (x,y) where $\nabla h = \lambda \nabla g$ & $g(x,y) = 3$

$$\langle -\tfrac{1}{2}x, -\tfrac{1}{2}y \rangle = \lambda \langle x, 1 \rangle$$

$$\begin{cases} -\frac{1}{2}x = \lambda x & ① \\ -\frac{1}{2}y = \lambda & ② \\ y + \frac{1}{2}x^2 = 3 & ③ \end{cases}$$

→ start ①: $\lambda x + \frac{1}{2}x = 0$ so $x(\lambda + \frac{1}{2}) = 0$

$\underline{x=0}$ or $\underline{\lambda = -\frac{1}{2}}$

Plug into ② & ③ $\begin{cases} -\frac{1}{2}y = \lambda \\ \underline{y = 3} \end{cases}$

$(0,3)$

$\begin{cases} -\frac{1}{2}y = -\frac{1}{2} & (2a) \\ y + \frac{1}{2}x^2 = 3 & (3a) \end{cases}$

$y = 1$ (Plug into 3a)

$1 + \frac{1}{2}x^2 = 3$

$x^2 = 4$

$x = \pm 2$

④ Evaluate

(x,y)	h
$(0,3)$	$7/4$
$(-2,1)$	$11/4$
$(2,1)$	$11/4$
	0

So the max height is $11/4$ meters.

include endpoints to be sure of global values : $16 = x^2 + y^2$ & $y + \frac{1}{2}x^2 = 3$
solve

Method of Lagrange Multipliers: To find the maximum and minimum values attained by a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = c$ and compute the value of f at these points.

If we have more than one constraint $g(x, y, z) = c_1, h(x, y, z) = c_2$, then find all points where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$ and $g(x, y, z) = c_1, h(x, y, z) = c_2$.

∇f in Span$(\nabla g, \nabla h)$

Example 72. Find the points on the surface $z^2 = xy + 4$ that are closest to the origin.

① Objective function: $-4 + z^2 - xy$? constraint — where x & $y = 0$?

$d = \sqrt{x^2+y^2+z^2}$ Actually use $f = d^2 = x^2+y^2+z^2$

② Constraint function: $\underbrace{z^2 - xy - 4}_{g(x,y,z)} = 0$

3) $\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2y, 2z\rangle = \lambda\langle -y, -x, 2z\rangle$

$g = c$ $\quad z^2 - xy - 4 = 0$

Algebra to solve added after class:

$2x = -\lambda y$ ①
$2y = -\lambda x$ ②
$2z = 2\lambda z$ ③
$z^2 - xy - 4 = 0$ ④

Starting w/ ③ b/c it involves fewest variables: $z - \lambda z = 0 \Rightarrow z(1-\lambda) = 0$
$z = 0$ or $\lambda = 1$

If $z = 0$:
$2x = -\lambda y$ (1a)
$2y = -\lambda x$ (2a)
$-xy - 4 = 0$ (3a)

Plug (1a) into (2a): $x = -\frac{1}{2}\lambda y$ so $2y = \frac{1}{2}\lambda^2 y$
$4y - \lambda^2 y = 0$
so $y(4 - \lambda^2) = 0$
so $y = 0$ or $\lambda = 2$ or $\lambda = -2$

Plug into 3a & solved 1a:
- $y = 0$: $-4 = 0$ ✗ impossible
- $\lambda = 2$: $xy + 4 = 0$, $x = -y$ → $-y^2 + 4 = 0$; $y = \pm 2$, $x = \mp 2$, $z = 0$
- $\lambda = -2$: $xy + 4 = 0$, $x = y$ → $y^2 + 4 = 0$ ✗ impossible

If $\lambda = 1$:
$2x = -y$ (1b)
$2y = -x$ (2b)
$z^2 - xy - 4 = 0$ (3b)

Plug (1b) into (2b): $y = -2x$ so $-4x = -x$
so $x = 0$
$y = 0$
Plug into 3b: $z^2 - 4 = 0 \rightarrow z = \pm 2$

Test pts	f	d
(2, -2, 0)	8	$\sqrt{8}$
(-2, 2, 0)	8	$\sqrt{8}$
(0, 0, 2)	4	2
(0, 0, -2)	4	2

So the closest points to the origin are $(0, 0, \pm 2)$ at distance 2.

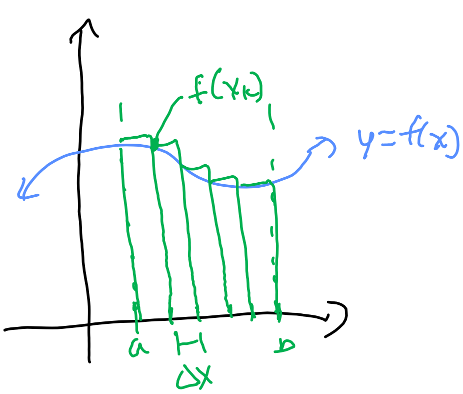
Daily Announcements & Reminders:

- HW 14.7 & 14.8 due tonight
- Do warmup on Ed

Goals for Today: Sections 15.1, 15.2

- Introduce double and iterated integrals for functions of two variables on rectangles
- Use Fubini's Theorem to change the order of integration of a iterated integral
- Be able to set up & evaluate a double integral over a general region
- Change the order of integration for general regions

Recall: Riemann sum and the definite integral from single-variable calculus.



$$\text{area} \approx \sum_{k=1}^{n} f(x_k)\Delta x$$

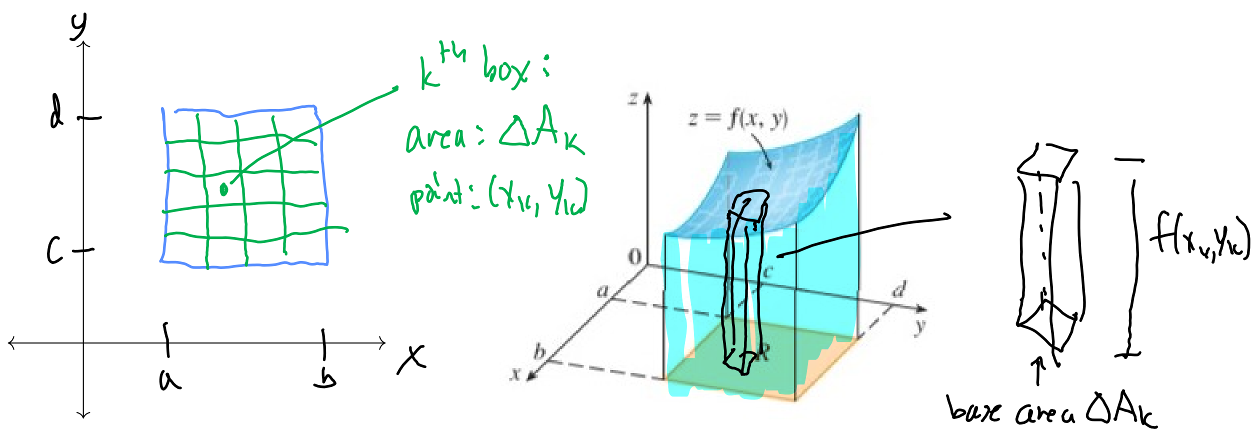
$$\text{area} = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

Double Integrals

Volumes and Double integrals Let R be the closed rectangle defined below:

$$R = \underbrace{[a,b]}_{x \text{ bounds}} \times \underbrace{[c,d]}_{y \text{ bounds}} = \{(x,y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$$

Let $f(x,y)$ be a function defined on R such that $f(x,y) \geq 0$. Let S be the solid that lies above R and under the graph f.



Question: How can we estimate the volume of S?

$$\text{Volume}(S) = \sum_{k=1}^{n} \underbrace{f(x_k, y_k)}_{\text{height of box}} \underbrace{\Delta A_k}_{\text{area of base of box}}$$

$$\Delta A_k = \Delta x_k \Delta y_k$$

Definition 73. The double integral of $f(x, y)$ over a rectangle R is

$$\iint_R f(x,y)\, dA = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

integrand

region of integration

$|P|$ is biggest size of all rectangles

if this limit exists.

then f is called integrable on R

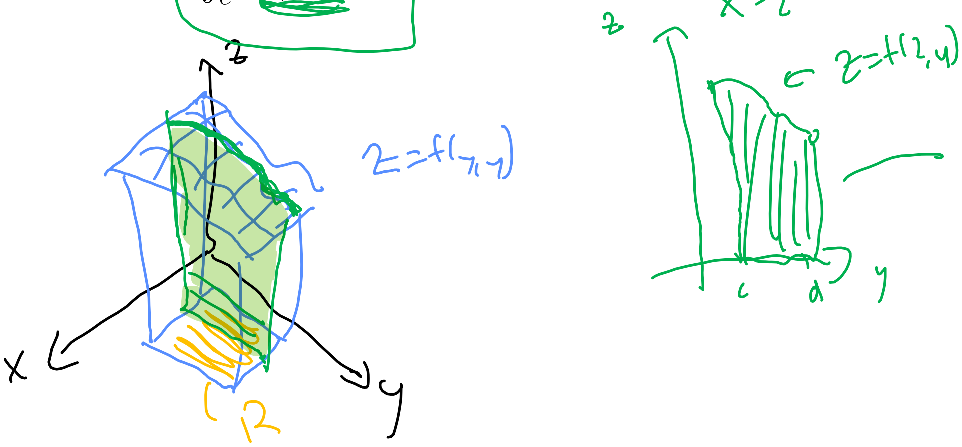
- If f is cts on R, the limit exists
- Some discontinuous f are integrable
- $\iint_R f(x,y)\, dA$ = signed volume between $z=f(x,y)$ & xy-plane above R

Question: How can we compute a double integral?

Answer: Iterated Integrals

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

What does $\int_c^d f(2, y)dy$ represent?



$\int_c^d f(2,y)\,dy$ is the cross-sectional area of the solid when $x=2$

What about $\underbrace{\int_c^d f(x,y)dy}$? $\rightarrow$ cross-sectional area of solid at each different x

Let $A(x) = \int_c^d f(x,y)dy$. Then, $= \int_c^d A(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$?

Volume $= \displaystyle\int_a^b A(x)dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$?

write differentials always!

This is called an iterated integral.

$f(x,y) = 6x^2y$

Example 74. Evaluate $\displaystyle\int_1^2 \int_3^4 6x^2y \; dy \; dx.$ $\leftarrow$ volume of solid below $z = 6x^2y$ above rectangle $[1,2]\times[3,4]$

treat $6x^2$ as constant

$$= \int_1^2 \left(3x^2 y^2 \Big|_{y=3}^{y=4} \right) dx$$

$$= \int_1^2 (3x^2 \cdot 16 - 3x^2 \cdot 9)\,dx$$

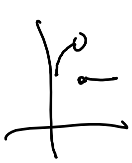
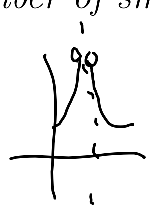
$$= \int_1^2 \underbrace{21x^2}_{A(x)}\,dx = 7x^3 \Big|_{x=1}^{x=2} = 56 - 7 = \boxed{49}$$

if there are y's left here, a mistake was made

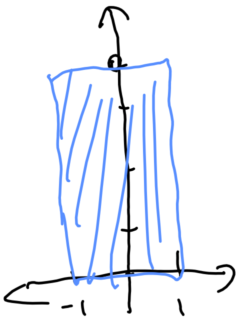
Theorem 75 (Fubini's Theorem)**.** *If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then*

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

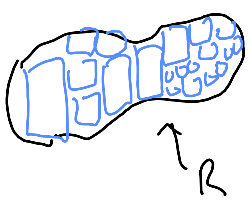
 ✓  ?

Example 76. Compute $\iint_R xe^{e^{e^y}}\,dA$, where R is the rectangle $[-1,1]\times[0,4]$.



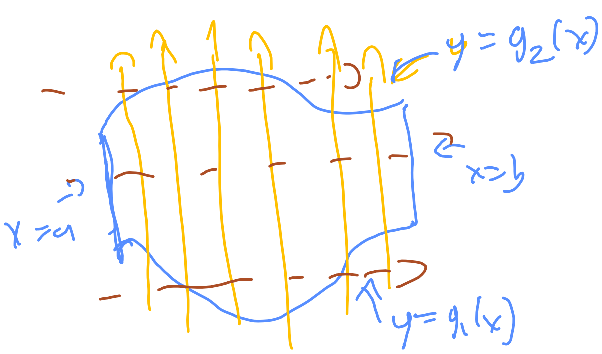
$$V = \int_{-1}^{1}\int_0^4 x e^{e^{e^y}}\,dy\,dx \leftarrow \text{HARD}$$
$$= \int_0^4 \int_{-1}^{1} x e^{e^{e^y}}\,dx\,dy$$
$$= \int_0^4 \left(\tfrac{1}{2} e^{e^{e^y}} x^2 \Big|_{-1}^{1}\right) dy$$
$$= \int_0^4 0\,dy = 0$$

Question: What if the region R we wish to integrate over is not a rectangle?



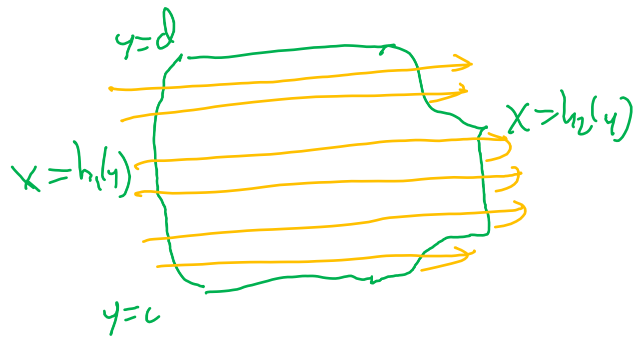
Answer: Repeat same procedure - it will work if the boundary of R is smooth and f is continuous.

Vertically simple



$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

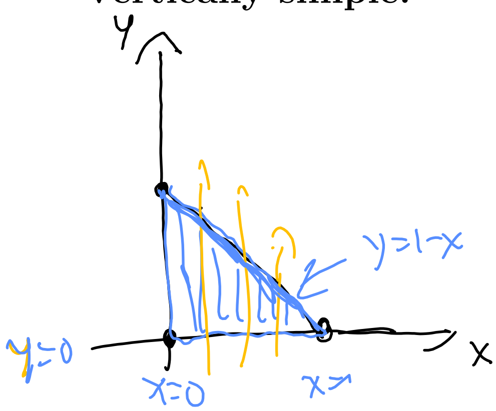
Horizontally simple



$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Example 77. Compute the volume of the solid whose base is the triangle with vertices $(0,0),(0,1),(1,0)$ in the xy-plane and whose top is $z = 2 - x - y$.

Vertically simple:



$$V = \iint_R 2-x-y \, dA$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} 2-x-y \, dy \, dx$$

$$= \int_0^1 (2-x)y - \frac{1}{2}y^2 \Big|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 (2-x)(1-x) - \frac{1}{2}(1-x)^2 \, dx$$

$$= \int_0^1 \frac{3}{2} - 2x + \frac{1}{2}x^2 \, dx$$

$$= \frac{3}{2}x - x^2 + \frac{1}{6}x^3 \Big|_0^1 = \boxed{\frac{2}{3}}$$

Horizontally simple:

Daily Announcements & Reminders:

Goals for Today: Sections 15.2, 15.3

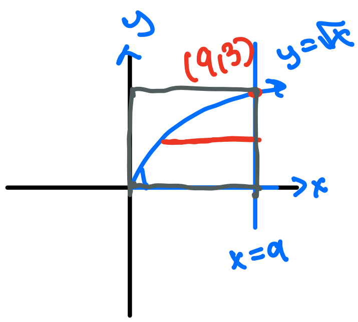
- Be able to set up & evaluate a double integral over a general region
- Change the order of integration for general regions
- Compute areas of general regions in the plane
- Compute the average value of a function of two variables

Example 78. Write the two iterated integrals for $\iint_R 1\ dA$ for the region R which is bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.

Not quite $y = \sqrt{x}$

$y^2 = x$

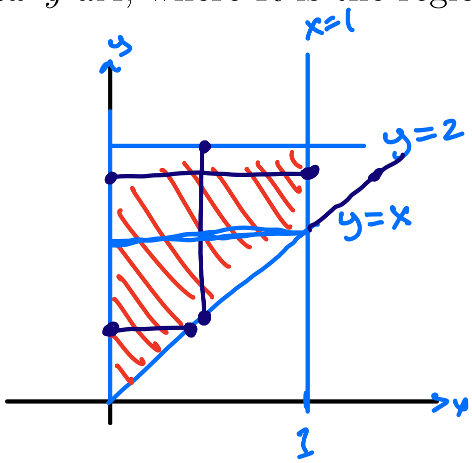
Hint: Sketch the region.



$$\int_0^9 \int_0^{\sqrt{x}} 1\, dy\, dx = \int_0^3 \int_{y^2}^9 1\ dx\, dy$$

outer-most integral should not have variables in the limits of integration.

Example 79. Set up an iterated integral to evaluate the double integral $\iint_R 6x^2y\ dA$, where R is the region bounded by $x = 0, x = 1, y = 2$, and $y = x$.



$$\int_0^1 \int_x^2 6x^2y\ dy\, dx$$

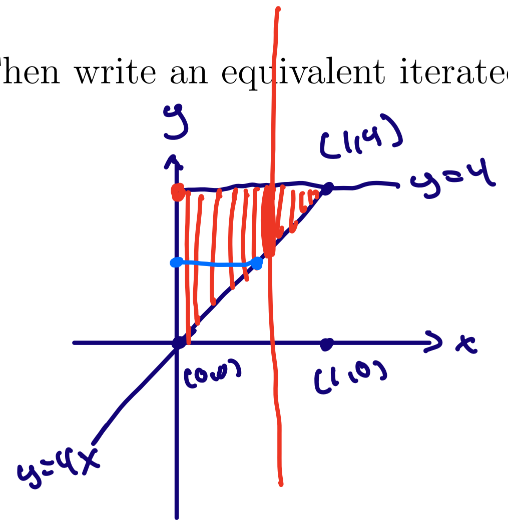
$$= \int_0^1 \int_0^y 6x^2y\ dx\, dy + \int_1^2 \int_0^1 6x^2y\ dx\, dy$$

Cut up the complicated region into simpler regions.

