MATH 2551-G Final Exam

Fall 2025

EXAM KEY

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 170 minutes to complete as many problems as you wish to attempt.
- You may not use electronic devices of any kind during the exam. You may not use any reference materials other than your single page of hand-written notes you brought to the exam.
- The Learning Targets covered by this exam are listed below.
- Show your work. Answers without work shown will receive a **Not Yet**
- Good luck! Write yourself a message of encouragement on the front page!

Learning Targets

- **G1: Lines and Planes.** I can describe lines using the vector equation of a line. I can describe planes using the general equation of a plane. I can find the equations of planes using a point and a normal vector. I can find the intersections of lines and planes. I can describe the relationships of lines and planes to each other. I can solve problems with lines and planes.
- **G2: Calculus of Curves.** I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.
- **G3: Geometry of Curves.** I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.
- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.
- **G5: Parameterization.** I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.
- **D1: Computing Derivatives.** I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.
- **D2: Tangent Planes and Linear Approximations.** I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.

- **D3: Optimization.** I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.
- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **I2: Iterated Integrals.** I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.
- **I3: Change of Variables.** I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.
- **V1: Line Integrals.** I can set up and evaluate scalar and vector field line integrals in two and three dimensions.
- **V2: Conservative Vector Fields.** I can test for conservative vector fields and find potential functions. I can state and apply the Fundamental Theorem of Line Integrals.
- **V3: Generalizations of the FTC.** I can state and apply Green's Theorem, Stokes' Theorem and the Divergence Theorem to solve problems in two and three dimensions. I can choose which theorem is appropriate for different integrals. I can compute curl and divergence of vector fields.
- **V4: Surface Integrals.** I can set up and compute surface integrals for scalar and vector valued functions.
- **A1: Interpreting Derivatives.** I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.
- **A2: Integral Applications.** I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.

Tasks

1. **[G1: Lines and Planes]** For parts (a) and (b), determine whether the statement is true or false and write T or F in the box provided. For parts (c) and (d), bubble in the multiple choice option that corresponds to your answer.

   (a) **T/F:** Any two distinct lines in 3-space determine a unique plane containing them.

   F

   (b) **T/F:** The equation $Ax + By + Cz + D = 0$ represents a line in space.

   F

   (c) Which line below is skew to the line given by the vector equation

   $$\mathbf{r}(t) = \langle 1, 2, 3\rangle + \langle 4, 5, 6\rangle t?$$

   ○ $\mathbf{r}(t) = \langle 1, 2, 3\rangle + \langle 8, 10, 12\rangle t$
   ○ $\mathbf{r}(t) = \langle 0, 0, 0\rangle + \langle 1, 1, 1\rangle t$
   ✓ $\mathbf{r}(t) = \langle 0, 0, 0\rangle + \langle 1, -1, 1\rangle t$
   ○ $\mathbf{r}(t) = \langle 5, 7, 9\rangle + \langle 4, 5, 6\rangle t$

   (d) Which plane below is **not** a plane containing the point $P = (10, 21, -42)$ and parallel to the vector $\langle 1, 2, 3\rangle$.

   ○ $-x - y + z = -73$
   ○ $2x + 2y - 2z = 146$
   ○ $-2x + 7y - 4z = 295$
   ✓ $x + 2y + 3z = -80$

2. **[G2: Calculus of Curves]** Suppose that a friendly monster is swimming around Loch Ness. Thirty minutes after it left its cave, its velocity vector is $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ miles per hour and its position is $\mathbf{i} - 10\mathbf{j} - \frac{1}{20}\mathbf{k}$ miles from the surface of the center of the loch.

(a) Find an equation of the tangent line to the path of the monster at this time. Think carefully about time.

Solution. At the thirty minute mark, the time is $t = \frac{1}{2}$ hours. Two possible equations for the tangent line are

$$\mathbf{r}_1(t) = \langle 1, -10, \frac{1}{20} \rangle + \langle 3, -2, -1 \rangle t$$

or

$$\mathbf{r}_2(t) = \langle 1, -10, \frac{1}{20} \rangle + \langle 3, -2, -1 \rangle (t - \frac{1}{2}),$$

depending on whether we want t to represent time since the monster left its cave ($\mathbf{r}_2$) or time since the thirty minute mark ($\mathbf{r}_1$).

(b) One hour after it left its cave, the position of the monster was $\frac{5}{2}\mathbf{i} - 11\mathbf{j} - \frac{1}{7}\mathbf{k}$ miles. Use this fact and your answer to part (a) to show that the monster was not swimming in a straight line for this half hour. Be sure to clearly explain your answer.

