

# MATH 2551-G Exam 3

Fall 2025

Full name: \_\_\_\_\_

GT ID: \_\_\_\_\_

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to complete as many problems as you wish to attempt.
- You may not use electronic devices of any kind during the exam. You may not use any reference materials other than your single page of hand-written notes you brought to the exam.
- The Learning Targets covered by this exam are listed below.
- Show your work. Answers without work shown will receive a **Not Yet**
- Good luck! Write yourself a message of encouragement on the front page!

## Learning Targets

- **G1: Lines and Planes.** I can describe lines using the vector equation of a line. I can describe planes using the general equation of a plane. I can find the equations of planes using a point and a normal vector. I can find the intersections of lines and planes. I can describe the relationships of lines and planes to each other. I can solve problems with lines and planes.
- **G2: Calculus of Curves.** I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.
- **G3: Geometry of Curves.** I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.
- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.
- **G5: Parameterization.** I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.
- **D1: Computing Derivatives.** I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.

- **D2: Tangent Planes and Linear Approximations.** I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.
- **D3: Optimization.** I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.
- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **I2: Iterated Integrals.** I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.
- **I3: Change of Variables.** I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.
- **V1: Line Integrals.** I can set up and evaluate scalar and vector field line integrals in two and three dimensions.
- **V2: Conservative Vector Fields.** I can test for conservative vector fields and find potential functions. I can state and apply the Fundamental Theorem of Line Integrals.
- **V3: Generalizations of the FTC.** I can state and apply Green's Theorem, Stokes' Theorem and the Divergence Theorem to solve problems in two and three dimensions. I can choose which theorem is appropriate for different integrals. I can compute curl and divergence of vector fields.
- **V4: Surface Integrals.** I can set up and compute surface integrals for scalar and vector valued functions.
- **A1: Interpreting Derivatives.** I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.
- **A2: Integral Applications.** I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.



2. [G2: Calculus of Curves] For parts (a)-(c), determine whether the statement is true or false and write T or F in the box provided. For part (d), bubble in the multiple choice option that corresponds to your answer.

(a) **T/F:** A smooth curve in the plane that never crosses itself has a single tangent line at a given point.

(b) **T/F:** Let  $\mathbf{r}(t)$  parameterize a curve  $C$  in space with  $\mathbf{r}(0) = \langle 1, 2, 3 \rangle$ . Then  $\mathbf{r}'(t)$  gives a direction vector for the tangent line to the curve at the point  $(1, 2, 3)$ .

(c) **T/F:** The solution to the initial value problem

$$\mathbf{r}''(t) = \sin(t)\mathbf{i} - 2\mathbf{j} + 16e^{4t}\mathbf{k}, \quad -\infty < t < \infty, \quad \mathbf{r}(0) = \langle 0, 0, 0 \rangle, \mathbf{r}'(0) = \langle 0, 0, 0 \rangle$$

is

$$\mathbf{r}(t) = -\sin(t)\mathbf{i} - t^2\mathbf{j} + e^{4t}\mathbf{k}$$

(d) Which statement below about motion in space is **not** true?

- A) A particle with no velocity at a given time may still be accelerating at that time.
- B) A particle moving with zero acceleration must be moving at constant speed.
- C) The path of a particle moving along a smooth curve can intersect itself.
- D) A particle moving at constant speed must have zero acceleration.

3. [**G3: Geometry of Curves**] For this problem, bubble in the multiple choice option that corresponds to your answer on each part.

(a) Which statement below about arc length is **not** true?

- The length of a curve  $C$  is always positive.
- The length of a curve  $C$  may be equal to the distance between its endpoints.
- The length of a curve  $C$  parameterized by arc length can be greater than 1.
- Every smooth curve can be parameterized by arc length.
- The length of a curve  $C$  depends on the parameterization we use to compute it.

(b) Which of the following expressions is the curvature of a curve  $x = g(y)$  in the  $xy$ -plane?

- A)  $|g''(y)|$
- B)  $\frac{|g''(y)|}{(1 + (g'(y))^2)^{3/2}}$
- C)  $\sqrt{1 + (g'(y))^2}$
- D)  $\frac{|g''(y)|}{\sqrt{1 + (g'(y))^2}}$

(c) Which of the following vectors could be the principal unit normal to a curve at a point  $P$  where the tangent line is

$$\mathbf{r}(t) = P + \langle 7, 1, -3 \rangle t?$$

- A)  $\frac{1}{\sqrt{50}} \langle 1, -7, 0 \rangle$
- B)  $\frac{1}{\sqrt{59}} \langle 3, 1, 7 \rangle$
- C)  $\frac{1}{\sqrt{59}} \langle 7, 1, -3 \rangle$
- D)  $\langle 0, 3, 1 \rangle$

4. [G4: Surfaces] For this problem, bubble in the multiple choice option that corresponds to your answer on each part.

