MATH 2551-G Exam 2

Fall 2025

EXAM KEY

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to complete as many problems as you wish to attempt.
- You may not use electronic devices of any kind during the exam. You may not use any reference materials other than your single page of hand-written notes you brought to the exam.
- The Learning Targets covered by this exam are listed below.
- Show your work. Answers without work shown will receive a **Not Yet**
- Good luck! Write yourself a message of encouragement on the front page!

Learning Targets

- **G1: Lines and Planes.** I can describe lines using the vector equation of a line. I can describe planes using the general equation of a plane. I can find the equations of planes using a point and a normal vector. I can find the intersections of lines and planes. I can describe the relationships of lines and planes to each other. I can solve problems with lines and planes.
- **G2: Calculus of Curves.** I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.
- **G3: Geometry of Curves.** I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.
- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.
- **G5: Parameterization.** I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.
- **D1: Computing Derivatives.** I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.

- **D2: Tangent Planes and Linear Approximations.** I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.
- **D3: Optimization.** I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.
- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **I2: Iterated Integrals.** I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.
- **I3: Change of Variables.** I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.
- **A1: Interpreting Derivatives.** I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.
- **A2: Integral Applications.** I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.

Tasks

1. [**G1: Lines and Planes**]

   (a) Which plane given below is **not** parallel to the plane $5x + 2y - 3z = 3$?

   i. $5x + 2y - 3z = 4$
   ii. $10x + 4y - 6z = 3$
   iii. The plane through the origin which is orthogonal to $\langle 5, 2, 3 \rangle$
   iv. The plane through the origin which is orthogonal to $\langle -5, -2, 3 \rangle$
   v. The plane containing the points $(3/5, 0, 0), (0, 3/2, 0)$, and $(0, 0, -1)$.

iii.

   (b) **T/F:** If two lines in $\mathbb{R}^3$ are each orthogonal to a third line, then they must be parallel to each other.

F

   (c) **T/F:** If two planes in $\mathbb{R}^3$ are parallel to a third plane, then they must be parallel to each other.

T

   (d) Which line given below is **not** the same line as the others?

   i. $\ell_1(t) = \langle 1, 2, 3 \rangle + t\langle 1, 0, -1 \rangle$
   ii. $\ell_2(t) = \langle 0, 2, 2 \rangle + t\langle 1, 0, -1 \rangle$
   iii. $\ell_3(s) = \langle 1, 2, 3 \rangle + s\langle 2, 0, -2 \rangle$
   iv. The line containing the points $(1, 2, 3)$ and $(0, 2, 4)$
   v. The line containing the points $(1, 2, 3)$ and $(2, 2, 2)$

ii.

2. **[G2: Calculus of Curves]**

(a) A helicopter with velocity vector $\mathbf{r}'(t)$ takes off from the top of a tower located at $(0, 4, 100)$ (in meters) at time $t = 0$ and flies around for 25 minutes. What is the height of the helicopter at time $t = 25$ minutes if $\int_0^{25} \mathbf{r}'(t)\,dt = 0$?

100 m

(b) Solve the initial-value problem

$$\mathbf{r}'(t) = \langle te^{t^2+1}, \cos(2t), t^2 - 3\rangle, \quad \mathbf{r}(0) = \langle 2, 3, 1\rangle.$$

Solution. We have

$$\begin{aligned}
\mathbf{r}(t) &= \langle 2, 3, 1\rangle + \int_0^t \langle Te^{T^2+1}, \cos(2T), T^2 - 3\rangle\, dT \\
&= \langle 2, 3, 1\rangle + \left\langle \frac{1}{2}e^{T^2+1}, \frac{1}{2}\sin(2T), \frac{1}{3}T^3 - 3T \right\rangle \Bigg|_0^t \\
&= \langle 2, 3, 1\rangle + \left\langle \frac{1}{2}e^{t^2+1} - \frac{1}{2}e, \frac{1}{2}\sin(2t), \frac{1}{3}t^3 - 3t \right\rangle \\
&= \left\langle \frac{1}{2}e^{t^2+1} + 2 - \frac{1}{2}e, \frac{1}{2}\sin(2t) + 3, \frac{1}{3}t^3 - 3t + 1 \right\rangle
\end{aligned}$$