Solution. If the monster swam in a straight line for this half hour, its position at this time would agree with the corresponding point in time on the tangent line from (a). Let's use $\mathbf{r}_2$ to compare: then we plug in $t = 1$ hour to see that the point on the tangent line at this time is

$$\mathbf{r}_2(1) = \langle 1, -10, \frac{1}{20} \rangle + \langle 3, -2, -1 \rangle (1 - \frac{1}{2}) = \left\langle \frac{5}{2}, -11, -\frac{9}{20} \right\rangle.$$

Since the actual position of the monster at this time is not the same as this, the monster could not have been swimming in a straight line for this half hour.

(c) If the monster maintains its swimming speed at the thirty minute mark of $\sqrt{14}$ miles per hour for the rest of its swim, is it possible for it to return to its cave? Be sure to clearly explain your answer.

Solution. Yes, it can. This constraint on its speed only restricts the magnitude of its velocity vector, not its direction. Therefore, it could turn around and swim directly back to its cave at this speed.

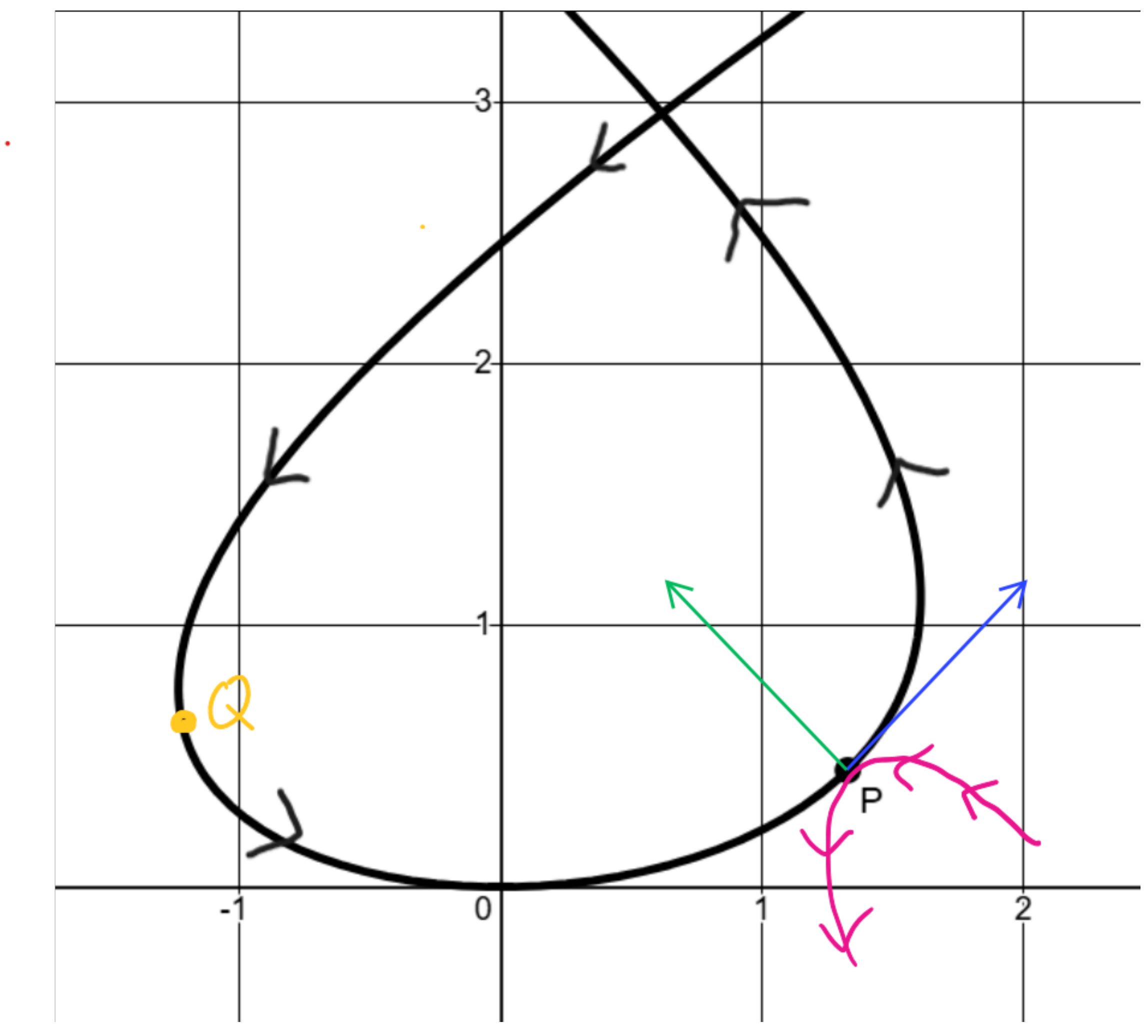
3. **[G3: Geometry of Curves]** Pictured below is a curve C. Sketch the following on this graph. Clearly indicate which vector is which and pay attention to scale.