Example 80. Sketch the region of integration for the integral

$$\int_0^1 \int_{4x}^4 f(x,y)\ dy\ dx.$$

Then write an equivalent iterated integral in the order $dx\ dy$.



$$\int_0^4 \int_0^{y/4} f(x,y)\,dx\,dy$$

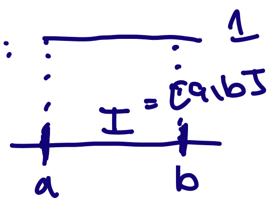
Area & Average Value

Two other applications of double integrals are computing the area of a region in the plane and finding the average value of a function over some domain.

Area: If R is a region bounded by smooth curves, then

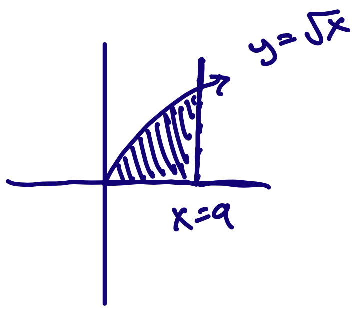
$$\text{Area}(R) = \iint_R 1\, dA$$

1-var:



$$\text{length}(I) = b-a = \int_a^b 1\, dx$$

Example 81. Find the area of the region R bounded by $y = \sqrt{x}, y = 0$, and $x = 9$.



$$\text{Area}(R) = \iint_R 1\, dA$$

$$= \int_0^9 \int_0^{\sqrt{x}} 1\, dy\, dx = \int_0^3 \int_{y^2}^9 1\, dx\, dy$$

18

Average Value: The average value of $f(x,y)$ on a region R contained in $\mathbb{R}^2$ is

$R = [a,b] \times [c,d]$

$$\frac{1}{d-c}\int_c^d \frac{\int_a^b f(x,y)\,dx}{b-a}\,dy \qquad f_{avg} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA$$

Recall: 1-var: $f(x)$ on $[a,b]$. $f_{avg} = \dfrac{\int_a^b f(x)\,dx}{b-a}$

$$\frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA$$

Example 82. Find the average temperature on the region R in the previous example if the temperature at each point is given by $T(x, y) = 4xy^2$.

$$T_{avg} = \frac{1}{\text{Area}(R)} \cdot \iint_R T(x,y)\, dA$$

$$= \frac{1}{18} \cdot \iint_R T(x,y)\, dA$$

$$= \frac{1}{18} \cdot \int_0^9 \int_0^{\sqrt{x}} 4xy^2 \, dy\, dx$$

$$= \frac{1}{18} \cdot \int_0^3 \int_{y^2}^9 4xy^2 \, dx\, dy$$

$$\frac{1}{18} \cdot \int_0^9 \frac{4}{3} x^{5/2} dx = \frac{1}{18}\left(\frac{4}{3} \cdot \frac{2}{7} \cdot 9^{7/2} \right)$$

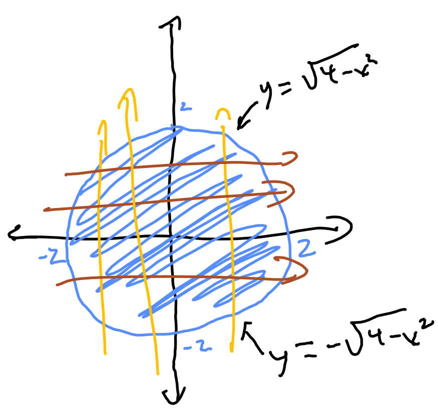
Daily Announcements & Reminders:

- HW 15.2 & 15.3 due tonight
- Exam 2 on T 10/22 (material from exam 1 to today)
- Fall Break next T, no class
- Progress report grades given:
  - $\geq 70\%$ midterm 1 $\Rightarrow$ S
  - $< 70\%$ midterm 1 $\Rightarrow$ U
- Do warmup on Ed

Goals for Today:

Sections 15.4, 15.5

- Introduce the polar coordinate system
- Convert double integrals to iterated polar integrals
- Compute iterated polar integrals
- Define triple integrals and compute basic triple integrals



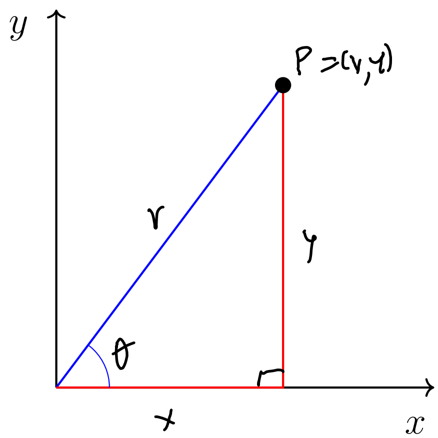
Horizontally or vertically simple?

$$A = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1 \, dy \, dx$$

$$= \int_{-2}^{2} y \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \underbrace{2\sqrt{4-x^2}}_{\text{trig sub}} \, dx$$

Polar Coordinates:



Cartesian coordinates: Give the distances in $\hat{i}$ and $\hat{j}$ directions from $(0,0)$

Polar coordinates: (r,θ)

- r = distance from $(0,0)$ to $P=(x,y)$
- θ = ccw angle between the ray $\vec{OP}$ and the positive x-axis

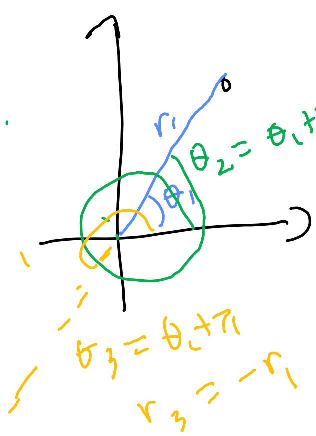
We can use trigonometry to go back and forth.

Polar to Cartesian: $(r,\theta) \to (x,y)$ $\qquad \cos\theta = \frac{x}{r}$

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Cartesian to Polar: $(x,y) \to$ many possible (r,θ)

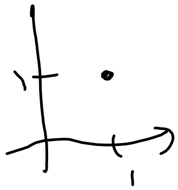
$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}$$



- In problems: $r \geq 0$, $0 \leq \theta \leq 2\pi$ (or any other interval of length 2π)

e.g. $-\pi \leq \theta \leq \pi$

Example 83. a) Find a set of polar coordinates for the point $(x, y) = (1, 1)$.



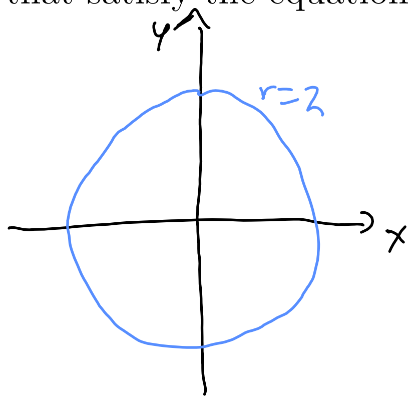
$r^2 = x^2 + y^2 = 1 + 1 = 2$

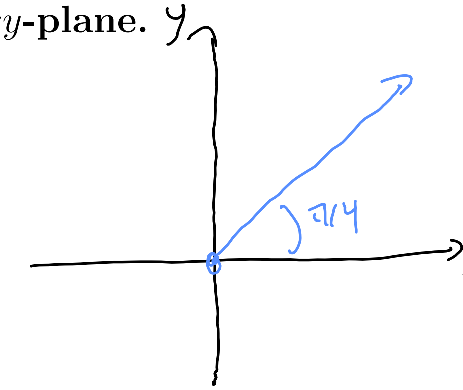
$r = \sqrt{2}$

$\tan\theta = \frac{y}{x} = \frac{1}{1} = 1$

$\theta = \pi/4$ $\quad (r, \theta) = (\sqrt{2}, \frac{\pi}{4})$

b) Graph the set of points (x, y) that satisfy the equation $r = 2$ and the set of points that satisfy the equation $\theta = \pi/4$ **in the xy-plane.**





$\theta = \pi/4$

$r > 0$

$y = x,\ x \geq 0$

c) Write the function $f(x, y) = \sqrt{x^2 + y^2}$ in polar coordinates.

$f = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}$

$= \sqrt{r^2(\cos^2\cancel{\theta} + \sin^2\theta)}$

$= \sqrt{r^2} = r$

$f = \sqrt{r^2}$

$= r$

d) [Poll] Write a Cartesian equation describing the points that satisfy $r = 2\sin(\theta)$.

$\underbrace{y = \frac{x^2}{\cos(\theta)}}_{\text{need only } x, y}$

$2y = \cancel{\sqrt{x^2 + y^2}}$

$2y = x^2 + y^2$

$x = r\cos\theta$

$\boxed{y = r\sin\theta}$

$x^2 + y^2 = r^2$

$\tan\theta = \frac{y}{x}$

$r = 2\sin\theta$

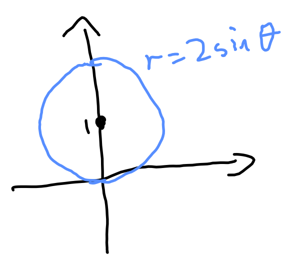
$r = \frac{2y}{r}$

$r^2 = 2y$

$x^2 + y^2 = 2y$

$x^2 + y^2 - 2y + 1 = 0 + 1$

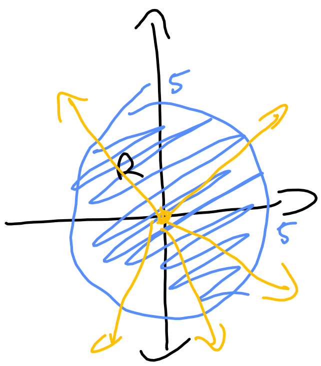
$x^2 + (y-1)^2 = 1$



15.4: Double Integrals in Polar Coordinates

Goal: Given a region R in the xy-plane described in polar coordinates and a function $f(r, \theta)$ on R, compute $\iint_R f(r, \theta)\, dA$.

Example 84. Compute the area of the disk of radius 5 centered at $(0, 0)$.



$$\text{Area} = \iint_R 1\, dA = \int_0^{2\pi}\int_0^5 1\, dr\, d\theta = \int_0^{2\pi} 5\, d\theta = 10\pi$$

WRONG

πr^2

$\text{Area} = 25\pi$

$0 \le r \le 5$

$0 \le \theta \le 2\pi$

"radially simple"

$\Delta A \ne \Delta\theta \Delta r$

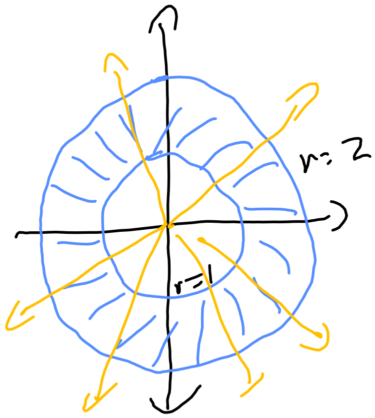
$\approx r \Delta\theta \Delta r$

$$\text{Area} = \int_0^{2\pi}\int_0^5 1 \cdot r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{2}r^2\Big|_0^5\, d\theta = \int_0^{2\pi} \frac{25}{2}\, d\theta = 25\pi \checkmark$$

Remember: In polar coordinates, the area form $dA =$ $r\, dr\, d\theta$

Example 85. Compute $\iint_D e^{-(x^2+y^2)}\,dA$ on the washer-shaped region $1 \le x^2+y^2 \le 4$.

$x^2+y^2=r^2$ $\quad D$



$1 \le r \le 2$

$0 \le \theta \le 2\pi$

$$\iint_D e^{-(x^2+y^2)}\,dA$$

$$= \int_0^{2\pi}\int_1^2 e^{-r^2}\, r\,dr\,d\theta$$

$u=-r^2$

$du=-2r\,dr$

$r=1 \to u=-1$

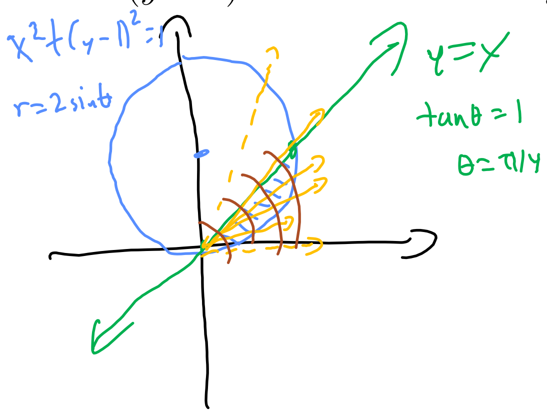
$r=2 \to u=-4$

$$= \int_0^{2\pi}\int_{-1}^{-4} e^{u}\left(-\tfrac{1}{2}\,du\right)d\theta$$

$$= \int_0^{2\pi} -\tfrac{1}{2}e^{u}\Big|_{-1}^{-4}\,d\theta$$

$$= \int_0^{2\pi} -\tfrac{1}{2}e^{-4}+\tfrac{1}{2}e^{-1}\,d\theta = \boxed{\pi\left(e^{-1}-e^{-4}\right)}$$

Example 86. Compute the area of the smaller region bounded by the circle $x^2 + (y-1)^2 = 1$ and the line $y = x$.



$0 \le r \le 2\sin\theta$

$0 \le \theta \le \pi/4$

$$\text{area} = \iint_R 1\,dA$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$= \int_0^{\pi/4}\int_0^{2\sin\theta} r\,dr\,d\theta \qquad \frac{r^2}{2}\Big|_0^{2\sin\theta}$$

$$= \int_0^{\pi/4} 2\sin^2\theta\,d\theta$$

$$= \int_0^{\pi/4} (1-\cos 2\theta)\,d\theta$$

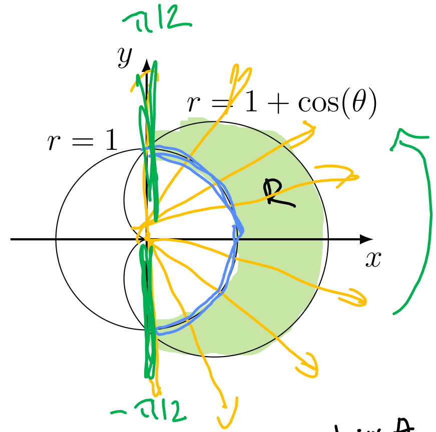
$$= \left(\theta-\tfrac{1}{2}\sin(2\theta)\right)\Big]_0^{\pi/4}$$

$$= \left[\left(\tfrac{\pi}{4}-\tfrac{1}{2}(1)\right)-(0-0)\right]$$

$$= \boxed{\tfrac{\pi}{4}-\tfrac{1}{2}}$$

$$= \int_0^{\sqrt{2}}\int_{\arcsin(\frac{r}{2})}^{\pi/4} 1\,dr\,d\theta$$

Example 87 (Poll)**.** Write an integral for the volume under $z = x$ on the region between the cardioid $r = 1 + \cos(\theta)$ and the circle $r = 1$, where $x \geq 0$.



$1 \leq r \leq 1 + \cos\theta$

$$\text{Volume} = \iint_R x \, dA = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} (r\cos\theta)\, r \, dr \, d\theta$$

Daily Announcements & Reminders:

- HW 15.4 due tonight
- Exam 2 on Tuesday
  - See Canvas announcement for details
- Do warmup poll on Ed

$-\sqrt{4-y^2} \le x \le \sqrt{4-y^2}$

$0 \le y \le 2$

Goals for Today: Sections 15.5, 15.6

- Learn how to write triple integrals as iterated integrals.
- Compute triple iterated integrals
- Change the order of integration in a triple iterated integral.
- Apply our work to find the mass and center of mass of objects in $\mathbb{R}^2$ and $\mathbb{R}^3$

15.5 & 15.6 Triple Integrals & Applications

Idea: Suppose D is a solid region in $\mathbb{R}^3$. If $f(x, y, z)$ is a function on D, e.g. mass density, electric charge density, temperature, etc., we can approximate the total value of f on D with a Riemann sum.

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k,$$

by breaking D into small rectangular prisms ΔV_k.

Taking the limit gives a

integrand

infinitesimal volume

triple integral : $\iiint_D f(x, y, z)\, dV$

region of integration

Important special case:

$$\iiint_D 1\, dV = \text{volume of } (D)$$

Again, we have Fubini's theorem to evaluate these triple integrals as iterated integrals.