(a) Which option below best describes the domain of the function  $f(x, y) = x^2 + y^2 + \sqrt{x - 2 + y^2}$ ?

- A) The part of the  $xy$ -plane on one side of a vertical line
- B) All of the  $xy$ -plane
- C) All of the  $xy$ -plane except for a disk about the origin
- D) The part of the  $xy$ -plane on one side of a parabola
- E) The part of the  $xy$ -plane on one side of a horizontal line

(b) The contours of the function  $f(x, y) = \arccos(x^2 + y^2)$  are best described as:

- A) lines
- B) circles
- C) sinusoidal curves
- D) parabolas

(c) The quadric surface defined by the equation  $z^2 = x^2 + y^2 + 4$  is a:

- A) ellipsoid
- B) elliptic paraboloid
- C) hyperbolic paraboloid
- D) hyperboloid of one sheet
- E) hyperboloid of two sheets
- F) cone

## 5. [G5: Parameterization]

(a) Fill in the circle next to **all** of the parameterizations  $\mathbf{r}(u, v)$  below corresponding to the surface which is the part of the elliptical paraboloid  $x = 4y^2 + z^2$  with  $0 \leq x \leq 4$ .

A)  $\mathbf{r}(u, v) = \langle u, v, 4u^2 + v^2 \rangle$ ,  $0 \leq 4u^2 + v^2 \leq 4$

B)  $\mathbf{r}(u, v) = \langle u \cos(v), u \sin(v), 4u^2 \cos^2(v) + u^2 \sin^2(v) \rangle$ ,  $0 \leq u \leq 4, 0 \leq v \leq 2\pi$

C)  $\mathbf{r}(u, v) = \langle u, \frac{\sqrt{u}}{2} \cos(v), \sqrt{u} \sin(v) \rangle$ ,  $0 \leq u \leq 4, 0 \leq v \leq 2\pi$

D)  $\mathbf{r}(u, v) = \langle 4v^2, v \cos(u), 2v \sin(u) \rangle$ ,  $0 \leq u \leq 2\pi, 0 \leq v \leq 1$

E)  $\mathbf{r}(u, v) = \langle 4u^2 + v^2, u, v \rangle$ ,  $0 \leq 4u^2 + v^2 \leq 4$

(b) Give a parameterization of the boundary of the surface in part (a) that is oriented clockwise around the positive  $x$ -axis when viewed looking down the positive  $x$ -axis toward the  $yz$ -plane. Be sure to give a domain.

6. **[D1: Computing Derivatives]** In this problem, you will work with the function  $g : \mathbb{R}^3 \rightarrow \mathbb{R}$  given by  $g(x, y, z) = x^4 + y^3 + z^2$  and the point  $P = (2, -4, 4)$  in the domain of  $g$ .
- (a) Suppose that you are only able to travel away from  $P$  in one of the following directions. Which direction (assuming you move with unit speed) will yield the greatest instantaneous decrease in  $g$ ?
- parallel to the  $x$ -axis, with  $x$  increasing
  - parallel to the  $y$ -axis, with  $y$  increasing
  - parallel to the  $z$ -axis, with  $z$  increasing
  - directly away from the origin
- (b) Justify your answer to part (a).

7. [D2: Tangent Planes and Linear Approximations] Let  $f(x, y, z) = yz + x^2e^{z-y}$ .
- (a) Find an equation of the tangent plane to the level surface  $f = 7$  at the point  $P = (\sqrt{e}, 3, 2)$ .
- (b) Find the linearization  $L(x, y, z)$  of  $f$  at  $P$ .
- (c) Use the linearization you found to approximate the value of  $f(\sqrt{e}, 3.1, 2.1)$ .

8. **[D3: Optimization]** Determine the largest value of the function  $f(x, y) = xy$  such that  $x^3 + y^3 = 2$ . Explain why  $f$  does not achieve a minimum subject to this constraint.