   (a) The unit tangent vector to C at P.

   (b) The principal unit normal vector to C at P.

   (c) Another curve passing through P which has a **parallel but different** unit tangent vector as C at P, has **higher** curvature at P and has a **different** principal unit normal vector at P.

   (d) A point on the graph where C has **greater** curvature than it does at P.



Solution. The unit tangent vector is drawn above in blue in the $\langle 1, 1\rangle$ direction. The principal unit normal vector is drawn above in green in the $\langle -1, 1\rangle$ direction. A possible other curve is drawn in magenta above, which has a sharper turn at P and therefore higher curvature, curves in the opposite direction from C at P, and is oriented opposite to C at P. A point Q where C has greater curvature than at P is indicated in yellow in the lower left corner of the loop.

4. **[G4: Surfaces]** Identify the contours, domain, and range of the function

$$f(x, y) = \sqrt{9 - x^2 - y^2}.$$

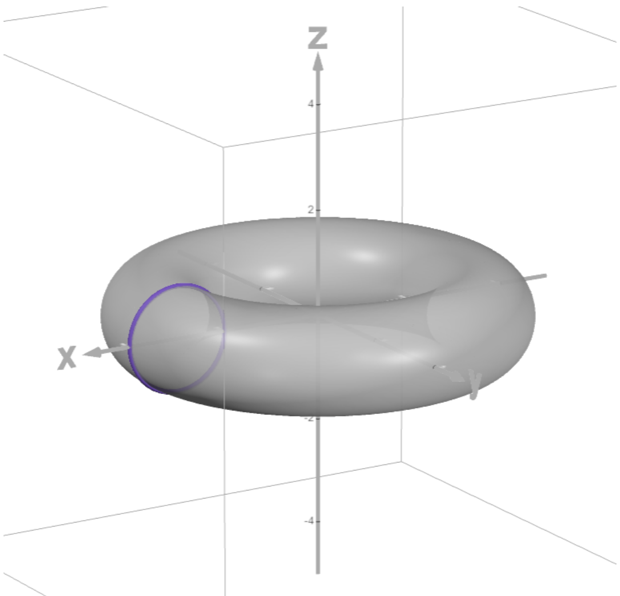
Solution. The contours of this function are circles centered at the origin with radius $\sqrt{9-k^2}$ for $0 \leq k \leq 3$ since for any k we have

$$k = \sqrt{9 - x^2 - y^2} \Longrightarrow x^2 + y^2 = 9 - k^2$$

and this has solutions when $0 \leq k \leq 3$. Correspondingly, the domain is the disk $x^2 + y^2 \leq 9$ and the range is $0 \leq z \leq 3$.

5. **[G5: Parameterization]** Find a parameterization of the torus S generated by revolving the circle of radius 1 in the xz-plane centered at the point $(3, 0, 0)$ about the z-axis. Be sure to clearly indicate the domain of your parameterization. A picture of this torus is provided below for reference.

   Hint: It might be helpful to use one parameter to parameterize a copy of this circle and a second parameter to describe its revolution about the z-axis, remembering that this revolution applied to each point on the circle generates a new circle with fixed radius r to the z-axis.



Solution. A parameterization of the circle in the xz-plane centered at $(3, 0, 0)$ with radius 1 is given by

$$\mathbf{r}_1(t) = \langle 3 + \cos(t), 0, \sin(t)\rangle, \quad 0 \leq t \leq 2\pi.$$

Revolving this circle about the z-axis generates circles of radius $r = 3 + \cos(t)$ in planes parallel to the xy-plane at height $z = \sin(t)$. A parameterization of these circles is given by

$$\mathbf{r}_2(u) = \langle r\cos(u), r\sin(u), z\rangle \quad 0 \leq u \leq 2\pi.$$

Therefore, a parameterization of the torus is given by

$$\mathbf{r}(t, u) = \langle (3 + \cos(t))\cos(u), (3 + \cos(t))\sin(u), \sin(t)\rangle, \quad 0 \leq t \leq 2\pi,\ 0 \leq u \leq 2\pi.$$

6. **[D1: Computing Derivatives]** In this problem, you will work with the function $g : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$g(x,y,z) = \begin{bmatrix} xy + z^2 \\ xyz + 2x + y \end{bmatrix}$$

(a) Find $Dg(1,2,3)$.