Other important spatial applications:

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint\limits_D \delta\, dV$ $\quad \delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint\limits_D x\,\delta\, dV, \qquad M_{xz} = \iiint\limits_D y\,\delta\, dV, \qquad M_{xy} = \iiint\limits_D z\,\delta\, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint\limits_R \delta\, dA$ $\quad \delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint\limits_R x\,\delta\, dA, \qquad M_x = \iint\limits_R y\,\delta\, dA$

Center of mass: $\bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$

Example 88. 1. **Mechanics:** Compute $\int_0^1 \int_0^{2-x} \underbrace{\int_0^{2-x-y}}\, dz\, dy\, dx$.

$$= \int_0^1 \int_0^{2-x} z \Big|_0^{2-x-y} dy\, dx = \int_0^1 \int_0^{2-x} 2-x-y \; dy\, dx$$

$$= \int_0^1 (2-x)y - \tfrac{1}{2}y^2 \Big|_0^{2-x} dx$$

$$= \int_0^1 (2-x)^2 - \tfrac{1}{2}(2-x)^2\, dx = \int_0^1 \tfrac{1}{2}(2-x)^2\, dx = -\tfrac{1}{6}(2-x)^3 \Big|_0^1$$

$$= -\tfrac{1}{6}(1)^3 + \tfrac{1}{6}(2^3)$$

$$= \tfrac{7}{6}$$

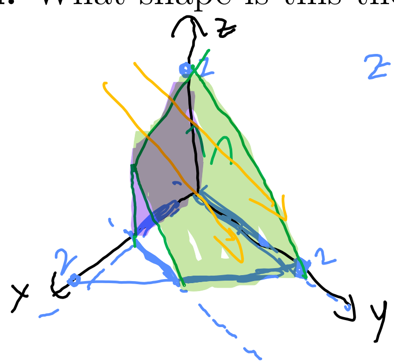
2. **Interpretation:** What shape is this the volume of?

Volume of

$0 \le z \le 2-x-y$

$0 \le y \le 2-x$

$0 \le x \le 1$



$z = 2-x-y$

$x+y+z=2$

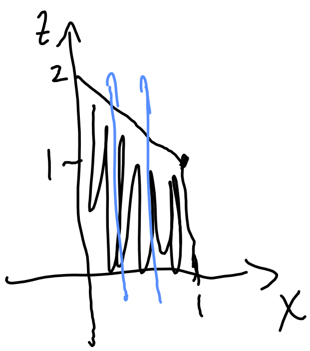
$y = 2-x-z$

3. **Rearrange:** Write an equivalent iterated integral in the order *dy dz dx*.

$$\int_0^1 \int_0^{2-x} \int_0^{2-x-z} dy\, dz\, dx$$

$\boxed{0 \le 2-x-z}$

$x+z \le 2$



We will think about converting triple integrals to iterated integrals in terms of the shadow of D on one of the coordinate planes.

Case 1: z-**simple**) region. If R is the shadow of D on the xy-plane and D is bounded above and below by the surfaces $z = h(x, y)$ and $z = g(x, y)$, then

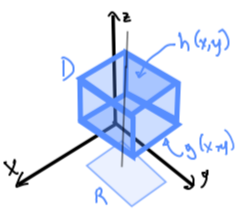
$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\, \underline{dz} \right) dy\, dx$$

← no z

or $dx\, dy$

↑ no z

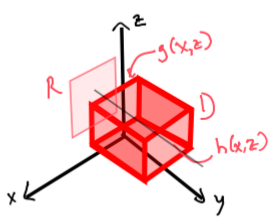
↑ no z, outer bound should be constants only



Case 2: y-**simple**) region. If R is the shadow of D on the xz-plane and D is bounded right and left by the surfaces $y = h(x, z)$ and $y = g(x, z)$, then

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(x,z)}^{h(x,z)} f(x,y,z)\, dy \right) dz\, dx$$

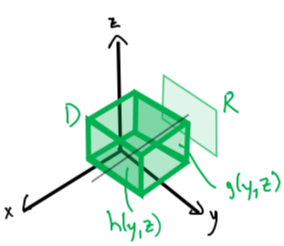
or $dx\, dz$



Case 3: x-**simple**) region. If R is the shadow of D on the yz-plane and D is bounded front and back by the surfaces $x = h(y, z)$ and $x = g(y, z)$, then

$$\iiint_D f(x,y,z)\, dV = \iint_R \left(\int_{g(y,z)}^{h(y,z)} f(x,y,z)\, dx \right) dz\, dy$$

or $dy\, dz$



If none of these: split into smaller pieces

$x\geq 0,\ y\geq 0,\ z\geq 0$

Example 89. Write an integral for the mass of the solid D in the first octant with $2y \leq z \leq 3 - x^2 - y^2$ with density $\delta(x, y, z) = x^2y + 0.1$ by treating the solid as a) z-simple and b) x-simple. Is the solid also y-simple?

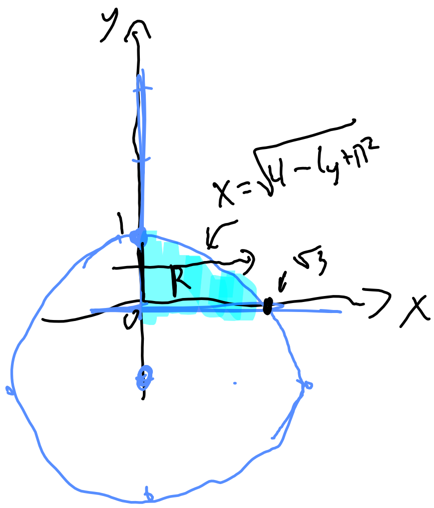
a) z-simple: $\text{mass} = \iiint_D \delta \, dV$

$= \iiint_D x^2y + 0.1 \, dV$

Sketch shadow R in xy-plane.

$x \geq 0,\ y \geq 0$

$= \int_0^1 \int_0^{\sqrt{4-(y+1)^2}} \int_{2y}^{3-x^2-y^2} x^2y + 0.1 \, dz\, dx\, dy$

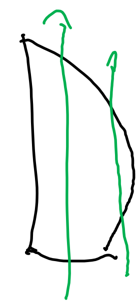


$2y \leq 3 - x^2 - y^2$

$x^2 + y^2 + 2y \leq 3$

$x^2 + y^2 + 2y + 1 \leq 4$

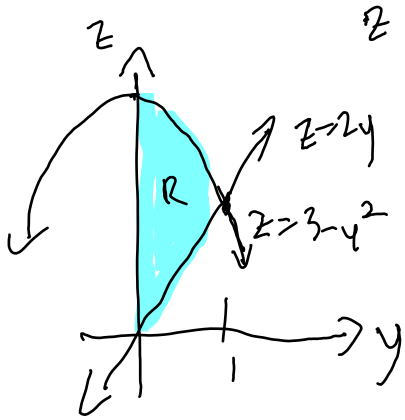
$x^2 + (y+1)^2 \leq 4$



b) x-simple $\quad \text{mass} = \int_0^1 \int_{2y}^{3-y^2} \int_0^{\sqrt{3-y^2-z}} x^2y + 0.1 \, dx\, dz\, dy$

$z = 3 - x^2 - y^2 \rightarrow x^2 = 3 - y^2 - z$

$x = \sqrt{3 - y^2 - z}$



$0 \leq y$

$0 \leq z$

$z \geq 2y$

$0 \leq \sqrt{3-y^2-z} \iff 0 \leq 3 - y^2 - z$

$z \leq 3 - y^2$

$2y = 3 - y^2$

$y^2 + 2y - 3 = 0$

$(y+3)(y-1) = 0$

c) Not y-simple.

Example 89 (cont.)

Rules for Triple Integrals for the Sketching Impaired (credit to Wm. Douglas Withers)

Basic

Rule 1: Choose a variable appearing exactly twice for the next integral.

Rule 2: After setting up an integral, cross out any constraints involving the variable just used.

Rule 3: Create a new constraint by setting the lower limit of the preceding integral less than the upper limit.

Intermediate

Rule 4: A square variable counts twice.

Rule 5: The region of integration of the next step must lie within the domain of any function used in previous limits.

Rule 6: If you do not know which is the upper limit and which is the lower, take a guess - but be prepared to backtrack.

Advanced

Rule 7: When forced to use a variable appearing more than twice, choose the most restrictive pair of constraints.

Rule 8: When unable to determine the most restrictive pair of constraints, set up the integral using each possible most restrictive pair and add the results.

Example 90. Set up an integral for the volume of the region D defined by

$$\cancel{x + y^2 \leq 8}, \quad \cancel{y^2 + 2z^2 \leq x}, \quad y \geq 0$$

Rule 1: x appears first ; choose it

$y^2+2z^2 \leq x \leq 8-y^2$

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-z^2}} \int_{y^2+2z^2}^{8-y^2} dx\, dy\, dz$$

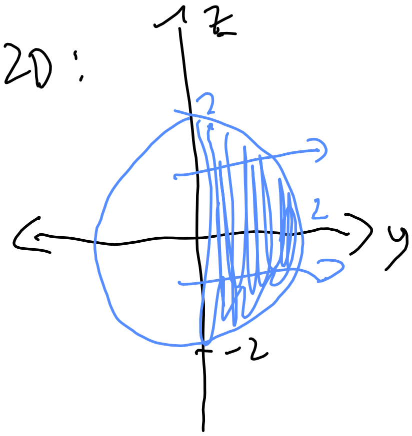
Rule 2: Cross out these constraints

Rule 3: $y \geq 0$ & add $y^2+2z^2 \leq 8-y^2$

$\Updownarrow$

$y^2+z^2 \leq 4$

Sketch 2D:



$0 \le y \le \sqrt{4-z^2}$

$-2 \le z \le 2$

Example 91. Set up a triple iterated integral for the triple integral of $f(x, y, z) = x^3y$ over the region D bounded by

$$x^2 + y^2 = 1, \quad \cancel{z = 0}, \quad \cancel{x + y + z = 2}.$$

After class: Rule 1/4: x appears 3 times, y appears 3 times, z appears 2 times

so choose z first

Rule 6: Guess $2-x-y \le z \le 0$ — so this is wrong

Rule 2: Cross out used constraints

Rule 3: Make new constraint: $2-x-y < 0$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\cancel{2-x-y}\ 0}^{\cancel{0}\ 2-x-y} x^3y \, dz \, dy \, dx$$

often, but not always, 0 is a lower bound

$x^2+y^2=1$

$y > 2-x$

But no part of this is bounded by the circle!

So then our constraint is

$x^2+y^2 \le 1 \qquad y < 2-x$

- concave up shapes are usually lower bounds
  concave down shapes are usually upper bounds

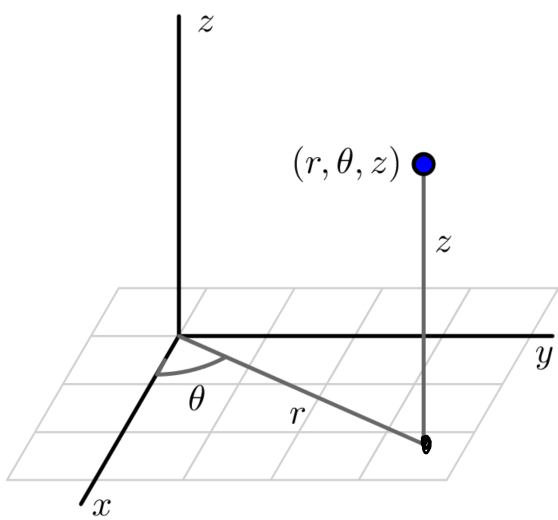
Daily Announcements & Reminders:

- HW 15.5 & 15.6 due tonight
- Exam grading ongoing; hope to release numeric grades F night
- Withdraw deadline is Saturday at 4pm
- Do warmup poll on Ed

Goals for Today: Section 15.7

- Be able to convert between Cartesian, cylindrical, and spherical coordinate systems in $\mathbb{R}^3$
- Compute triple integrals expressed in cylindrical coordinates
- Compute triple integrals expressed in spherical coordinates

Cylindrical Coordinate System



Cylindrical to Cartesian:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

Cartesian to Cylindrical:

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$$

For uniqueness:

- $r \geq 0$, θ in an interval of size 2π
  e.g. $[0, 2\pi]$ or $[-\pi, \pi]$

Example 92. a) Find cylindrical coordinates for the point with Cartesian coordinates $(-1, \sqrt{3}, 3)$.

$x \quad y \quad z$

$(r, \theta, z) = (2, \frac{2\pi}{3}, 3)$

r: $r^2 = (-1)^2 + (\sqrt{3})^2$
$= 1 + 3 = 4$

θ: $\tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$
$\theta = 2\pi/3$

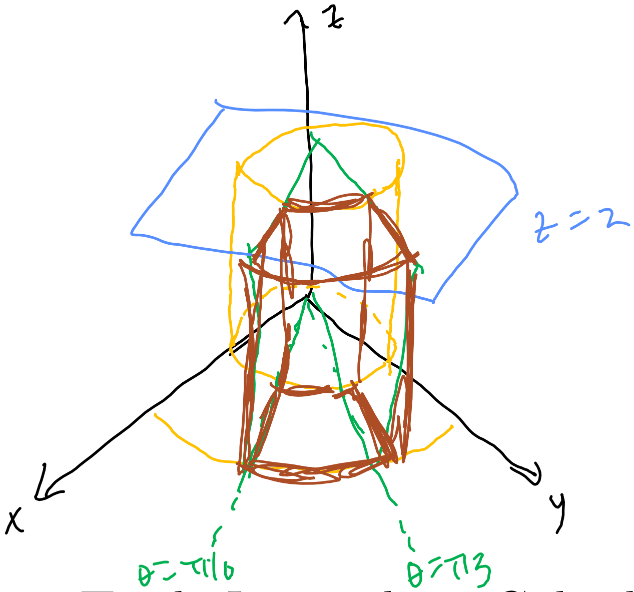
b) Find Cartesian coordinates for the point with cylindrical coordinates $(2, 5\pi/4, 1)$.

$r \quad \theta \quad z$

$(x, y, z) = (2\cos(5\pi/4), 2\sin(5\pi/4), 1)$
$= (-\frac{2}{\sqrt{2}}, -\frac{2}{\sqrt{2}}, 1)$

Example 93. In xyz-space sketch the *cylindrical box*

$$B = \{(r, \theta, z) \mid 1 \le r \le 2,\ \pi/6 \le \theta \le \pi/3,\ 0 \le z \le 2\}.$$



$r = 1$: cylinder of radius 1

$\theta = \pi/6$: vertical plane through z-axis

$z = 2$: horizontal plane

∘ other good regions: (circular) paraboloids, cones, spheres ok

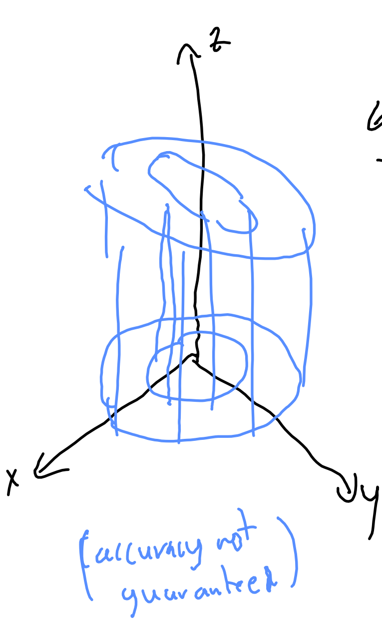
Triple Integrals in Cylindrical Coordinates

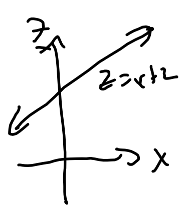
We have $dV =$ $r\,dz\,dr\,d\theta$

Example 94. Set up a iterated integral in cylindrical coordinates for the volume of the region D lying below $z = x + 2$, above the xy-plane, and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

D: $0 \le z \le x + 2$ $1 \le x^2 + y^2 \le 4$; $0 \le z \le r\cos\theta + 2$

$1 \le r^2 \le 4 \iff 1 \le r \le 2$



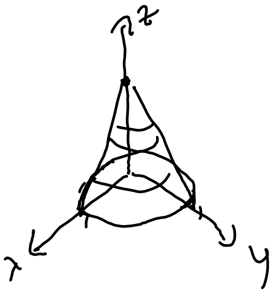


$$V = \iiint_D 1\, dV$$

$$= \int_0^{2\pi} \int_1^2 \int_0^{r\cos\theta + 2} r\, dz\, dr\, d\theta$$

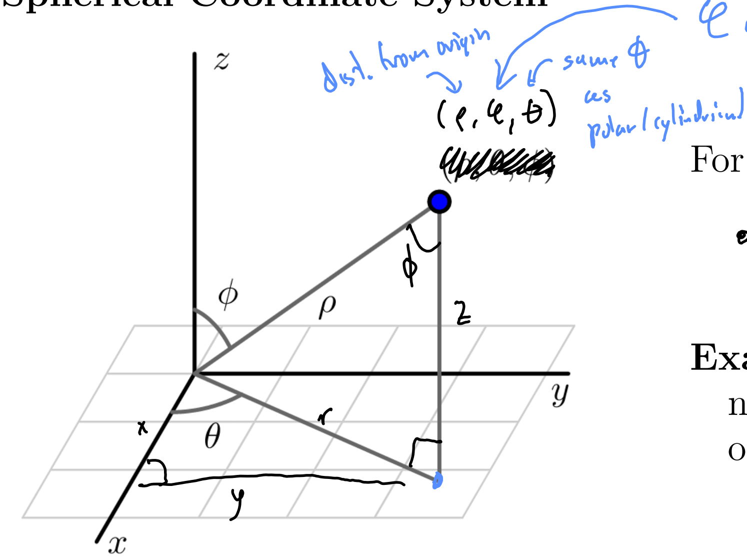
Example 95 (Poll)**.** Suppose the density of the cone defined by $r = 1 - z$ with $z \geq 0$ is given by $\delta(r, \theta, z) = z$. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.

$$\text{mass} = \iiint_D (\text{density})\, dV$$
$$= \int_0^{2\pi} \int_0^1 \int_0^{1-r} zr \, dz \, dr \, d\theta$$
$$= \int_0^{2\pi} \int_0^1 \int_0^{1-z} zr \, dr \, dz \, d\theta$$
$$= \int_0^1 \int_0^{1-r} \int_0^{2\pi} zr \, d\theta \, dz \, dr$$



Spherical Coordinate System

φ or ϕ : angle down from pos. z-axis



For uniqueness:

- $\rho \geq 0$, φ in $[0, \pi]$, θ in $[0, 2\pi]$

Example 96. a) Find spherical coordinates for the point with Cartesian coordinates $(-2, 2, \sqrt{8})$.
$\quad x \quad y \quad z$

$$\rho = \sqrt{(-2)^2 + (2)^2 + (\sqrt{8})^2} = \sqrt{4+4+8} = 4$$
$$\tan\theta = \frac{2}{-2} = -1 \qquad \theta = 3\pi/4$$
$$\tan\varphi = \frac{\sqrt{x^2+y^2}}{z} = \frac{\sqrt{8}}{\sqrt{8}} = 1 \qquad \varphi = \pi/4$$

b) Find Cartesian coordinates for the point with spherical coordinates $(2, \pi/2, \pi/3)$.
$\rho \quad \varphi \quad \theta$

$$x = 2\sin(\pi/2)\cos(\pi/3) = 1$$
$$y = 2\sin(\pi/2)\sin(\pi/3) = \sqrt{3}$$
$$z = 2\cos(\pi/2) = 0$$

Spherical to Cartesian:

$$x = \rho \sin(\varphi)\cos(\theta)$$
$$y = \rho \sin(\varphi)\sin(\theta)$$
$$z = \rho \cos(\varphi)$$

Cartesian to Spherical:

$$\rho^2 = x^2 + y^2 + z^2$$
$$\tan(\theta) = \frac{y}{x}$$
$$\tan(\varphi) = \frac{\sqrt{x^2+y^2}}{z}$$

Example 97. In *xyz*-space sketch the *spherical box*

$$B = \{(\rho, \varphi, \theta) \mid 1 \le \rho \le 2,\ 0 \le \varphi \le \pi/4,\ \pi/6 \le \theta \le \pi/3\}.$$

- spheres / cones / planes through z axis are good!