9. **[I1: Double & Triple Integrals]** Write an integral for the volume of the finite region bounded by the planes

$$z = x, \quad x + z = 5, \quad z = y, \quad y = 10, \quad \text{and } z = 0.$$

10. [**I2: Iterated Integrals**] Credit for this learning target may be earned either by completing the problem below or by completing successfully the V4 problem.

Compute

$$\int_0^3 \int_0^1 \int_0^{\sqrt{y}} 2xyze^{x^2} dx dy dz.$$

11. **[I3: Change of Variables]** Make an appropriate linear change of variables  $(x, y) = T(u, v)$  to rewrite

$$\iint_R (9x^2 + 24xy + 16y^2)(x - 5y) \, dx \, dy$$

as an integral in  $u$  and  $v$  where  $R$  is the parallelogram in the fourth quadrant bounded by the lines

$$3x + 4y = 17, \quad 3x + 4y = 77, \quad x - 5y = 227, \quad x - 5y = 314.$$

Do **not** evaluate the integral.

12. [**V1: Line Integrals**] Consider the curve  $C$  which is the line segment from the point  $(1, 1)$  to the point  $(4, 2)$ . Use any method you like to answer the following questions.

(a) Compute  $\int_C x - y \, ds$ .

(b) Compute  $\int_C (x\mathbf{i} - y\mathbf{j}) \cdot d\mathbf{r}$ .

13. [**V2: Conservative Vector Fields**] Let

$$\mathbf{F}(x, y) = \langle 2x \sin(y) + y \cos(x), x^2 \cos(y) + \sin(x) \rangle \quad \mathbf{G}(x, y) = \langle x^2 \sin(y), y \cos(x) \rangle.$$

In this problem you will work with these vector fields and the line segment  $C$  beginning  $(\pi, \pi/2)$  and ending at  $(\pi/2, \pi)$ .

- (a) One of these two fields is conservative. Identify which one and find a potential function for that field. Clearly explain how you know the other field is not conservative.

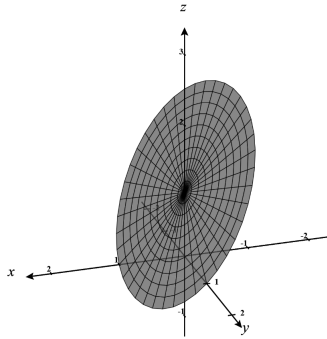
- (b) Compute the work done by the conservative field along the curve  $C$ . Fully simplify your answer.

14. [V3: Generalizations of the FTC] Consider the vector field

$$\mathbf{F}(x, y, z) = \langle z, x, y \rangle,$$

the surface  $S$  which is the part of the plane  $x + y + z = 1$  inside the cylinder  $x^2 + y^2 = 1$ , oriented with normal in the  $\langle 1, 1, 1 \rangle$  direction, and the curve  $C$  which is the boundary of  $S$ . The curve and surface are pictured below.

- (a) Give a parameterization of  $C$  which is oriented compatibly with  $S$ .



- (b) Apply Stokes' Theorem with your parameterization from (a) to compute the flux of the curl of  $\mathbf{F}$  across  $S$ .

15. [V4: Surface Integrals] Compute the flux of the vector field  $\mathbf{F} = \langle zx, z, zy \rangle$  across the surface  $S$  consisting of the portion of the paraboloid  $z = 1 - x^2 - y^2$  with  $z \geq 0$ , oriented with normal vectors away from the origin. Use any method you like.

Depending on your method, it may be helpful that the following are parameterizations of this surface or that at every point on the disk  $x^2 + y^2 \leq 1$  in the  $xy$ -plane, you have  $\mathbf{F} = \langle 0, 0, 0 \rangle$ .