Solution. We have

$$Dg(x,y,z) = \begin{bmatrix} y & x & 2z \\ yz+2 & xz+1 & xy \end{bmatrix}.$$

Therefore

$$Dg(1,2,3) = \begin{bmatrix} 2 & 1 & 6 \\ 8 & 4 & 2 \end{bmatrix}.$$

(b) Suppose you follow up this transformation g by applying a further transformation $h : \mathbb{R}^2 \to \mathbb{R}^2$ which at the point $(11,10)$ transforms vectors by multiplying them by the matrix

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

Find the total derivative of the composite function $f = h \circ g$ at the point $(1,2,3)$.

Solution. We have

$$Dh(11,10) = Dh(g(1,2,3)) = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore,

$$Df(1,2,3) = Dh(g(1,2,3))Dg(1,2,3) = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 \\ 8 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -2 & 10 \\ 2 & 1 & 6 \end{bmatrix}.$$

(c) Compute $Df_{\mathbf{u}}(1,2,3)$ where $\mathbf{u} = \frac{1}{3}\langle 2, -2, 1\rangle$.

Solution.

$$\begin{aligned} Df_{\mathbf{u}}(1,2,3) &= Df(1,2,3)\mathbf{u} \\ &= \frac{1}{3}\begin{bmatrix} -4 & -2 & 10 \\ 2 & 1 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \\ &= \langle 2, \frac{8}{3} \rangle. \end{aligned}$$

7. **[D2: Tangent Planes and Linear Approximations]** Suppose that $f(x, y, z)$ is a differentiable function and the plane $2x - y + 3z = -3$ is tangent to the level surface of f containing the point $(2, 1, -2)$. Use this information to answer the following questions.

   (a) Which one of these vectors could be $\nabla f(2, 1, -2)$?

   ◯ (A) $\langle 2, 1, -2 \rangle$

   ✓ (B) $\langle 4, -2, 6 \rangle$

   ◯ (C) $\langle 3, 3, -1 \rangle$

   ◯ (D) $\langle -2, -1, 3 \rangle$

   (b) Suppose we know that the linear approximation of f at $(2, 1, -2)$ can be used to estimate $f(2.1, 1, -1.9) \approx 6$. Use your answer to (a) to determine $f(2, 1, -2)$ if ∇f is the vector you chose.

Solution. There are four possible correct answers depending on the vector chosen for ∇f in (a). The general solution is as follows. From the linearization formula, we have:

$$\begin{aligned} f(2.1, 1, -1.9) &\approx 6 \\ &= L(2.1, 1, -1.9) \\ &= f(2, 1, -2) + \nabla f(2, 1, -2) \cdot \langle 2.1 - 2, 1 - 1, -1.9 + 2 \rangle. \end{aligned}$$

Now we solve for $f(2, 1, -2)$ to get:

$$f(2, 1, -2) = 6 - \nabla f(2, 1, -2) \cdot \langle 0.1, 0, 0.1 \rangle.$$

Using each possible $\nabla f(2, 1, -2)$ from (a):

- For $\langle 2, -1, 2 \rangle$, we have $f(2, 1, -2) = 6$.
- For $\langle 4, -2, 6 \rangle$, we have $f(2, 1, -2) = 5$.
- For $\langle 3, 3, -1 \rangle$, we have $f(2, 1, -2) = 5.8$.
- For $\langle -2, -1, 3 \rangle$, we have $f(2, 1, -2) = 5.9$.

8. **[D3: Optimization]** Find and classify all of the critical points of the function

$$f(x,y) = 2x^2 - 2x^2y + y^2 - 7.$$

Solution. First, find the critical points by locating zeros of the derivative:

$$\begin{bmatrix} 4x - 4xy & -2x^2 + 2y \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

From the first equation we have $x(1-y) = 0$, so either $x = 0$ or $y = 1$. If $x = 0$, then from the second equation we have $2y = 0$ so $y = 0$. This gives the critical point $(0,0)$. If $y = 1$, then from the second equation we have $-2x^2 + 2 = 0$ so $x = \pm 1$. This gives the critical points $(1,1)$ and $(-1,1)$.
Now we classify these three critical points using the second derivative test. We have

$$Hf(x,y) = \begin{bmatrix} 4 - 4y & -4x \\ -4x & 2 \end{bmatrix}.$$

At $(0,0)$:

$$\det(Hf(0,0)) = 8 > 0, \quad f_{xx}(0,0) = 4 > 0$$

so f has a local minimum at $(0,0)$.
At $(1,1)$:

$$\det(Hf(1,1)) = -4 < 0$$

so f has a saddle point at $(1,1)$.
At $(-1,1)$:

$$\det(Hf(-1,1)) = -4 < 0$$

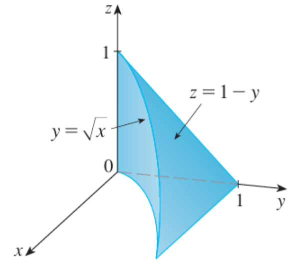
so f has a saddle point at $(-1,1)$.