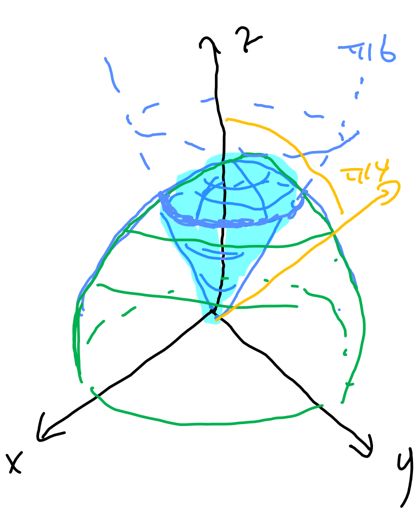
Triple Integrals in Spherical Coordinates

$\sqrt{x^2+y^2} = \rho \sin\varphi$

$x^2+y^2 = \rho^2 \sin^2\varphi$

We have $dV =$ $\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

Example 98. Write an iterated integral for the volume of the "ice cream cone" D bounded above by the sphere $x^2+y^2+z^2 = 1$ and below by the cone $z = \sqrt{3}\sqrt{x^2+y^2}$.



Convert bounds:

Sphere: $x^2+y^2+z^2=1$

$\rho^2 = 1$

$\rho = 1$

cone: $z = \sqrt{3}\sqrt{x^2+y^2}$

$\cancel{\rho}\cos\varphi = \sqrt{3}\,\cancel{\rho}\sin\varphi$

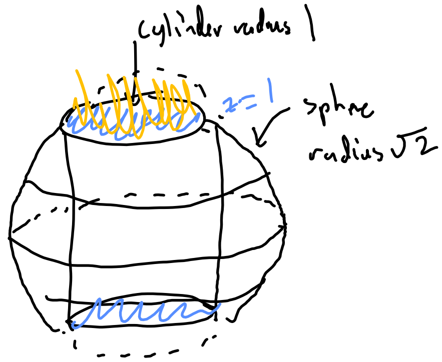
$\tan\varphi = \frac{1}{\sqrt{3}}$

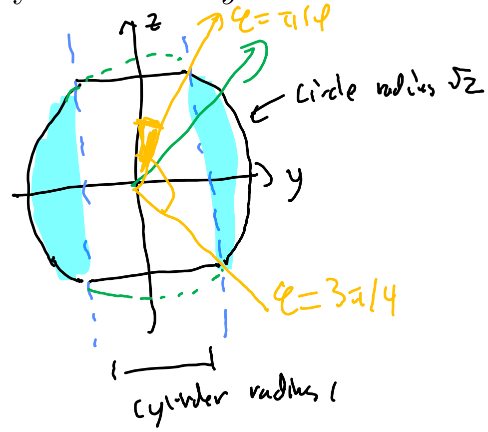
$\varphi = \pi/6$

$\frac{1}{\sqrt{3}} = \frac{\sqrt{x^2+y^2}}{z} = \tan\varphi$

$$V = \int_0^{2\pi} \int_0^{\pi/6} \int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Example 99 (Poll). Write an iterated integral for the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.





Spherical coordinates

$$V = \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_{\csc(\varphi)}^{\sqrt{2}} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Convert bounds:

sphere: $\rho = \sqrt{2}$

cylinder: $\rho^2 \sin^2\varphi = 1$

$\rho \sin\varphi = 1$

$\rho = \csc\varphi$

① 2 bounds w/ ρ so do ρ first

② To find φ: set ρ-bounds equal & solve

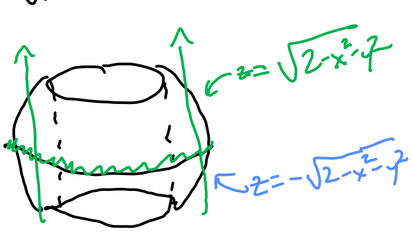
$\sqrt{2} = \csc(\varphi)$

$\sin(\varphi) = \frac{1}{\sqrt{2}}$

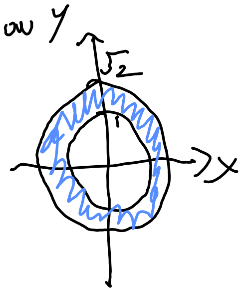
$\varphi = \pi/4, \; 3\pi/4$

(Post class) Cylindrical coordinates

We have



with shadow



so $V = \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} r \, dz \, dr \, d\theta$

Daily Announcements & Reminders:

- HW 15.7 due tonight
- Quiz 7 in studio tomorrow on 15.5-15.7
  L.O. I3, I4, I5
- Exam 2 will be released this afternoon with regrades open until 5pm 11/5
  - median 79%
- Do warmup poll on Ed →

Volume bounded above by $x^2+y^2+z^2=2 \to \rho = \sqrt{2}$

below by $z=1 \to \rho\cos\varphi = 1 \to \rho = \sec\varphi$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\varphi}^{\sqrt{2}} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Set equal to get φ: $\sqrt{2} = \sec\varphi$

$\cos\varphi = \frac{1}{\sqrt{2}}$

$\varphi = \pi/4$

Goals for Today: Section 15.8

- Change variables in multiple integrals
- Identify choices for changing variables in a given integration problem

Thinking about single variable calculus: Compute $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

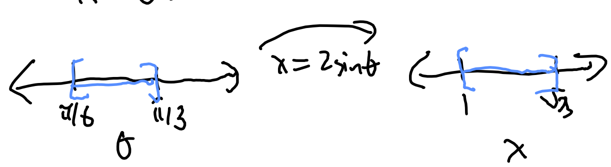
① Identify substitution: $x = 2\sin\theta$

② Compute derivative: $dx = \underbrace{2\cos\theta}\, d\theta$

③ Find new region of integration:

$x = 1 \to 1 = 2\sin\theta \to \frac{1}{2} = \sin\theta \to \theta = \pi/6$

$x = \sqrt{3} \to \sqrt{3} = 2\sin\theta \to \frac{\sqrt{3}}{2} = \sin\theta \to \theta = \pi/3$



④ Plug in:

$$\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \int_{\pi/6}^{\pi/3} \frac{1}{\underbrace{\sqrt{4-4\sin^2\theta}}_{=2\cos\theta}}\, 2\cos\theta \, d\theta = \int_{\pi/6}^{\pi/3} d\theta = \boxed{\pi/6}$$

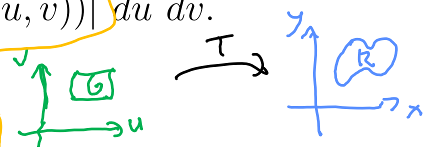
$\swarrow \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}$ (like $x = 2\sin\theta$)

Theorem 100 (Substitution Theorem). *Suppose* $\mathbf{T}(u,v)$ *is a one-to-one, differentiable transformation that maps the region* G *in the* uv*-plane to the region* R *in the* xy*-plane. Then*

↑ like $[\pi/6, \pi/3]$ ↑ like $[1, \sqrt{3}]$

EASIER

$$\underbrace{\iint_R f(x,y)\,dx\,dy}_{\text{HARD}} = \iint_G f(\mathbf{T}(u,v))\,|\det(D\mathbf{T}(u,v))|\,du\,dv.$$



Jacobian (determinant)

- measure area scaling

e.g. if $|\det(D\mathbf{T}(u,v))| = 2$

then Area(R) = 2 · Area(G)

Example 101. Evaluate $\displaystyle\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy$ via the transformation $x = u + v$, $y = 2v$.

1. **Find T:**

$$\vec{T}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix} = \begin{bmatrix} u+v \\ 2v \end{bmatrix}$$

$\begin{cases} x = u+v \\ y = 2v \end{cases}$ → solve for u, v

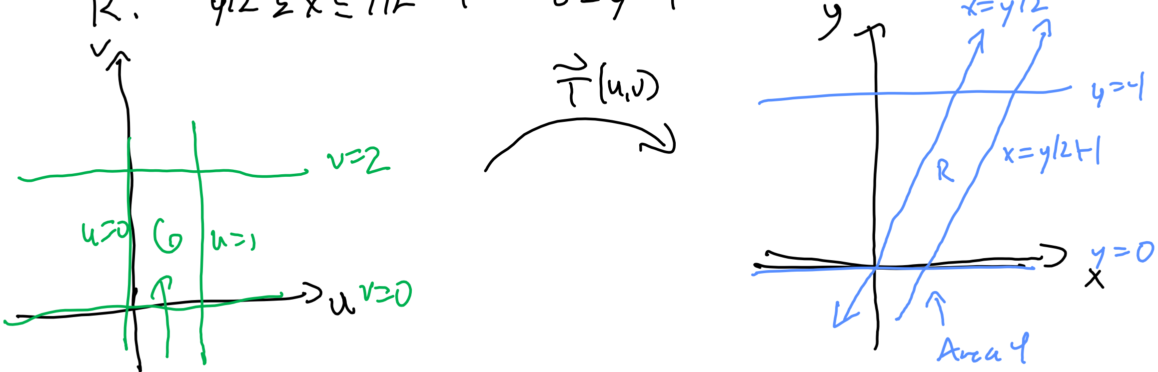
often $\vec{T}^{-1}(x,y)$ is given

$$u = \frac{2x-y}{2} \qquad v = \frac{y}{2}$$

↑ if given these, invert system & solve for x, y to get $\vec{T}$

2. **Find G and sketch:**

R: $y/2 \le x \le y/2 + 1$ $0 \le y \le 4$



$y = 0 \to 2v = 0 \to v = 0$

$y = 4 \to 2v = 4 \to v = 2$

$x = y/2 \to u + v = \frac{2v}{2} \to u = 0$

$x = y/2 + 1 \to u + v = \frac{2v}{2} + 1 \to u = 1$

3. **Find Jacobian:** $\vec{T}(u,v) = \begin{bmatrix} u+v \\ 2v \end{bmatrix}$

$$D\vec{T}(u,v) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{so} \quad |\det(D\vec{T}(u,v))| = \left| \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \right| = |2-0| = 2$$

4. **Convert and use theorem:**

$$\int_0^4 \int_{y/2}^{y/2+1} \frac{(2x-y)}{2}\,dx\,dy = \iint_G f(T(u,v))\,|\det(DT(u,v))|\,dA$$

$$= \int_0^2 \int_0^1 \frac{2(u+v)-2v}{2} \cdot 2 \quad du\,dv$$

$$= \int_0^2 \int_0^1 2u\,du\,dv = 2$$

Example 102. a) [Poll] Find the Jacobian of the transformation

$$x = u + (1/2)v, \ \ y = v.$$

$\vec{T} = \begin{bmatrix} u + \frac{1}{2}v \\ v \end{bmatrix} \Rightarrow D\vec{T} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \to |\det D\vec{T}| = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = |1-0| = 1$

b) [Poll] Which transformation(s) seem suitable for the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 \underline{(2x-y)} e^{(2x-y)^2} \, dx \, dy?$$

i) $u = x, v = y$ ← does nothing

ii) $u = \sqrt{x^2+y^2}, v = \arctan(y/x)$ ← polar, not helpful

Good

iii) $u = 2x - y,\ v = y^3$

iv) $u = y,\ v = 2x - y$

v) $u = 2x - y,\ v = y$

Better (b/c of bounds)

vi) $u = e^{(2x-y)^2},\ v = y^3$

* hard to invert / strands $(2x-y)$ factor

Theorem 103 (Derivative of Inverse Coordinate Transformation)**.** *If* $\mathbf{T}(u, v)$ *is a one-to-one differentiable transformation that maps a region* G *in the uv-plane to a region* R *in the xy-plane and* $T(u_0, v_0) = (x_0, y_0)$*, then we have*

$$|\det(D\mathbf{T}(u_0, v_0))| = \frac{1}{|\det(D\mathbf{T}^{-1}(x_0, y_0))|}$$

Example 104. Let's evaluate $\iint_R \frac{y(x+y)}{x^3}$ where R is the region in the xy-plane bounded by $y = x, y = 3x, y = 1 - x$, and $y = 2 - x$. Consider the coordinate transformation $u = x + y, v = y/x$.

$$\vec{T}^{-1}(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} x+y \\ y/x \end{bmatrix}$$

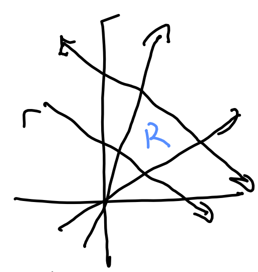
1. **Find the rectangle G in the uv plane that is mapped to R**

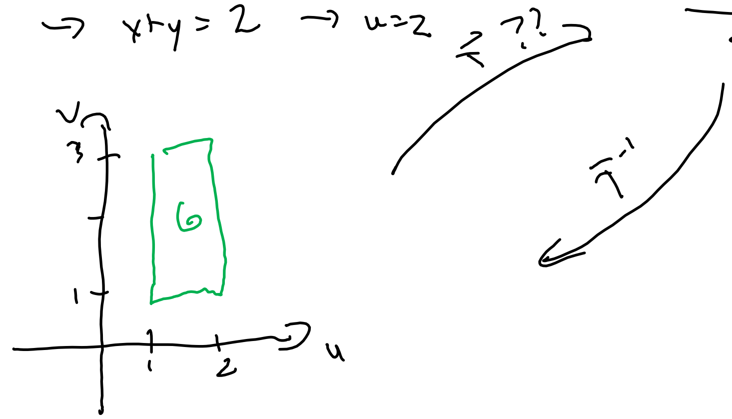
$y = x \to y/x = 1 \to v = 1$

$y = 3x \to y/x = 3 \to v = 3$

$y = 1 - x \to x + y = 1 \to u = 1$

$y = 2 - x \to x + y = 2 \to u = 2$





2. **Evaluate $f(\mathbf{T}(u,v))|\det(D\mathbf{T}(u,v))|$ in terms of u and v without directly solving for T using the theorem above**

We know: $|\det(DT(u,v))| = \dfrac{1}{|\det(DT^{-1}(x,y))|}$ and $DT^{-1} = \begin{bmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}$

$$= \frac{1}{\left|\frac{1}{x} + \frac{y}{x^2}\right|} = \frac{1}{\frac{x+y}{x^2}} = \frac{x^2}{x+y}$$

So $f(T(u,v))\,|\det(DT(u,v))| = \dfrac{y\cancel{(x+y)}}{x\,\cancel{x^3}} \cdot \dfrac{\cancel{x^2}\,1}{\cancel{(x+y)}} = \dfrac{y}{x} = v$

3. **Use the Substitution Theorem to compute the integral.**

$$\iint_R f(x,y)\, dA = \iint_G f(T(u,v))\, |\det(\vec{D}T(u,v))\, dA$$

$$= \int_1^3 \int_1^2 v \, du\, dv$$

Daily Announcements & Reminders:

- HW 15.8 due tonight
- Review graded Exam 2
  - regrade requests should reference a specific correct thing you did which did not receive credit.
- Do warmup poll on Ed

Goals for Today: Section 16.1, 16.2

- Define a line integral for a scalar function $f(x, y)$ or $f(x, y, z)$
- Compute line integrals using parameterizations

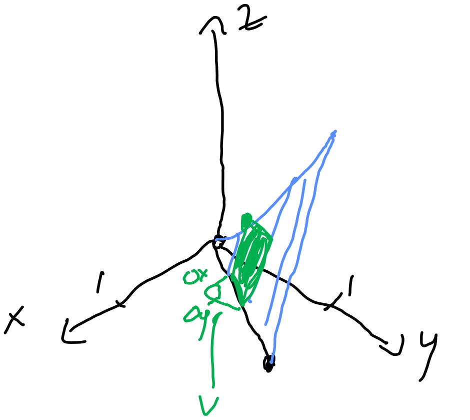
Unit 4: Vector Calculus

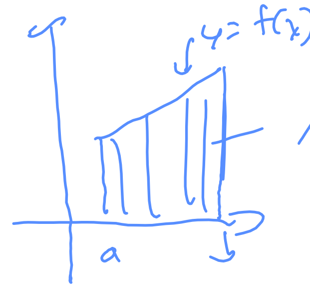
Goals:

- Extend 1D / 2D integrals to 1D/2D objects living in higher-dimensional space
- Extend the Fundamental Theorem of Calculus in new ways

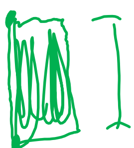
We will use tools from everything we have covered so far to do this: parameterizations, derivatives and gradients, and multiple integrals.