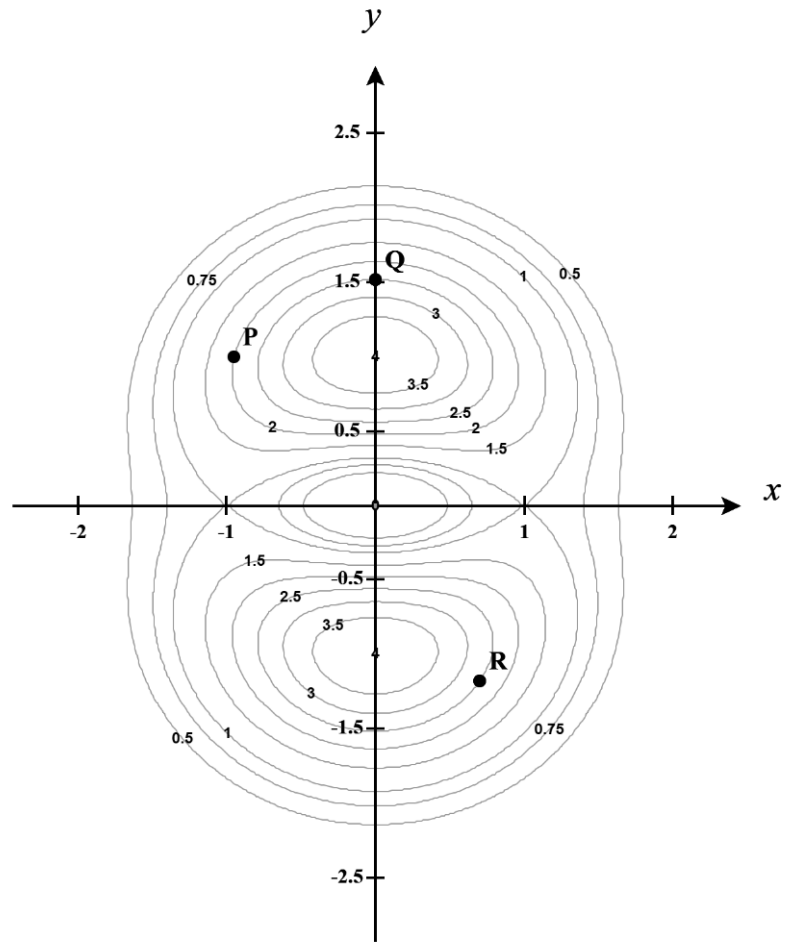
$$\begin{aligned} \mathbf{r}_1(u, v) &= \langle u, v, 1 - u^2 - v^2 \rangle, \quad u^2 + v^2 \leq 1 \\ \mathbf{r}_2(r, \theta) &= \langle r \cos(\theta), r \sin(\theta), 1 - r^2 \rangle, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi \\ \mathbf{r}_3(\varphi, \theta) &= \left\langle \frac{(-\cos(\varphi) + \sqrt{\cos^2(\varphi) + 4\sin^2(\varphi)}) \cos(\theta)}{2\sin(\varphi)}, \frac{(-\cos(\varphi) + \sqrt{\cos^2(\varphi) + 4\sin^2(\varphi)}) \sin(\theta)}{2\sin(\varphi)}, \right. \\ &\quad \left. \frac{(-\cos(\varphi) + \sqrt{\cos^2(\varphi) + 4\sin^2(\varphi)}) \cos(\varphi)}{2\sin^2(\varphi)} \right\rangle, \quad 0 \leq \varphi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq 2\pi \end{aligned}$$

16. [A1: Interpreting Derivatives] In this problem, you will analyze the derivatives of a height function  $h(x, y)$  measured in inches. Here positive  $x$  is east and positive  $y$  is north, both measured in feet. A contour plot for  $h$  is shown below.

- (a) Do you move up, down, or stay level moving north from the point  $P$ ?

- (b) Determine the sign  $(+, -, 0)$  of the rate of change of  $h$  at  $Q$  towards  $(-1, 0)$ .

- (c) Draw a vector which points in the direction of greatest rate of change of height at the point  $R$ .



- (d) Explain the meaning of the fact that

$$D_{\mathbf{u}}h(-2, 0) \approx 0.422$$

if  $\mathbf{u}$  is the unit vector in the  $\langle 1, 1 \rangle$  direction.

## 17. [A2: Integral Applications]

- (a) Give a possible interpretation of the statement

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = -200$$

where  $S$  is the surface of a net. Be specific about the physical meaning you choose.

- (b) Suppose that
- $\delta(x, y) = x^2 + (2 - y) + 100$
- is the density (in
- $\text{kg/m}^2$
- ) of a lamina occupying the region
- $R$
- in the
- $xy$
- plane bounded by the lines
- $x = 0$
- ,
- $x = 4$
- ,
- $y = 0$
- , and
- $y = 2$
- .

Explain which quarter of the rectangle contains the center of mass of the lamina. Justify your answer.

- (c) A drone moves through the air (position measured in meters) and experiences a drag force in Newtons given by
- $\mathbf{F}(x, y)$
- . We compute that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 50 \text{ J}$$

where  $C$  is a closed path taken by the drone. Explain the meaning of this calculation, including units.

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