9. **[I1: Double & Triple Integrals]**

In this problem you will work with the region D with $0 \leq z \leq 1 - y$, $\sqrt{x} \leq y \leq 1$, and $0 \leq x \leq 1$ to write a triple integral

$$\iiint_D f(x, y, z)\, dV.$$

A sketch of this region is shown to the right.



(a) Write an iterated integral expression for this region in the order $dy\, dz\, dx$. You may need to split the integral.

Solution.

$$\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dz\, dx$$

(b) Write an iterated integral expression for this region in the order $dx\, dz\, dy$. You may need to split the integral.

Solution.

$$\int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z)\, dx\, dz\, dy$$

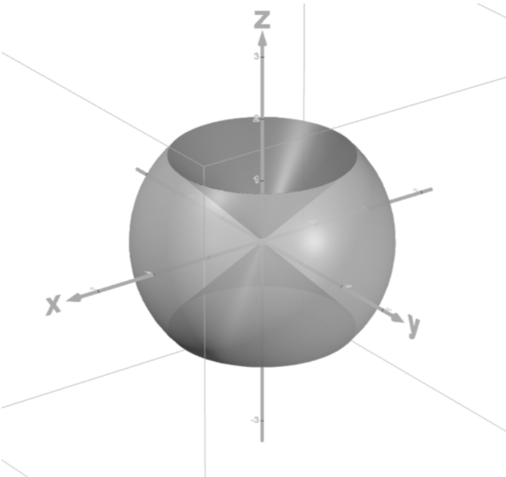
10. **[I2: Iterated Integrals]** Your friend has successfully written an iterated integral in spherical coordinates for the function $f(x, y, z) = x^2$ over the unit sphere $x^2+y^2+z^2 = 1$ below but has forgotten how to integrate trig functions. Help them evaluate this integral.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} (\rho^2 \sin^2(\varphi)\cos^2(\theta))\rho^2 \sin(\varphi)\ d\rho\ d\varphi\ d\theta.$$

Solution. Let I be the given integral. Then

$$\begin{aligned}
I &= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^4 \sin^3(\varphi)\cos^2(\theta)\ d\rho\ d\varphi\ d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \frac{\rho^5}{5}\bigg|_0^1 \sin^3(\varphi)\cos^2(\theta)\ d\varphi\ d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi} \sin^3(\varphi)\cos^2(\theta)\ d\varphi\ d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi} \sin(\varphi)(1-\cos^2(\varphi))\cos^2(\theta)\ d\varphi\ d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\int_1^{-1} -(1-u^2)\cos^2(\theta)\ du\ d\theta \qquad (u=\cos(\varphi),\ du = -\sin(\varphi)\ d\varphi) \\
&= \frac{1}{5}\int_0^{2\pi} (u-\frac{1}{3}u^3)\bigg|_{-1}^1 \cos^2(\theta)\ d\theta \\
&= \frac{4}{15}\int_0^{2\pi} \cos^2(\theta)\ d\theta \\
&= \frac{2}{15}\int_0^{2\pi} 1+\cos(2\theta)\ d\theta \\
&= \frac{2}{15}\ (\theta+\frac{1}{2}\sin(2\theta))\bigg|_0^{2\pi} \\
&= \frac{4\pi}{15}.
\end{aligned}$$

11. **[I3: Change of Variables]** Another friend has forgotten how to convert to spherical coordinates. Help them write an integral for the function $f(x, y, z) = xyz$ over the region E which lies inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cone $z^2 = x^2 + y^2$, shown below. Your answer should be an iterated integral in spherical coordinates.