Example 105. Suppose we build a wall whose base is the straight line from $(0,0)$ to $(1,1)$ in the xy-plane and whose height at each point is given by $h(x,y) = 2x + y^2$ meters. What is the area of this wall?

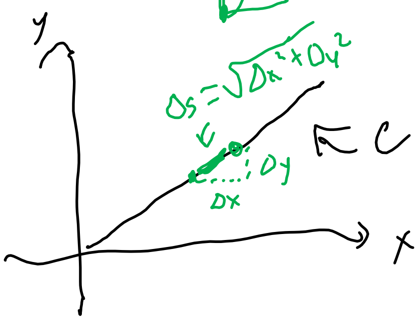




$$\text{Area} = \int_a^b f(x)\,dx$$



$$h(x_0,y_0) = 2x_0 + y_0^2$$



$$\text{Area} \approx \sum_{k=1}^{n} h(x_k, y_k)\sqrt{\Delta x^2 + \Delta y^2}$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} h(x_k, y_k)\sqrt{\Delta x^2 + \Delta y^2}$$
$$= \int_a^b h(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

1) Parameterize C: $\vec{r}(t) = \langle 1,1\rangle t + \langle 0,0\rangle$, $0 \le t \le 1$

$\vec{r}\,'(t) = \langle 1,1\rangle$

2) Substitute:
$$\int_0^1 \left(2(t) + (t)^2\right)\sqrt{1^2+1^2}\,dt = \int_0^1 2\sqrt{2}\,t + \sqrt{2}\,t^2\,dt = \boxed{4\sqrt{2}/3}$$

Definition 106. The **line integral** of a scalar function $f(x,y)$ over a curve C in $\mathbb{R}^2$ is

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\sqrt{\Delta x_k^2 + \Delta y_k^2}$$
$$= \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

where $\vec{r}(t)$ parameterizes C with $a \le t \le b$

$ds = \|\vec{r}\,'(t)\|\,dt$

What things can we compute with this?

- If $f = 1$: $\int_C 1\,ds$ = arc length of C $= \int_a^b \|\vec{r}\,'(t)\|\,dt$
- If $f = \delta$ is a density function: $\int_C \delta\,ds$ = mass of object along C w/ density δ
- If f is a height: $\int_C f\,ds$ = area of the surface above C with height f

Strategy for computing line integrals:

1. Parameterize the curve C with some $\mathbf{r}(t)$ for $a \leq t \leq b$
2. Compute $ds = \|\mathbf{r}'(t)\|\ dt$
3. Substitute: $\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\ dt$
4. Integrate

Example 107. [Poll] Compute $\int_C 2x + y^2\ ds$ along the curve C given by $\mathbf{r}(t) = 10t\mathbf{i} + 10t\mathbf{j}$ for $0 \leq t \leq \frac{1}{10}$.

1) Parameterize C. Given in problem.

2) $ds = \|\vec{r}\,'(t)\|\,dt$ $\qquad \vec{r}\,'(t) = \langle 10, 10 \rangle$

$= \sqrt{10^2+10^2}\ dt$

$= 10\sqrt{2}\ dt$

3) $\int_C 2x+y^2\ ds = \int_0^{1/10} \left(2(10t) + (10t)^2\right) \cdot 10\sqrt{2}\ dt$

$= \int_0^{1/10} 10\sqrt{2}\left(20t + 100t^2\right) dt$

$= 10\sqrt{2}\left(10t^2 + \frac{100}{3}(t^3)\right)\Big|_0^{1/10}$

$= 10\sqrt{2}\left(\frac{10}{100} + \frac{1}{30}\right)$

$= \sqrt{2}\left(1 + \frac{1}{3}\right)$

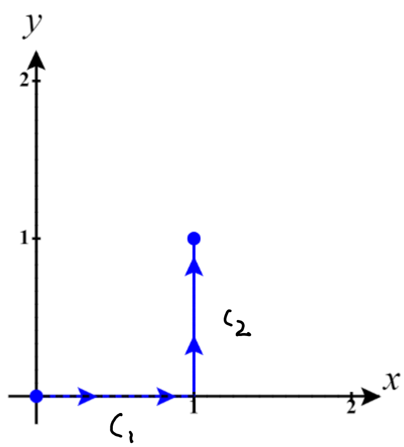
$= \frac{4\sqrt{2}}{3}$

- If $\vec{r}_1(t)$ & $\vec{r}_2(t)$ parameterize same curve C then

$\int_{a_1}^{b_1} f(\vec{r}_1(t))\|\vec{r}_1\,'(t)\|\,dt = \int_{a_2}^{b_2} f(\vec{r}_2(t))\|\vec{r}_2\,'(t)\|\,dt$

- $\int_{-C} f(x,y)\,ds = -\int_C f(x,y)\,ds$ ($-C$ is C w/ opposite orientation)

Example 108. Compute $\int_C 2x + y^2 \; ds$ along the curve C pictured below.



1) Parameterize C

$$\int_C 2x+y^2\,ds = \int_{C_1} 2x+y^2\,ds + \int_{C_2} 2x+y^2\,ds$$

$C_1: \vec{r}(t) = \langle 1,0\rangle t + \langle 0,0\rangle \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 1,0\rangle$

2) $\|\vec{r}\,'(t)\| = 1$

$C_2: \vec{r}_2(t) = \langle 0,1\rangle t + \langle 1,0\rangle \quad 0 \le t \le 1$

$\vec{r}_2\,'(t) = \langle 0,1\rangle$

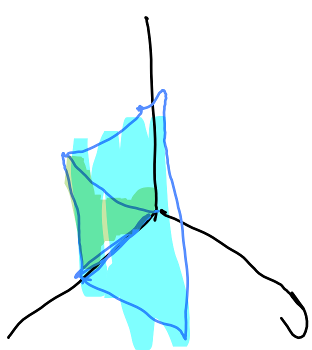
2) $\|\vec{r}\,'(t)\| = 1$

3) $$\int_C 2x+y^2\,ds = \int_0^1 \left(2(t) + 0^2\right)\cdot 1\,dt + \int_0^1 \left(2(1)+t^2\right)\cdot 1\,dt$$

$$= t^2\Big|_0^1 + 2t + \frac{t^3}{3}\Big|_0^1$$

$$= 1 + 2 + \frac{1}{3}$$

$$= \boxed{\frac{10}{3}}$$



- most line integrals are path-dependent

Example 109 (Poll)**.** Let C be a curve parameterized by $\mathbf{r}(t)$ from $a \le t \le b$. Select all of the true statements below.

✓ to fix: $a-4 \le t \le b-4$

a) $\mathbf{r}(t+4)$ for $a \le t \le b$ is also a parameterization of C with the same orientation

False

b) $\mathbf{r}(2t)$ for $a/2 \le t \le b/2$ is also a parameterization of C with the same orientation

True

1 3

c) $\mathbf{r}(-t)$ for $a \le t \le b$ is also a parameterization of C with the opposite orientation

False

↑ goes from $\vec{r}(-a)$ to $\vec{r}(-b)$

-1 -3

d) $\mathbf{r}(-t)$ for $-b \le t \le -a$ is also a parameterization of C with the opposite orientation

True

e) $\mathbf{r}(b-t)$ for $0 \le t \le b-a$ is also a parameterization of C with the opposite orientation

True

Parameterizations to know

1) Lines/line segment

2) circles/ellipses

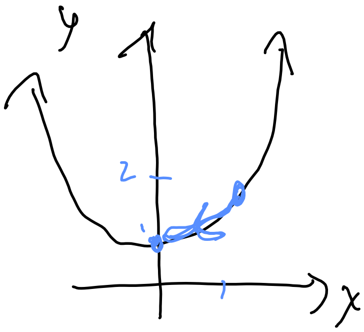
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\vec{r}(t) = \langle a\cos(t), b\sin(t) \rangle$$

ccw; $\alpha \le t \le \beta$

3) segments $y = f(x)$, $x = f(y)$

Example 110. Find a parameterization of the curve C that consists of the portion of the curve $y = x^2+1$ from $(1, 2)$ to $(0, 1)$ and use it to write the integral $\int_C x^2+y^2\, ds$ as an integral with respect to your parameter.



1) $\vec{r}_1(t) = \langle t, t^2+1 \rangle \quad 0 \le t \le 1$ has wrong orientation

so use

$\vec{r}(t) = \vec{r}_1(-t) = \langle -t, (-t)^2+1 \rangle, \quad -1 \le t \le 0$

2) $\|\vec{r}\,'(t)\| = \sqrt{(-1)^2 + (2t)^2}$

3) $\int_C x^2+y^2\, ds = \int_{-1}^{0} \left[(-t)^2 + (t^2+1)^2 \right] \sqrt{1+4t^2}\, dt$

Daily Announcements & Reminders:

- HW 16.1 due tonight
- Quiz 8 in studio tomorrow, L.O. I6 & V1
- Do warmup poll on Ed ⟶
- Go vote

Goals for Today: Section 16.2

- Define and explore vector fields
- Define tangential and normal line integrals for vector fields
- Apply vector line integrals to problems involving work, flow, and flux
- Compute vector line integrals using parameterizations

Vector Fields:

Definition 111. A vector field is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ which associates a vector to every point in its domain.

w/ $\mathbb{R}^2 \to \mathbb{R}$

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$

$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

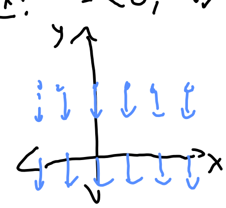
Examples:

- Velocity field of a flowing fluid
- Force field (gravity, electromagnetic)
- Slope field for a D.E
- Given any $f : \mathbb{R}^n \to \mathbb{R}$: $\nabla f = \langle f_x, f_y, f_z \rangle$
- Tangent vectors to a curve
  Normal vectors to a surface

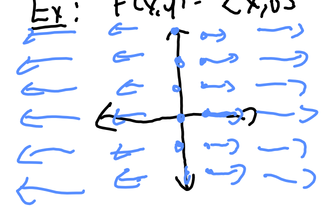
Graphically: For each point (a, b, c) in the domain of $\mathbf{F}$, draw the vector $\mathbf{F}(a, b, c)$ with its base at (a, b, c).

Tools: CalcPlot3d
Field Play

Ex: $\vec{F}(x,y) = \langle 0, -1 \rangle$

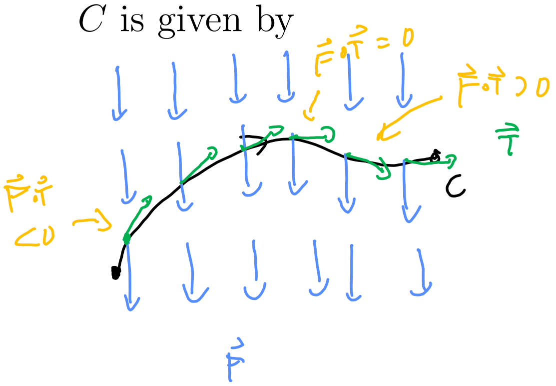


Ex: $\vec{F}(x,y) = \langle x, 0 \rangle$



Idea: In many physical processes, we care about the total sum of the strength of that part of a field that lies either in the direction of a curve or perpendicular to that curve.

1. The work done by a field $\mathbf{F}$ on an object moving along a curve C is given by

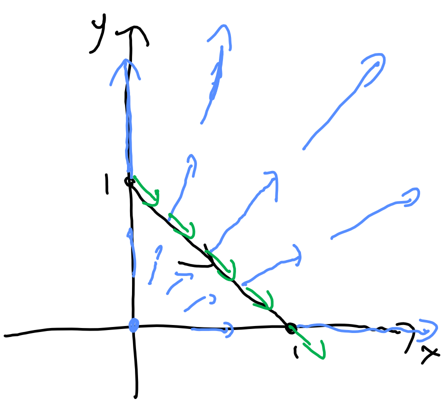


component of $\vec{F}$ along C : $\vec{F}\cdot\vec{T}$

work done: $\int_C \underbrace{\vec{F}\cdot\vec{T}} \, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}} \cancel{\|\vec{r}\,'(t)\|} \, dt$

$= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\, dt$

Example 112. Work Done by a Field. Suppose we have a force field $\mathbf{F}(x, y) = \langle x, y\rangle$ N. Find the work done by $\mathbf{F}$ on a moving object from $(0, 1)$ to $(1, 0)$ in a straight line, where x, y are measured in meters.



Guess negative or 0

1) Parameterize C: $\vec{r}(t) = \langle 1, -1\rangle t + \langle 0, 1\rangle \quad 0 \le t \le 1$

2) Find $\vec{r}\,'(t)$: $\vec{r}\,'(t) = \langle 1, -1\rangle$

3) Substitute:

work done $= \int_C \vec{F}\cdot\vec{T}\, ds = \int_0^1 \langle t, 1-t\rangle \cdot \langle 1, -1\rangle \, dt$

$= \int_0^1 t + (t-1)\, dt$ ← not a vector

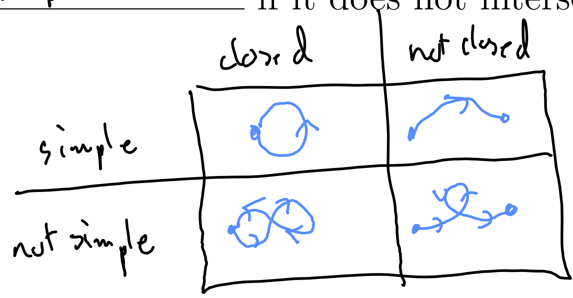
$= \int_0^1 2t - 1 \, dt$

$= t^2 - t \Big|_0^1 = 0$

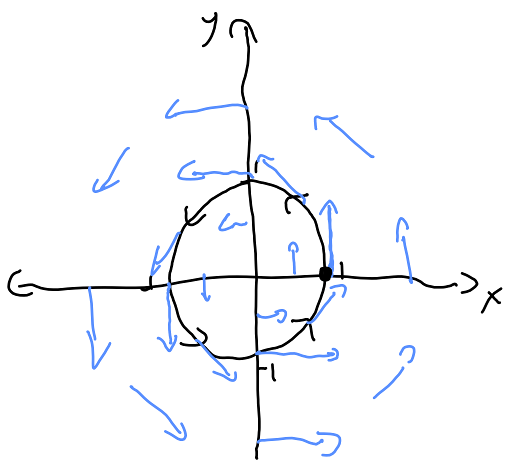
2. ~~1~~. The ___flow___ along a curve C of a velocity field **F** for a fluid in motion is given by

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

When C is ___closed___, this is called ___circulation___. C is called ___simple___ if it does not intersect itself.



Example 113. Flow of a Velocity Field. Find the circulation of the velocity field $\mathbf{F}(x, y) = \langle -y, x\rangle$ cm/s around the unit circle, parameterized counterclockwise.



1) Parametrize C: $\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$ $\quad 0 \le t \le 2\pi$

2) Find $\vec{r}'(t)$: $\quad \vec{r}'(t) = \langle -\sin(t), \cos(t)\rangle$

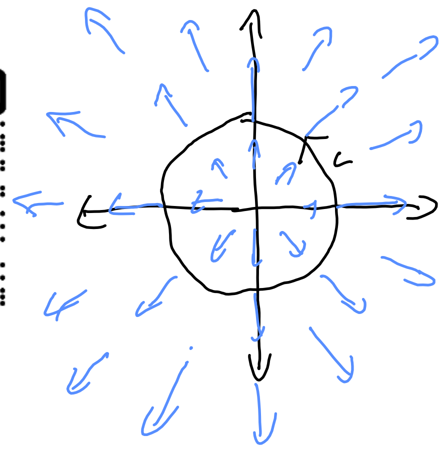
3) Substitute:
$$\vec{F}(\vec{r}(t)) = \langle -\sin(t), \cos(t)\rangle$$

$$\text{Circulation} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \langle -\sin(t), \cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle\,dt$$
$$= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt$$
$$= 2\pi \text{ cm}^2/\text{s}$$

Q: What is the clockwise circulation of $\vec{F}$ around unit circle?

0, 2π, $\boxed{-2\pi}$

Example 114. [Poll] What is the circulation of $\mathbf{F}(x, y) = \langle x, y \rangle$ around the unit circle, parameterized counterclockwise?



$\vec{F} \cdot \vec{T} = 0$ everywhere

1) $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle \quad 0 \le t \le 2\pi$

2) $\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$

3) circulation $= \int_C \vec{F} \cdot \vec{T}\, ds$

$$= \int_0^{2\pi} \langle \cos(t), \sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle\, dt$$

$$= \int_0^{2\pi} 0\, dt$$

$$= 0$$

Strategy for computing tangential component line integrals

e.g. work, flow, circulation integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \le t \le b$ for the curve C.

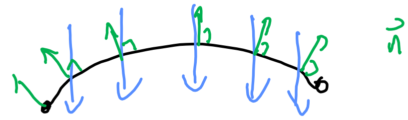
2. Compute $\mathbf{r}'(t)$.

3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$

4. Integrate

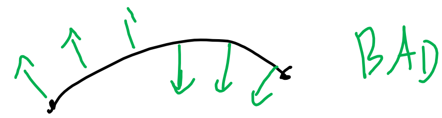
Idea: flux across a plane curve of a 2D-vector field measures the flow of the field across that curve (instead of along it).

We compute this with the integral



$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds.$$

unit normal



The sign of the flux integral tells us whether the net flow of the field across the curve is in the direction of $\vec{n}$ or in the opposite direction.

We can choose **n** to be either of

usual

$$\vec{n} = \frac{\langle y'(t), -x'(t)\rangle}{\|\vec{r}\,'(t)\|} \quad \text{or} \quad \frac{\langle -y'(t), x'(t)\rangle}{\|\vec{r}\,'(t)\|}$$

$$\int_C \vec{F}\cdot\vec{n}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \langle y'(t), -x'(t)\rangle\, dt$$

Example 115. Flux of a Velocity Field. Compute the flux of the velocity field $\mathbf{v} = \langle 3 + 2y - y^2/3, 0\rangle$ cm/s across the quarter of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1$ in the first quadrant.

Next time

Strategy for computing normal component line integrals

e.g. flux integrals

1. Find a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ for the curve C.

2. Compute $x'(t)$ and $y'(t)$ and determine which normal to work with.

3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{n}\; ds = \pm \int_a^b P(\mathbf{r}(t))y'(t) - Q(\mathbf{r}(t))x'(t)\; dt$ (sign based on choice of normal)

4. Integrate

Daily Announcements & Reminders:

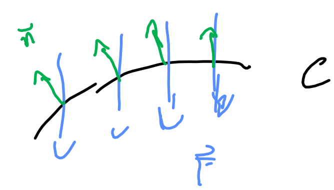
- HW 16.2 due tonight
- Exam 3 is on 11/26; remember that travel for break is <u>not</u> an excused absence
- Do warmup problem on Ed

Goals for Today: Section 16.3

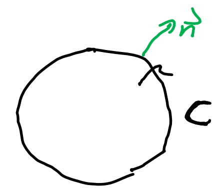
- Define conservative vector fields and recognize examples from physics
- Learn how to check if a field is conservative
- Compute potential functions
- Apply the Fundamental Theorem of Line Integrals to compute line integrals of conservative vector fields

Strategy for computing normal component line integrals

e.g. flux integrals

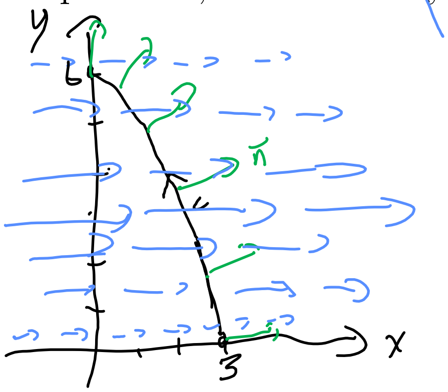


1. Find a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ for the curve C.
2. Compute $x'(t)$ and $y'(t)$ and determine which normal to work with.
3. Substitute: $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \pm \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t) \rangle\, dt$ (sign based on choice of normal)
   ↑ default
4. Integrate



Example 115. Flux of a Velocity Field. Compute the flux of the velocity field $\mathbf{v} = \langle 3 + 2y - y^2/3, 0\rangle$ cm/s across the quarter of the ellipse $\frac{x^2}{9} + \frac{y^2}{36} = 1$ in the first quadrant, oriented away from the origin.

horizontal



1) Parametorize C: $\vec{r}(t) = \langle 3\cos(t), 6\sin(t)\rangle$, $0 \le t \le \pi/2$

2) Get normal: $\vec{r}\,'(t) = \langle -3\sin(t), 6\cos(t)\rangle$

$\vec{n} = \langle y', -x'\rangle$

$= \langle 6\cos(t), 3\sin(t)\rangle$

3) Substitute:

$$\text{flux} = \int_0^{\pi/2} \langle 3 + 12\sin(t) - 12\sin^2(t), 0\rangle \cdot \langle 6\cos(t), 3\sin(t)\rangle\, dt$$

$$= \int_0^{\pi/2} 18\cos(t) + 72\sin(t)\cos(t) - 72\sin^2(t)\cos(t)\, dt$$

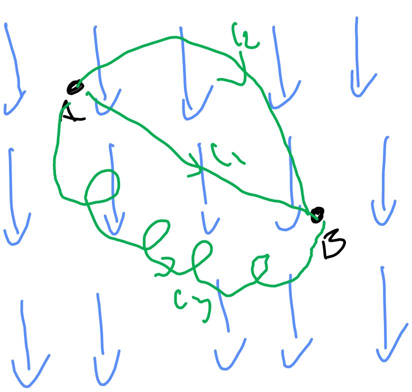
$u = \sin(t)$

$= 30$

Path-Indepedence and Conservative Vector Fields

Definition 116. A vector field $\mathbf{F}$ is **path independent** on an open region D if $\int_C \vec{F}\cdot\vec{T}\,ds$ is the same for all paths C in the region that have the same endpoints.

$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds = \int_{C_3} \vec{F}\cdot\vec{T}\,ds = \ldots,$$



Q: How do you tell if $\vec{F}$ is path-independent?

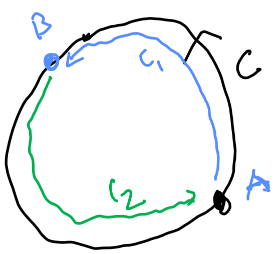
- gravitational/electrostatic/spring force fields are path ind.

When **F** is path independent, we can use the simplest path from point A to point B to compute a line integral, and will often denote the line integral with points as bounds, e.g.

$$\int_{(0,1,2)}^{(3,1,1)} \mathbf{F} \cdot \mathbf{T}\, ds \qquad \text{or} \qquad \int_{(a,b)}^{(c,d)} \mathbf{F} \cdot d\mathbf{r}.$$

Example 117. If C is any closed path and **F** is path independent on a region containing C, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$



Simple path: $\bullet\, c \to \vec{r}\,'(t) = \langle 0,0\rangle$

$\int_{C_1} \vec{F} \cdot d\vec{r} = - \int_{C_2} \vec{F} \cdot d\vec{r}$

Question: Given **F**, how do we tell if it is path independent on a particular region?

For example, is $\mathbf{F}(x, y) = \langle x, y\rangle$ a path independent vector field on its domain?

$\int_C \langle x,y\rangle \cdot d\vec{r} = 0$ Don't know yet.

C unit circle

Example 118 (Poll)**.** Last time, we saw that if C is the unit circle about the origin, oriented counterclockwise, then $\int_C \langle -y, x\rangle \cdot d\underline{\mathbf{r}} = 2\pi$. From this, we can conclude:

$\vec{F} = \langle -y, x\rangle$ is <u>not</u> path independent

b/c C is a closed curve but

$\int_C \vec{F} \cdot \vec{T}\, ds = 2\pi \neq 0.$

A different idea: Suppose $\mathbf{F}$ is a gradient vector field, i.e. $\mathbf{F} = \nabla f$ for some function of multiple variables f. f is called a potential for $\mathbf{F}$. In this case we also say that $\mathbf{F}$ is **conservative.**

$\vec{F} = \nabla f$: there is some f s.t. $\vec{F} = \langle f_x, f_y \rangle$

E.g. Is $\vec{F} = \langle x, y \rangle$ conservative?

Need f s.t. $f_x = x$ $\boxed{f_y = y}$

$f = \int f_x \, dx = \int x \, dx = \frac{1}{2}x^2 + C(y)$

$\frac{\partial}{\partial y}\left(\frac{1}{2}x^2 + C(y)\right) = C'(y) = y$

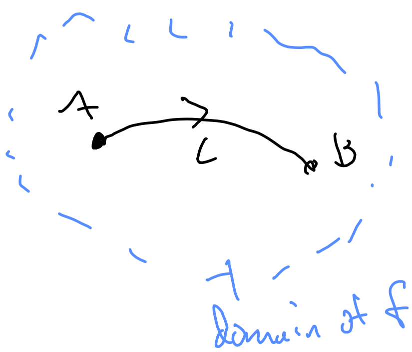
$C(y) = \int C'(y)\, dy = \int y \, dy = \frac{1}{2}y^2 + C$

$\vec{F} = \langle x, y \rangle$ is ∇f for $f = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C$

Theorem 119 (Fundamental Theorem of Line Integrals)**.** *If C is a smooth curve from the point A to the point B in the domain of a function f with continuous gradient on C, then*

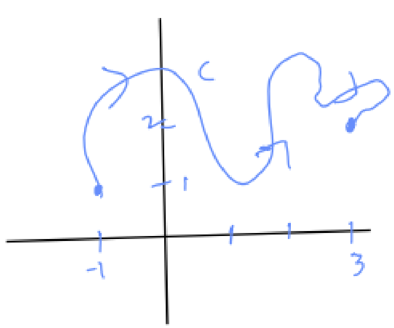
$$\int_C \nabla f \cdot \mathbf{T} \, ds = f(B) - f(A)$$

FTC: $\int_a^b f'(x)\,dx = f(b) - f(a)$



- conservative fields are path-independent

Example 120. Compute $\int_C \langle x, y \rangle \cdot d\mathbf{r}$ for the curve C shown below from $(-1, 1)$ to $(3, 2)$.



- On Tuesday, this was impossible.

$\int_C \langle x, y \rangle \cdot d\vec{r} = f(3,2) - f(-1,1)$

$\vec{F} = \langle x, y \rangle = \nabla f$; $f = \frac{1}{2}x^2 + \frac{1}{2}y^2$

$= \frac{1}{2}(9) + \frac{1}{2}(4) - \frac{1}{2}(1) - \frac{1}{2}(1)$

$= \frac{11}{2}$

Q: Is $\vec{F} = \langle ax, by \rangle$ cons? A: yes

$\vec{F} = \langle P(x), Q(y) \rangle$

Ex: $\vec{F} = \langle -y, x \rangle$

$f = \int -y \, dx = -xy + C_1(y)$

&

$f = \int x \, dy = xy + C_2(x)$

incompatible

$\vec{F} = \langle x, y \rangle$

$f = \int x \, dx = \frac{1}{2}x^2 + C_1(y)$

$f = \int y \, dy = \frac{1}{2}y^2 + C_2(x)$

compatible

$f = \frac{1}{2}x^2 + \frac{1}{2}y^2$

It follows that **every conservative field is path independent.**

In fact, by carefully constructing a potential function, we can show the converse is also true: every path independent $\vec{F}$ is cons.

$$f(x,y) = \int_{(a,b)}^{(x,y)} \vec{F}\cdot\vec{T}\,ds$$

This leads to a better way to test for path-independence and a way to apply the FToLI.

Curl Test for Conservative Fields: Let $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ be a vector field defined on a **simply-connected** region. If $\operatorname{curl}\mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 0,0,0\rangle$, then $\mathbf{F}$ is conservative.

$$\vec{F} = \langle P, Q, 0\rangle$$

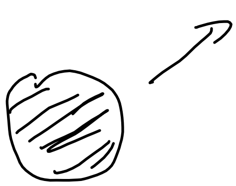
- If $\mathbf{F}$ is a 2-d vector field, $\operatorname{curl}\mathbf{F} =$ $\langle 0, 0, Q_x - P_y\rangle = \langle 0,0,0\rangle$ i.e. $\boxed{Q_x = P_y}$
- This is also called the **mixed-partials test**, because

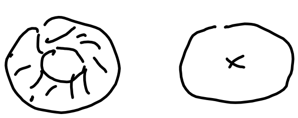
$$R_y = Q_z$$
$$P_z = R_x$$
$$Q_x = P_y$$

simply-connected

"no holes"

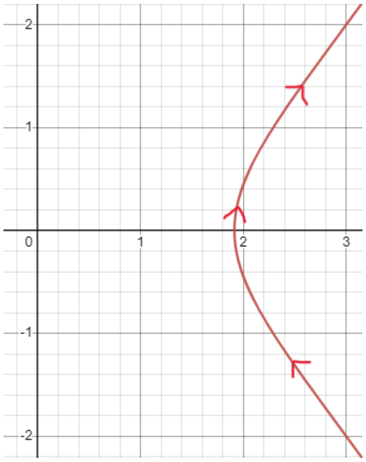
ex: $\mathbb{R}^2$, & $\mathbb{R}^3$, are simply connected





not simply connected

Example 121. Evaluate $\int_C (10x^4 - 2xy^3)\ dx - 3x^2y^2\ dy$ where C is the part of the curve $x^5 - 5x^2y^2 - 7x^2 = 0$ from $(3, -2)$ to $(3, 2)$.



Daily Announcements & Reminders:

- 16.3 HW due tonight
- Quiz 9 in studio tomorrow
  - L.O. V2 & V3
- Exam 3 in two weeks
- Do warmup on Ed

Goals for Today: Section 16.4

- Define the divergence and curl of a vector field
- Interpret divergence and curl geometrically
- Apply Green's Theorem to compute line integrals over the boundary of a simply-connected region
  - $\vec{F} = \nabla f$ on simply-connected region $\Leftrightarrow$ curl $\vec{F} = \vec{0}$

Ex: (part of 120)

If possible, find a potential function for

$$\vec{F} = (10x^4 - 2xy^3)\vec{i} - 3x^2y^2\vec{j}.$$

$$f = \int f_x\,dx = \int 10x^4 - 2xy^3\,dx = 2x^5 - x^2y^3 + C(y)$$

$$f = \int f_y\,dy = \int -3x^2y^2\,dy = -x^2y^3 + C_2(x)$$

$$\boxed{f = 2x^5 - x^2y^3 + C}$$

- $C_2(x) = 2x^5$
- $C_1(y) = 0$

Useful notation: $\boxed{\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle}$

So if $f(x,y,z)$ is a function of three variables, $\nabla f = \left\langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f) \right\rangle$

If $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a vector field:

- $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(P) + \frac{\partial}{\partial y}(Q) + \frac{\partial}{\partial z}(R)$

- $\nabla \times \mathbf{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

Instructions → NOT functions

$= \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle$

• divergence of $\vec{F}$ or div $\vec{F}$

• works in any $\mathbb{R}^n$

• curl of $\vec{F}$, curl$(\vec{F})$ only $\mathbb{R}^3$

Ex: $\vec{F} = \langle xy, 2y^2, x+z \rangle$

$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(x+z)$

$= y + 4y + 1 = 5y + 1$

$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2y^2 & x+z \end{vmatrix}$

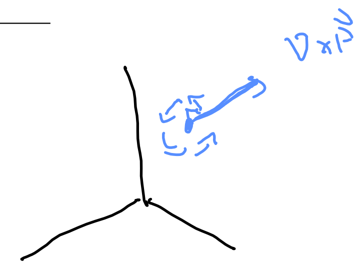
$= \left\langle \frac{\partial}{\partial y}(x+z) - \frac{\partial}{\partial z}(2y^2), -\left(\frac{\partial}{\partial x}(x+z) - \frac{\partial}{\partial z}(xy)\right), \frac{\partial}{\partial x}(2y^2) - \frac{\partial}{\partial y}(xy) \right\rangle$

$= \langle 0, -(1-0), 1-x \rangle = \langle 0, -1, -x \rangle$

How do we measure the change of a vector field?

1. **Curl** (in $\mathbb{R}^3$)

- Tells us circulation density
- Measures instantaneous / local circulation
- Is a vector
- Direction gives RHR compatible axis of rotation
- Magnitude gives rate of rotation



- curl $\mathbf{F}$ = $\nabla \times \vec{F}$

$(\nabla \times F) \cdot \vec{k}$

- If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$: we use $\nabla \times \mathbf{F} = \nabla \times \langle P, Q, 0\rangle = \langle 0, 0, \underbrace{Q_x - P_y}\rangle$

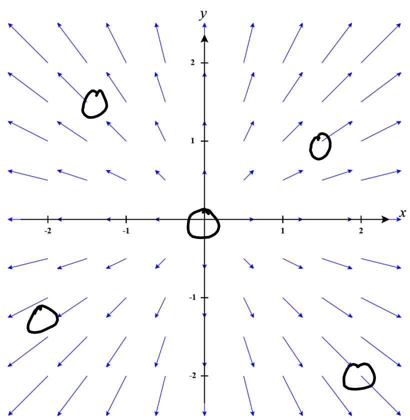
scalar curl

- If $\nabla \times \vec{F} = \vec{0}$, $\vec{F}$ is conservative; $\vec{F} = \nabla f$ & $\vec{F}$ is irrotational

2. **Divergence** (in any $\mathbb{R}^n$)

- Tells us flux density
- Measures local expansion / compression
- Is a scalar
- div $\mathbf{F}$ = $\nabla \cdot \vec{F} = P_x + Q_y + R_z$
- If $\nabla \cdot \vec{F} = 0$, $\vec{F}$ is incompressible and $\vec{F} = \nabla \times \vec{G}$ for some vector field $\vec{G}$

Example 122. Let $\mathbf{F}(x, y) = \langle x, y \rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.

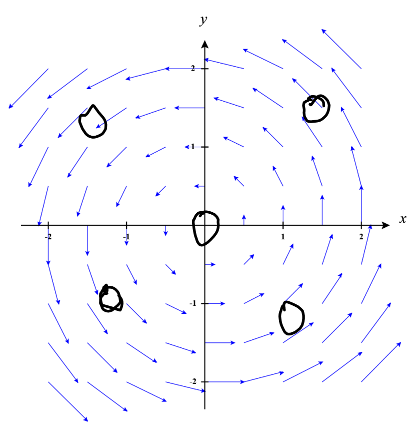


- div $\vec{F} > 0$ at all points
- curl $\vec{F} \cdot \vec{k} = 0$ at all points

$\text{div } \vec{F} = 1 + 1 = 2$

$\text{curl } \vec{F} \cdot \vec{k} = Q_x - P_y = 0 - 0 = 0$

Example 123 (Poll)**.** Let $\mathbf{F}(x, y) = \langle -y, x \rangle$. Based on the visualization of this vector field below, what can we say about the sign (+,-,0) of the divergence and scalar curl of this vector field? Verify by computing the divergence and scalar curl.