Solution. Since E lies inside the sphere $x^2 + y^2 + z^2 = 4$ or $\rho = 2$, we have $0 \le \rho \le 2$. Since E lies outside the cone $z^2 = x^2 + y^2$ we have

$$z^2 \le x^2 + y^2 \implies \rho^2 \cos^2(\varphi) \le \rho^2 \sin^2(\varphi) \implies \tan(\varphi) \ge 1 \implies \pi/4 \le \varphi \le 3\pi/4.$$

Finally, since the shape is radially symmetric about the z-axis, we have $0 \le \theta \le 2\pi$. Now we convert the function to spherical coordinates:

$$xyz = (\rho \sin(\varphi) \cos(\theta))(\rho \sin(\varphi) \sin(\theta))(\rho \cos(\varphi)) = \rho^3 \sin^2(\varphi) \cos(\varphi) \cos(\theta) \sin(\theta).$$

Therefore, the integral is given by

$$\int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_0^2 \rho^5 \sin^3(\varphi) \cos(\varphi) \cos(\theta) \sin(\theta)\ d\rho\ d\varphi\ d\theta.$$

12. **[V1: Line Integrals]** Let C be the curve

$$\mathbf{r}(t) = \langle e^t, 2e^t, 4t \rangle, \quad 0 \le t \le 1.$$

Write integrals in terms of the parameter t for the following line integrals along C. You do not need to evaluate these integrals.

(a) $\int_C x\ dx + z\ dy - 2y^2\ dz$

Solution. This integral is $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \langle x, z, -2y^2 \rangle$. Since $\mathbf{r}'(t) = \langle e^t, 2e^t, 4 \rangle$, we have

$$\begin{aligned}\int_C x\ dx + z\ dy - 2y^2\ dz &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^1 \langle e^t, 4t, -8e^{2t} \rangle \cdot \langle e^t, 2e^t, 4 \rangle\ dt \\ &= \int_0^1 e^{2t} + 8te^t - 32e^{2t}\ dt \\ &= \int_0^1 -31e^{2t} + 8te^t\ dt.\end{aligned}$$

(b) $\int_C \sqrt{x^2 + y^2}\ ds$

Solution. From above, we have

$$\|\mathbf{r}'(t)\| = \sqrt{(e^t)^2 + (2e^t)^2 + 4^2} = \sqrt{5e^{2t} + 16}.$$

Therefore,

$$\begin{aligned}\int_C \sqrt{x^2 + y^2}\ ds &= \int_0^1 f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\ dt \\ &= \int_0^1 \sqrt{(e^t)^2 + (2e^t)^2}\sqrt{5e^{2t} + 16}\ dt \\ &= \int_0^1 \sqrt{5}e^t\sqrt{5e^{2t} + 16}\ dt.\end{aligned}$$

13. [**V2: Conservative Vector Fields**] For part (a), bubble in the multiple choice option that corresponds to your answer. For parts (b)-(e), determine whether the statement is true or false and write T or F in the box provided.

(a) If possible, find a potential function for the vector field

$$\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

with domain $x, y > 0$.

✓ $f(x,y) = \arctan(y/x)$

◯ $f(x,y) = \arctan(y/x) + \arctan(x/y)$

◯ $f(x,y) = -\frac{1}{2}\ln(x^2+y^2)$

◯ $f(x,y) = \frac{1}{2}\ln(x^2+y^2)$

◯ This vector field is not conservative.

(b) **T/F:** There exists a non-constant vector field $\mathbf{F}(x,y)$ with both $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$.

T

(c) **T/F:** Every vector field $\mathbf{F}$ defined on all of $\mathbb{R}^2$ is conservative.

F

(d) **T/F:** If f has continuous partial derivatives on $\mathbb{R}^3$ and C is any circle, then

$$\int_C \nabla f \cdot d\mathbf{r} = 0.$$

T

(e) **T/F:** If $\mathbf{F}$ is a conservative vector field, then the value of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of C.

T

14. **[V3: Generalizations of the FTC]**

(a) Consider $\mathbf{F}(x,y,z) = \langle 20z^{24}, x^{2025}, 2026y\rangle$ and S be the surface which is the part of the paraboloid $z = x^2 + y^2$ between $z = 0$ and $z = 100$, oriented away from the z-axis. For each integration theorem or strategy below, bubble in "Yes" if it could be used to compute the flux of $\mathbf{F}$ across S and "No" otherwise.

- ◯ Yes ✓ No Fundamental Theorem of Line Integrals
- ◯ Yes ✓ No Green's Theorem
- ✓ Yes ◯ No Stokes' Theorem
- ◯ Yes ✓ No Divergence Theorem
- ✓ Yes ◯ No Parameterization and direct calculation

(b) Consider $\mathbf{G}(x,y,z) = \langle 20x^{24}, y^{2025}, 2026z\rangle$ and S be the surface of the ellipsoid $x^2+2y^2+3z^2 = 4$. For each integration theorem or strategy below, bubble in "Yes" if it could be used to compute the flux of $\mathbf{G}$ across S and "No" otherwise.