$\text{div } \vec{F} = 0$

$\text{curl } \vec{F} \cdot \vec{k} > 0$

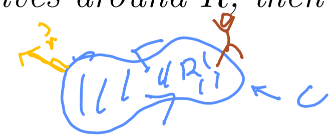
$\text{div } \vec{F} = P_x + Q_y = 0 + 0 = 0$

$\text{curl } \vec{F} \cdot \vec{k} = Q_x - P_y = 1 - (-1) = 2$

Question: How is this useful?

Answer: We can relate rates of change of vector field inside a region to the behavior of the vector field on the boundary of the region.

Theorem 124 (Green's Theorem)**.** *Suppose C is a piecewise smooth, simple, closed curve enclosing on its left a region R in the plane with outward oriented unit normal* $\mathbf{n}$. *If $\mathbf{F} = \langle P, Q \rangle$ has continuous partial derivatives around R, then*



a) Circulation form:

$$\int_C \mathbf{F} \cdot \mathbf{T}\ ds = \int_C P\ dx + Q\ dy = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\ dA = \iint_R Q_x - P_y\ dA$$

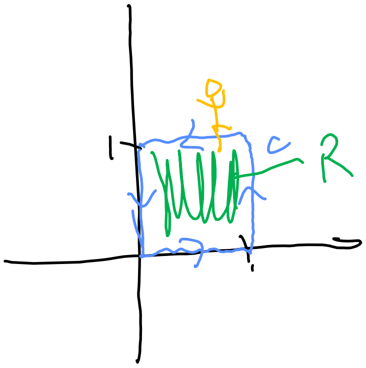
b) Flux form:

$$\int_C \mathbf{F} \cdot \mathbf{n}\ ds = \int_C P\ dy - Q\ dx = \iint_R (\nabla \cdot \mathbf{F})\ dA = \iint_R P_x + Q_y\ dA$$

a) Total circulation of $\vec{F}$ along boundary of a region is the double integral of the local circulation inside the region

b) Total flux of $\vec{F}$ out of a region is the double integral of the local flux on the inside of the region

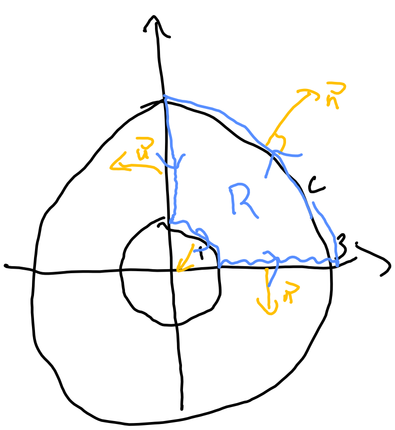
Example 125. Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\ ds$ for the vector field $\mathbf{F} = \langle -y^2, xy \rangle$ where C is the boundary of the square bounded by $x = 0, x = 1, y = 0,$ and $y = 1$ oriented counterclockwise.



- Direct calculation needs 4 parameterizations
- FToLI: $Q_x - P_y = y - (-2y) = 3y$
- Green's Thm: curl $\vec{F} = \langle 0, 0, Q_x - P_y \rangle$

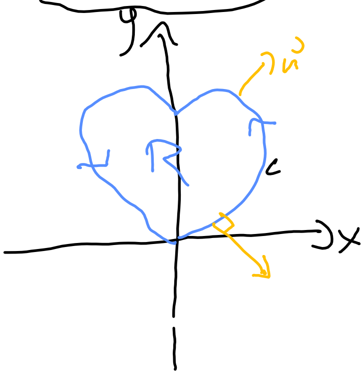
$$\int_C \vec{F} \cdot \vec{T}\, ds = \iint_R \text{curl}\,\vec{F} \cdot \vec{k}\, dA$$
$$= \int_0^1 \int_0^1 3y\, dy\, dx$$
$$= \frac{3}{2}$$

Example 126. Compute the flux out of the region R which is the portion of the annulus between the circles of radius 1 and 3 in the first ~~octant~~ quadrant for the vector field $\mathbf{F} = \langle \frac{1}{3}x^3, \frac{1}{3}y^3 \rangle$.



$$\int_C \vec{F} \cdot \vec{n}\, ds \overset{\text{Green's Thm}}{=\!=} \iint_R \nabla \cdot \vec{F}\, dA$$
$$= \iint_R P_x + Q_y\, dA$$
$$= \iint_R x^2 + y^2\, dA$$
$$= \int_0^{\pi/2} \int_1^3 r^2 \cdot r\, dr\, d\theta$$
$$= \frac{\pi}{2} \cdot \frac{1}{4} r^4 \Big|_1^3 = \frac{80\pi}{8} = \boxed{10\pi}$$

Example 127. Let R be the region bounded by the curve $\mathbf{r}(t) = \langle \sin(2t), \sin(t)\rangle$ for $0 \le t \le \pi$. Find the area of R, using Green's Theorem applied to the vector field $\mathbf{F} = \frac{1}{2}\langle x, y\rangle$.



$$\text{area}(R) = \iint_R 1\, dA = \iint_R \operatorname{div} \vec{F}\, dA$$

$$\operatorname{div} \vec{F} = \frac{1}{2} + \frac{1}{2} = 1$$

$$\overset{\text{Green's Thm}}{=} \int_C \vec{F} \cdot \vec{n}\, ds$$

1) $\vec{r}\,'(t) = \langle \overset{x'(t)}{2\cos(2t)}, \overset{y'(t)}{\cos(t)} \rangle$

2) Use: $\langle \cos(t), -2\cos(2t)\rangle$ for normal

3) Substitute:

$$= \int_0^\pi \frac{1}{2}\langle \sin(2t), \sin(t)\rangle \cdot \langle \cos(t), -2\cos(2t)\rangle\, dt$$

$$= \int_0^\pi \frac{1}{2}\sin(2t)\cos(t) - \sin(t)\cos(2t)\, dt$$

$$= \int_0^\pi \sin(t)\cos^2(t) - \sin(t)\left(\cos^2(t) - \sin^2(t)\right) dt$$

$$= \int_0^\pi \sin^3(t)\, dt$$

$$= \int_0^\pi \sin(t)\left(1 - \cos^2(t)\right) dt$$

$$= -\cos(t) + \frac{1}{3}\cos^3(t) \Big|_0^\pi$$

$$= -(-1) + \frac{1}{3}(-1)^3 - \left(-1 + \frac{1}{3}(1)^3\right)$$

$$= 1 - \frac{1}{3} + 1 - \frac{1}{3} = \boxed{\frac{4}{3}}$$

Note: This is the idea behind the operation of the measuring instrument known as a planimeter.

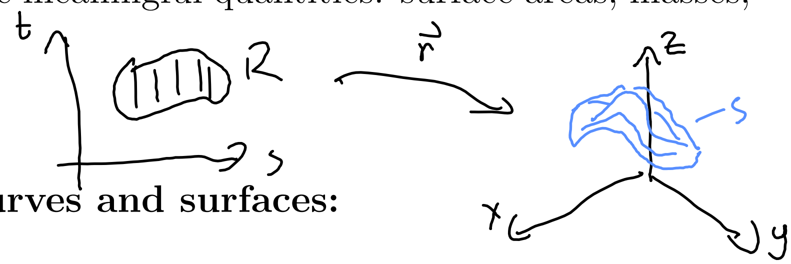
Daily Announcements & Reminders:

- HW 16.4 due tonight
- Office hours at 3 pm today instead of right after class
- Do warmup problem on Ed

Goals for Today:

Sections 16.5/16.6

- Describe surfaces in $\mathbb{R}^3$ with a parameterization
- Define and compute surface integrals
- Use surface integrals to compute meaningful quantities: surface areas, masses, flux, etc.



Different ways to think about curves and surfaces:

	Curves	Surfaces
Explicit:	$y = f(x)$ $y=\sqrt{4-x^2}$	$z = f(x, y)$ $z = 4-\frac{1}{4}x^2-\frac{1}{4}y^2$
Implicit:	$F(x, y) = 0$ $x^2+y^2=4$	$F(x, y, z) = 0$ $x^2+y^2+z^2=4$
Parametric Form:	$\mathbf{r}(t) = \langle x(t), y(t)\rangle$ $\vec{r}(t)=\langle 2\cos(t), 2\sin(t)\rangle$ $0\le t\le 2\pi$	$\vec{r}(s,t)=\langle x(s,t), y(s,t), z(s,t)\rangle$ $\vec{r}:\mathbb{R}^2\to\mathbb{R}^3$ $s\begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix}+t\begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix}=\vec{r}(s,t)$

- You have already done this; plane is: (through origin)

Example 128. Give parameteric representations for the surfaces below. S

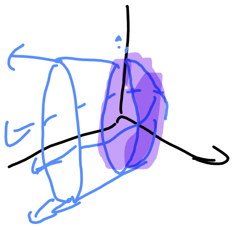
Goal: $\vec{r}(s,t)$ $(\mathbb{R}^2 \to \mathbb{R}^3)$ with domain R s.t. $\vec{r}$ describes S

a) $x = y^2 + \frac{1}{2}z^2 - 2$

$\vec{r}(s,t) = \langle s^2 + \frac{1}{2}t^2 - 2, s, t \rangle \quad s,t \in \mathbb{R}$

$\vec{r}_2(s,t) = \langle t^2 + \frac{1}{2}s^2 - 2, t, s \rangle$

$\vec{r}_3(s,t) = \langle s^2 - 2, s\cos(t), \sqrt{2}s \cdot \sin(t) \rangle \quad s \geq 0, \; 0 \leq t \leq 2\pi$

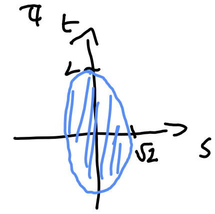


b) The portion of the surface $x = y^2 + \frac{1}{2}z^2 - 2$ which lies behind the yz-plane.

$\vec{r}(s,t) = \langle s^2 + \frac{1}{2}t^2 - 2, s, t \rangle \qquad s^2 + \frac{1}{2}t^2 - 2 < 0 \Rightarrow \frac{s^2}{2} + \frac{t^2}{4} < 1$

$\vec{r}_3(s,t) = \langle s^2 - 2, s\cos(t), \sqrt{2}s \cdot \sin(t) \rangle$

$0 \leq s \leq \sqrt{2} \qquad 0 \leq t \leq 2\pi$



c) $x^2 + y^2 + z^2 = 9$ Think in spherical coords: $\rho^2 = 9 \Rightarrow \rho = 3$

$\vec{r}(\varphi, \theta) = \langle 3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi \rangle$

$0 \leq \varphi \leq \pi$

$0 \leq \theta \leq 2\pi$

d) $x^2 + y^2 = 25 \xrightarrow{\text{cylindrical}} r = 5$

$\vec{r}(\theta, z) = \langle 5\cos\theta, 5\sin\theta, z \rangle \qquad 0 \leq \theta \leq 2\pi, \; z \in \mathbb{R}$

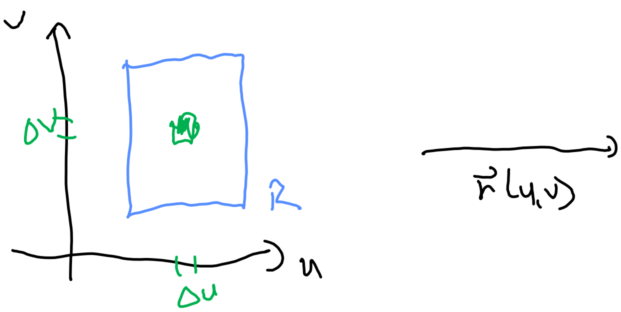
What can we do with this? $\vec{r}(u,v)$ is a parameterization of S

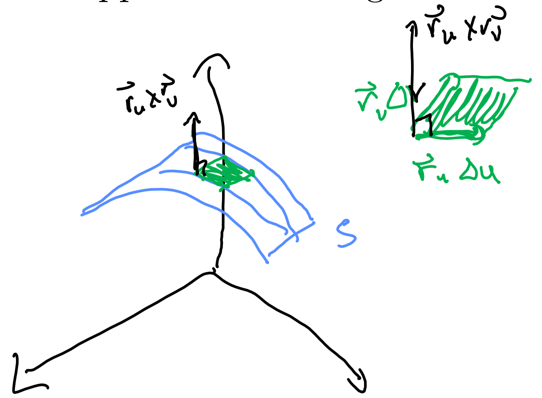
If our parameterization is **smooth** ($\mathbf{r}_u, \mathbf{r}_v$ not parallel in the domain), then:

- $\mathbf{r}_u \times \mathbf{r}_v$ is normal vector to the surface S

- A rectangle of size $\Delta u \times \Delta v$ in the uv-domain is mapped to a ~~rectangle~~ parallelogram of size $\|\vec{r}_u \times \vec{r}_v\| \, \Delta u \, \Delta v$ on the surface in $\mathbb{R}^3$.

$d\sigma = \|\vec{r}_u \times \vec{r}_v\| \, dA$





- Thus, Area(S) = $\displaystyle\iint_S d\sigma = \iint_R \|\vec{r}_u \times \vec{r}_v\| \, dA$

surface integral double integral

Example 129 (Poll)**.** Find the area of the portion of the cylinder $x^2 + y^2 = 25$ between $z = 0$ and $z = 1$. 10π

1) Parameterize:

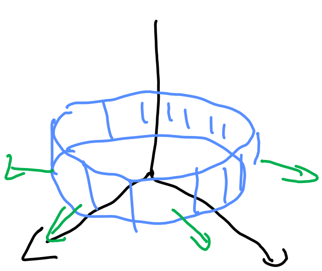
$$\vec{r}(u,v) = \langle 5\cos(u), 5\sin(u), v \rangle \qquad 0 \le u \le 2\pi, \quad 0 \le v \le 1$$

2) Compute $\|\vec{r}_u \times \vec{r}_v\|$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5\sin(u) & 5\cos(u) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 5\cos(u), 5\sin(u), 0 \rangle \quad \vec{n}$$

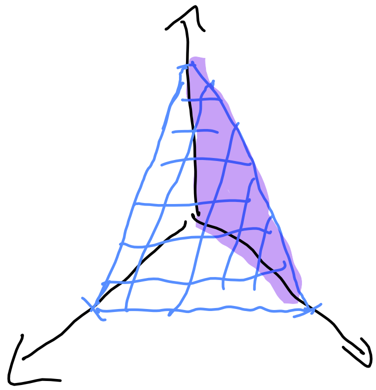
$\|\vec{r}_u \times \vec{r}_v\| = 5$

3) Compute: $SA = \displaystyle\int_0^{2\pi}\int_0^1 5 \, dv \, du = 10\pi$



Example 130. Suppose the density of a thin plate S in the shape of the portion of the plane $x + y + z = 1$ in the first octant is $\delta(x, y, z) = 6xy$. Find the mass of the plate.

kg/m^2



$$\text{mass} = \iint_S \delta(x,y,z)\, d\sigma$$

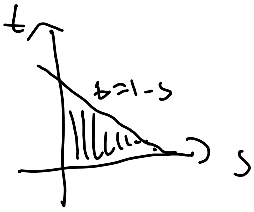
1) Parameterize S: $x = 1 - y - z$

$$\vec{r}(s,t) = \langle 1-s-t,\ s,\ t \rangle$$

$s \geq 0$

$t \geq 0$

$1 - s - t \geq 0$



$0 \leq t \leq 1-s$

$0 \leq s \leq 1$

2) Find $\|\vec{r}_s \times \vec{r}_t\|$:

$$\vec{r}_s \times \vec{r}_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

$$\|\vec{r}_s \times \vec{r}_t\| = \sqrt{3}$$

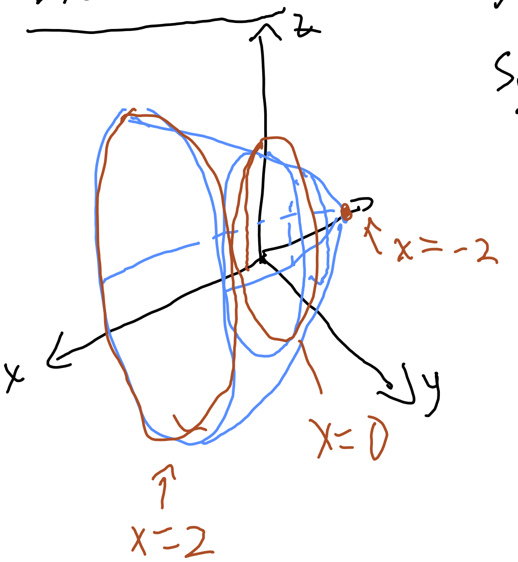
3) Substitute:

$$\iint_S \delta(x,y,z)\, d\sigma = \iint_R \delta(\vec{r}(s,t))\, \|\vec{r}_s \times \vec{r}_t\|\, dA$$

$$= \int_0^1 \int_0^{1-s} 6(1-s-t)s \cdot \sqrt{3}\, dt\, ds$$

$$= \sqrt{3}/4 \text{ kg}$$

Ex 128 a) $x = y^2 + \frac{1}{2}z^2 - 2$ $\qquad \vec{r}(s,t) = \langle s^2 + \frac{1}{2}t^2 - 2, s, t \rangle$

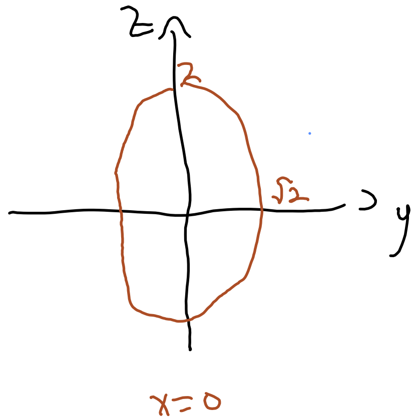


Systematic construction of

$$\vec{r}_3(s,t) = \langle s^2 - 2, s\cdot\cos(t), \sqrt{2}s\cdot\sin(t)\rangle$$

$$s \geq 0$$
$$0 \leq t \leq 2\pi$$

In $x = k$: $(k \geq -2)$



$$k = y^2 + \frac{1}{2}z^2 - 2$$
$$k + 2 = y^2 + \frac{1}{2}z^2$$
$$1 = \frac{y^2}{k+2} + \frac{z^2}{2(k+2)}$$

minor radius $\sqrt{k+2}$ $\qquad$ major radius $\sqrt{2(k+2)}$

$$\vec{r}_4(k,\theta) = \langle k, \sqrt{k+2}\cos\theta, \sqrt{2}\sqrt{k+2}\sin\theta\rangle$$
$$k \geq -2, \quad 0 \leq \theta \leq 2\pi$$

$s = \sqrt{k+2}$ $\qquad s \geq \sqrt{-2+2} = 0$ $\qquad \theta = t$

$s^2 = k+2 \rightarrow k = s^2 - 2$

$$\vec{r}_3(s,t) = \langle s^2 - 2, s\cdot\cos(t), \sqrt{2}s\cdot\sin(t)\rangle$$
$$s \geq 0$$
$$0 \leq t \leq 2\pi$$

Daily Announcements & Reminders:

- HW 16.5 due tonight, 16.6 on R, 16.7 & 8 next T
- Quiz 10 tomorrow on 16.4/16.5
  - L.O. V4, V5
- Exam 3 next Tuesday
  - see Canvas announcement
- Do warmup on Ed Discussion

Goals for Today: Section 16.6/16.7

- Compute flux surface integrals
- Interpret the physical significance of flux surface integrals
- Introduce and apply Stokes' Theorem for surface integrals

Goal: If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, find the total flux of $\mathbf{F}$ through a surface S.