- ◯ Yes ✓ No Fundamental Theorem of Line Integrals
- ◯ Yes ✓ No Green's Theorem
- ◯ Yes ✓ No Stokes' Theorem
- ✓ Yes ◯ No Divergence Theorem
- ✓ Yes ◯ No Parameterization and direct calculation

(c) Consider $\mathbf{H}(x,y,z) = \langle y^{24}, 24xy^{23}, 2026z\rangle$ and C be the curve which is the ellipse in the plane $x + y + z = 1$ that lies on the cylinder $x^2 + y^2 = 1$ with counterclockwise orientation. For each integration theorem or strategy below, bubble in "Yes" if it could be used to compute the circulation of $\mathbf{H}$ around C and "No" otherwise.

- ✓ Yes ◯ No Fundamental Theorem of Line Integrals
- ◯ Yes ✓ No Green's Theorem
- ✓ Yes ◯ No Stokes' Theorem
- ◯ Yes ✓ No Divergence Theorem
- ✓ Yes ◯ No Parameterization and direct calculation

15. [**V4: Surface Integrals**] Compute the flux of the vector field

$$\mathbf{F}(x, y, z) = 7xy^2\mathbf{i} + 4yz^2\mathbf{j} + z^3\mathbf{k}$$

across the surface S of the solid bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = -1$ and $x = 1$ using any method.

Solution. This is simplest to do using the Divergence Theorem, since S is a closed surface. We have

$$\nabla \cdot \mathbf{F} = 7y^2 + 4z^2 + 3z^2 = 7y^2 + 7z^2.$$

Using cylindrical coordinates with $y = r\cos(\theta)$, $z = r\sin(\theta)$, and $x = x$, we have

$$\begin{aligned}
\text{flux} &= \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma \\
&= \iiint_E \nabla \cdot \mathbf{F}\, dV \\
&= \int_0^{2\pi} \int_0^2 \int_{-1}^1 7r^3\, dx\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 14r^3\, dr\, d\theta \\
&= \int_0^2 28\pi r^3\, dr \\
&= 7\pi r^4 \Big|_0^2 \\
&= 112\pi.
\end{aligned}$$

It is possible but more difficult to compute this flux using a direct parameterization of the surface. This requires splitting the integral into three parts: one each over the curved surface of the cylinder and the two flat disks where $x = -1$ and $x = 1$.
On the disk at $x = 1$ we have $\mathbf{F} = \langle 7y^2, 4yz^2, z^3\rangle$ and $\mathbf{n} = \langle 1, 0, 0\rangle$, so the flux is given by

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_D 7y^2\, dA$$

where D is the disk $y^2 + z^2 \leq 4$. In polar coordinates, this is

$$\int_0^{2\pi} \int_0^2 7r^2\cos^2(\theta) r\, dr\, d\theta = \int_0^{2\pi} 28\cos^2(\theta)\, d\theta = 28\pi.$$

On the disk at $x = -1$ we have $\mathbf{F} = \langle -7y^2, 4yz^2, z^3\rangle$ and $\mathbf{n} = \langle -1, 0, 0\rangle$, so the flux is given by the same integral as above, which is 28π.
Finally, we can parameterize the curved surface of the cylinder as

$$\mathbf{r}(x, \theta) = \langle x, 2\cos(\theta), 2\sin(\theta)\rangle, \quad -1 \leq x \leq 1,\ 0 \leq \theta \leq 2\pi.$$

Solution. Now we have

$$\mathbf{r}_x = \langle 1, 0, 0 \rangle, \quad \mathbf{r}_\theta = \langle 0, -2\sin(\theta), 2\cos(\theta) \rangle.$$

Therefore a normal vector $\mathbf{r}_x \times \mathbf{r}_\theta$ is given by

$$\mathbf{r}_x \times \mathbf{r}_\theta = \langle 0, -2\cos(\theta), -2\sin(\theta) \rangle.$$