Note: If the flux is positive, that means the net movement of the field through S is in the direction of the normal vectors to S

If $\mathbf{r}(u, v)$ is a smooth parameterization of S with domain R, we have

unit normal

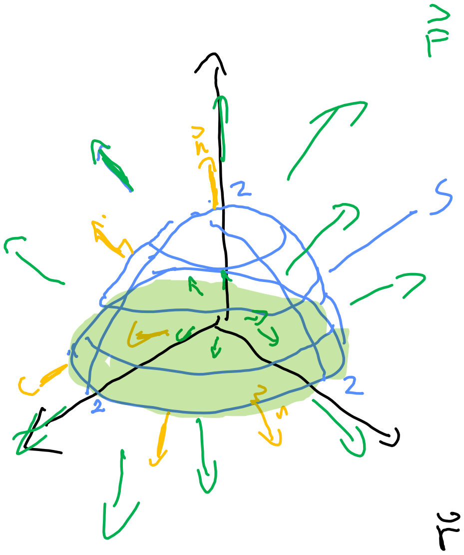
vector, not magnitude

$$\text{flux of } \mathbf{F} \text{ through } S = \iint_S \underbrace{(\mathbf{F}\cdot\mathbf{n})}_{\text{normal component of } \vec{F} \text{ to } S}\, d\sigma = \underbrace{\iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dA}.$$

$\vec{r}_u \times \vec{r}_v$ is normal to S

so $\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$

Example 131. Find the flux of $\mathbf{F} = \langle x, y, z\rangle$ through the upper hemisphere of $x^2+y^2+z^2=4$, oriented away from the origin.

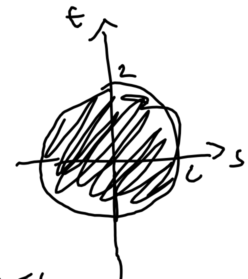


1) Parameterize S

a) $z = \sqrt{4-x^2-y^2}$

$\vec{r}(s,t) = \langle s, t, \sqrt{4-s^2-t^2}\rangle$

$s^2+t^2 \le 4$



b) $\rho = 2$; $2\cos\varphi > 0 \iff 0 \le \varphi \le \pi/2$

$\vec{r}(\varphi,\theta) = \langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi\rangle$

$0 \le \varphi \le \pi/2, \quad 0 \le \theta \le 2\pi$

2) Using b) Compute $\vec{r}_\varphi \times \vec{r}_\theta$:

$\vec{r}_\varphi = \langle 2\cos\varphi\cos\theta, 2\cos\varphi\sin\theta, -2\sin\varphi\rangle$

$\vec{r}_\theta = \langle -2\sin\varphi\sin\theta, 2\sin\varphi\cos\theta, 0\rangle$

$\vec{r}_\varphi \times \vec{r}_\theta = \langle 4\sin^2\varphi\cos\theta, -(-4\sin^2\varphi\sin\theta), 4\sin\varphi\cos\varphi\rangle$

3) Substitute: $\vec{F}(\vec{r}(\varphi,\theta)) = \langle 2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi\rangle$

$\vec{F}(\vec{r}(\varphi,\theta))\cdot(\vec{r}_\varphi\times\vec{r}_\theta) = 8\sin^3\varphi\cos^2\theta + 8\sin^3\varphi\sin^2\theta + 8\sin\varphi\cos^2\varphi$

$= 8\sin^3\varphi + 8\sin\varphi\cos^2\varphi$

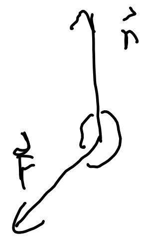
$= 8\sin\varphi\,(\underbrace{\sin^2\varphi+\cos^2\varphi}_{1})$

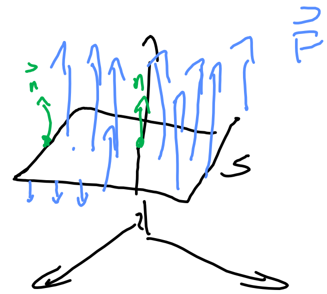
$\text{flux} = \iint_S \vec{F}\cdot\hat{n}\,d\sigma = \int_0^{2\pi}\int_0^{\pi/2} 8\sin\varphi\,d\varphi\,d\theta$

$= 2\pi\cdot(-8\cos\varphi)\Big|_0^{\pi/2} = \boxed{16\pi}$

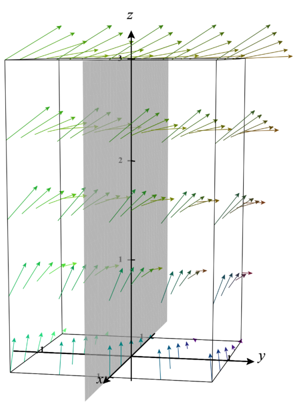
Example 132 (Poll)**.** Suppose S is a smooth surface in $\mathbb{R}^3$ and $\mathbf{F}$ is a vector field in $\mathbb{R}^3$. **True or False:** If $\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma > 0$, then the angle between $\mathbf{F}$ and $\mathbf{n}$ is acute at all points on S.

some but not all needed





Example 133 (Poll)**.** Based on the plot of the vector field $\mathbf{F}$ and the surface S below, oriented in the positive y-direction, is the flux integral $\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma$ positive, negative, or zero?



At all points $\vec F\cdot\vec n \geq 0$

Recall: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field, we defined its:

1. *divergence:* $\nabla\cdot\mathbf{F} = P_x + Q_y + R_z$

2. *curl:* $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$

$P, Q, R: \mathbb{R}^3 \to \mathbb{R}$

Example 134 (Poll). Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field in $\mathbb{R}^3$ with continuous partial derivatives. Compute the divergence of the curl of $\mathbf{F}$, i.e. $\nabla \cdot (\nabla \times \mathbf{F})$.

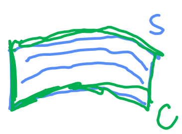
$$\nabla \cdot (\nabla \times \vec{F}) = \nabla \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$

$$= 0$$

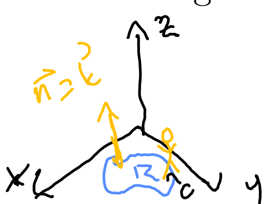
Theorem 135 (Stokes' Theorem). *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T} \, ds.$$



"flux of curl of $\vec{F}$ through S" = "circulation of $\vec{F}$ around the boundary of S"

- If S is a region R in the xy-plane, then we get:



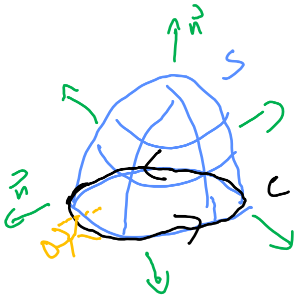
$$\iint_R (\nabla \times \vec{F}) \cdot \vec{k} \, dA = \int_C \vec{F} \cdot \vec{T} \, ds \qquad \text{Green's Thm!}$$

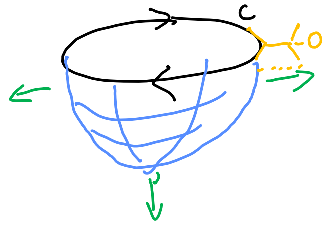
- An **oriented surface** is one where the normal vectors are consistent on all pts
    - Möbius strip is non-orientable

- S and C are oriented compatibly if:

  walking along C in its orientation with your head is in the direction of the normal $\vec{n}$ to S results in S being on your left

unit tangent, principal unit normal (Ch. 13)
$\vec{T} \times \vec{N}$ is parallel to $\vec{n}$ ⟺
$\vec{n}$ normal to S





← no boundary!

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = \int_C \vec{F} \cdot \vec{T}\, ds = 0$$

↑

S no boundary (closed)

(e.g. sphere)

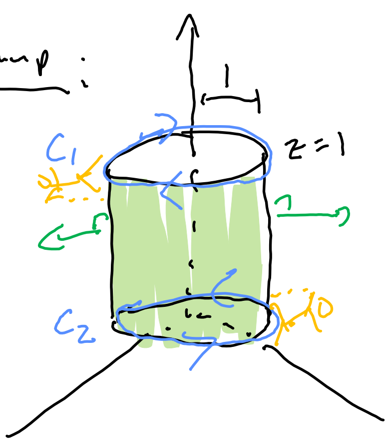
Daily Announcements & Reminders:

- HW 16.6 due tonight, 16.7, 16.8, Review Sets due Tuesday
  - Chapter X Practice due Tues Dec 3
- Exam 3 on T, see Canvas for details
- Final Exam on Monday 12/9, 8–10:50 am in lecture hall
  - 3 parts, corresponding to midterms
  - all optional, replace corresponding midterm if higher grade
- Do warmup poll

Goals for Today: Section 16.7/16.8

- Apply Stokes' Theorem to flux integral problems.
- Use Stokes' Theorem to simplify flux integrals
- Introduce and apply the Divergence Theorem to flux integral problems

Warmup:

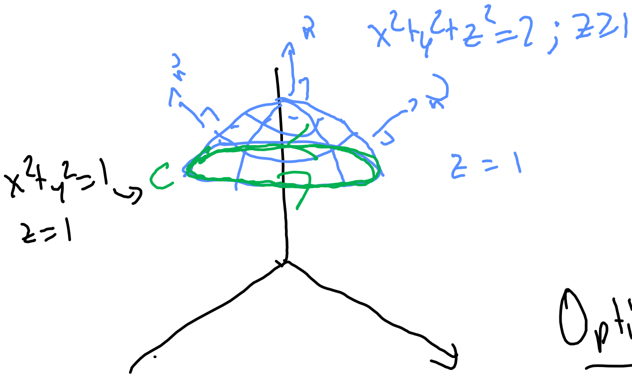


S and C are oriented compatibly if walking on C w/ head in direction of $\vec{n}$ to S results in left hand being over C

Theorem 136 (Stokes' Theorem)**.** *Let S be a smooth oriented surface and C be its compatibly oriented boundary. Let* **F** *be a vector field with continuous partial derivatives. Then*

$$\iint_S \underbrace{(\nabla \times \mathbf{F})}_{\text{this has divergence } 0} \cdot \mathbf{n}\, d\sigma = \int_C \mathbf{F} \cdot \mathbf{T}\, ds. = \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \int_{C_2} \vec{F} \cdot \vec{T}\, ds$$

Example 137 (DD)**.** Let $\mathbf{F} = \langle -y, x+(z-1)x^{x\sin(x)}, x^2+y^2\rangle$. Find $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma$ over the surface S which is the part of the sphere $x^2+y^2+z^2=2$ above $z=1$, oriented away from the origin.



Option 1) Parametrize S, Compute $\nabla\times\vec{F}$, compute $\vec{r}_s\times\vec{r}_t$, Substitute

HARD

Option 2: Use Stokes' Theorem

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \int_C \vec{F}\cdot\vec{T}\,ds$$

- Orient C: CCW
- Parametrize C: $\vec{r}(t) = \langle \cos(t), \sin(t), 1\rangle \quad 0\leq t\leq 2\pi$

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 0\rangle$$

$$\vec{F}(\vec{r}(t)) = \langle -\sin(t), \cos(t) + \cancel{(1-1)x^{x\sin(x)}}^{\,0}, 1\rangle$$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \int_C \vec{F}\cdot\vec{T}\,ds = \int_0^{2\pi} \sin^2(t)+\cos^2(t)\,dt$$

$$= \boxed{2\pi}$$

Question: What can we say if two different surfaces S_1 and S_2 have the same oriented boundary C?

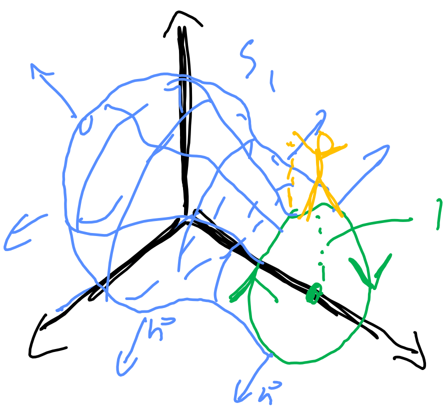
$$\iint_{S_1} (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \int_C \vec{F}\cdot\vec{T}\,ds = \iint_{S_2} (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma$$

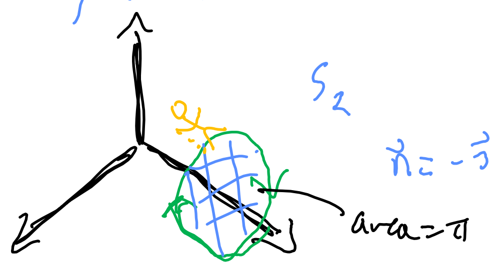
on S_2 $y=1$

Example 138. Suppose curl $\mathbf{F} = \langle y^{y^y}\sin(z^2), (y-1)e^{x^{x^x}} + 2, -ze^{x^{x^x}}\rangle$. Compute the net flux of the curl of $\mathbf{F}$ over the surface pictured below, which is oriented outward and whose boundary curve is a unit circle centered on the y-axis in the plane $y = 1$.

from theorem

$$\nabla\cdot\langle y^{y^y}\sin(z^2), (y-1)e^{x^{x^x}}+2, -ze^{x^{x^x}}\rangle$$
$$= 0 + e^{x^{x^x}} - e^{x^{x^x}} = 0$$





$$\iint_{S_1} \underbrace{(\nabla\times\vec{F})\cdot\vec{n}\,d\sigma}_{\text{hard on many levels}} = \int_C \vec{F}\cdot\vec{T}\,ds = \iint_{S_2} (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma$$

don't know $\vec{F}$

$$= \iint_{S_2} (\nabla\times\vec{F})\cdot\langle 0,-1,0\rangle\,d\sigma$$
$$= \iint_{S_2} 0 + (0+2)\cdot(-1) + 0\,d\sigma$$
$$= \iint_{S_2} -2\,d\sigma$$
$$= -2\boxed{\iint_{S_2} d\sigma} \leftarrow \text{area}$$
$$= \boxed{-2\pi}$$

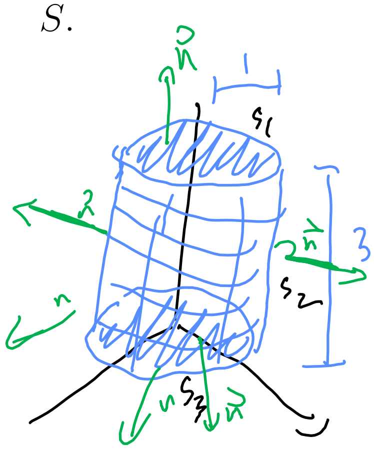
Theorem 139 (Divergence Theorem)**.** *Let S be a closed surface oriented outward, D be the volume inside S, and $\mathbf{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV.$$

"net flux of $\vec{F}$ out of S" = "sum of local flux of F inside surface S"

$\boxed{\text{S MUST BE CLOSED}}$

Example 140. Let $\mathbf{F} = \langle y^{1234}e^{\sin(yz)}, y - x^{z^x}, z^2 - z\rangle$ and S be the surface consisting of the portion of cylinder of radius 1 centered on the z-axis between $z = 0$ and $z = 3$, together with top and bottom disks, oriented outward. Find the flux of $\mathbf{F}$ through S.



$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_{S_1} \vec{F}\cdot\vec{n}\,d\sigma + \iint_{S_2} \vec{F}\cdot\vec{n}\,d\sigma + \iint_{S_3} \vec{F}\cdot\vec{n}\,d\sigma$$

$$= \iiint_D \nabla\cdot\vec{F}\,dV$$

$$= \iiint_D 0 + 1 + (2z-1)\,dV$$

$$= \iiint_D 2z\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^3 2zr\,dz\,dr\,d\theta$$

$$= 2\pi\cdot\frac{1}{2}\cdot 3^2 = \boxed{9\pi}$$