This vector points inward toward the x-axis, so we take the negative to get the outward normal vector. Now, we compute

$$\begin{aligned}
\iint_{\text{cylinder}} \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \int_0^{2\pi} \int_{-1}^{1} \mathbf{F}(\mathbf{r}(x,\theta)) \cdot (-\mathbf{r}_x \times \mathbf{r}_\theta)\ dx\ d\theta \\
&= \int_0^{2\pi} \int_{-1}^{1} \langle 28x\cos^2(\theta), 32\cos(\theta)\sin^2(\theta), 8\sin^3(\theta) \rangle \cdot \langle 0, 2\cos(\theta), 2\sin(\theta) \rangle\ dx\ d\theta \\
&= \int_0^{2\pi} \int_{-1}^{1} 64\cos^2(\theta)\sin^2(\theta) + 16\sin^4(\theta)\ dx\ d\theta \\
&= \int_0^{2\pi} 128\cos^2(\theta)\sin^2(\theta) + 32\sin^4(\theta)\ d\theta \\
&= \int_0^{2\pi} 32(1-\cos(2\theta))(1+\cos(2\theta)) + 8(1-\cos(2\theta))^2\ d\theta \\
&= \int_0^{2\pi} 32(1-\cos^2(2\theta)) + 8(1 - 2\cos(2\theta) + \cos^2(2\theta))\ d\theta \\
&= \int_0^{2\pi} 32 - (16 + 16\cos(4\theta)) + 8 - 16\cos(2\theta) + 4(1+\cos(4\theta))\ d\theta \\
&= \int_0^{2\pi} 28 - 16\cos(2\theta) - 4\cos(4\theta)\ d\theta \\
&= 28\theta - 8\sin(2\theta) - \sin(4\theta)\Big|_0^{2\pi} \\
&= 56\pi.
\end{aligned}$$

Adding the three fluxes together, we find the total flux across S to be

$$28\pi + 28\pi + 56\pi = 112\pi.$$

16. **[A1: Interpreting Derivatives]** Let $D(x, y)$ represent the population density of Antarctic krill (in thousands of individuals per cubic meter) in a research zone, where x and y are coordinates measured in kilometers from a central research buoy.

(a) What is the meaning of the statement $D_y(5, -2) = -15$? Be as specific as possible, and carefully consider the units of your answer.

Solution. The statement $D_y(5, -2) = -15$ means that at the location 5 kilometers east and 2 kilometers south of the research buoy, the population density of Antarctic krill is decreasing by 15 thousand individuals per cubic meter for every kilometer you move north.

(b) A specific location $(12, 8)$ represents the "center of the swarm," where the krill population density is at a global maximum for the region. What will the value of $\nabla D(12, 8)$ be? Explain your reasoning.

Solution. Since the point $(12, 8)$ is the location of a global maximum for the population density function $D(x, y)$, the gradient vector at that point must be the zero vector.

(c) A whale is feeding at the location with coordinates $(10, -4)$. Via its echolocation, it can tell that the krill density is increasing by 5 thousand individuals per cubic meter per kilometer to its north and decreasing by 2 thousand individuals per cubic meter per kilometer to its west. In which direction should it swim to most rapidly increase the density of krill it encounters? Explain your answer.

Solution. The whale should swim in the direction of the gradient vector at this point. The given information tells us that $D_y(10, -4) = 5$ and $D_x(10, -4) = 2$. Therefore, the gradient vector at this point is

$$\nabla D(10, -4) = \langle 2 \text{ km East}, 5 \text{ km North} \rangle.$$

The whale should swim in the direction of this vector to most rapidly increase the density of krill it encounters.
Justified solutions mentioning the gradient and saying something like "Northeast" or "north-northeast" are fine also.

17. [**A2: Integral Applications**] For parts (a), (c), and (d), bubble in the multiple choice option that corresponds to your answer. For part (b), determine whether the statement is true or false and write T or F in the box provided.

(a) The integral below could describe the mass of:

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{4}\frac{1}{x^2+y^2+z^2+1}\,dz\,dy\,dx$$

- ✓ a solid cone that gets lighter away from the origin
- ◯ a solid cone that is equally heavy at all points
- ◯ a solid cone that gets heavier away from the origin
- ◯ a solid ball that gets lighter away from the origin
- ◯ a solid ball that gets heavier away from the origin

(b) **T/F:** If $\mathbf{F}$ is a friction force and $\mathbf{F}\cdot\mathbf{T} > 0$ at every point on a curve C, then the net work done by $\mathbf{F}$ along the curve C is positive.

T

(c) If $\mathbf{F}$ is a vector field representing the velocity of a fluid in feet per second and C is a curve in the fluid with position measured in feet in space, then the units of the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ are:

- ◯ ft
- ◯ ft/s
- ◯ s
- ✓ ft^2/s
- ◯ s^{-1}

(d) If $\mathbf{F}$ is a vector field representing the velocity of a fluid in feet per second and C is a curve in the fluid with position measured in feet in space, then the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ computes:

- ◯ the total distance traveled along C
- ◯ the total velocity of the fluid along C
- ◯ the work done by the fluid along C
- ✓ the flow of the fluid along C
- ◯ the time it takes for one cubic foot of fluid to travel along C

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