3. **[G3: Geometry of Curves]** Consider the curve parameterized by

$$\mathbf{r}(t) = \langle \sqrt{2}e^t, e^t \sin(t), e^t \cos(t) \rangle, \quad t \in \mathbb{R}.$$

(a) Compute the unit tangent vector $\mathbf{T}(t)$.

Solution. We have $\mathbf{r}(t) = e^t \langle \sqrt{2}, \sin(t), \cos(t) \rangle$, so by the product rule

$$\begin{aligned} \mathbf{r}'(t) &= e^t \langle \sqrt{2}, \sin(t), \cos(t) \rangle + e^t \langle 0, \cos(t), -\sin(t) \rangle \\ &= e^t \langle \sqrt{2}, \sin(t) + \cos(t), \cos(t) - \sin(t) \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} \|\mathbf{r}'(t)\| &= e^t \sqrt{2 + (\cos(t) + \sin(t))^2 + (\cos(t) - \sin(t))^2} \\ &= e^t \sqrt{2 + \cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t) + \cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t)} \\ &= e^t \sqrt{2 + 2\cos^2(t) + 2\sin^2(t)} \\ &= e^t \sqrt{4} = 2e^t \end{aligned}$$

Combining these, we get that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{2} \langle \sqrt{2}, \sin(t) + \cos(t), \cos(t) - \sin(t) \rangle.$$

(b) Compute the curvature $\kappa(t)$.

Solution. First we compute $\mathbf{T}'(t) = \frac{1}{2} \langle 0, -\sin(t) + \cos(t), -\sin(t) - \cos(t) \rangle$. Then we have

$$\begin{aligned} |\mathbf{T}'(t)| &= \frac{1}{2} \sqrt{(-\sin(t) + \cos(t))^2 + (-\sin(t) - \cos(t))^2} \\ &= \frac{1}{2} \sqrt{\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t) + \cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t)} \\ &= \frac{1}{2} \sqrt{2\cos^2(t) + 2\sin^2(t)} \\ &= \frac{\sqrt{2}}{2}. \end{aligned}$$

Therefore

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{2}/2}{2e^t} = \frac{\sqrt{2}}{4e^t}.$$

4. **[G4: Surfaces]**

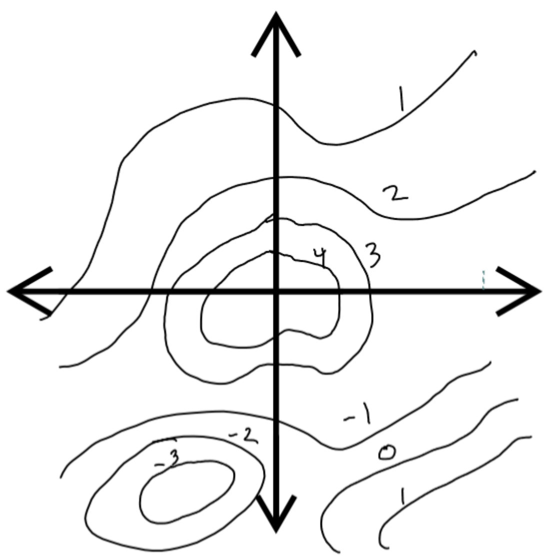
(a) Find and sketch the domain of the function

$$f(x,y) = \frac{\sqrt{4-(x-1)^2-(y-1)^2}}{x^2 e^{y-5}}.$$

Be sure to clearly indicate which points are included or excluded.

Solution. The domain is $\{(x,y) \mid (x-1)^2 + (y-1)^2 \leq 4, x \neq 0\}$. The numerator requires that $(x-1)^2 + (y-1)^2 \leq 4$, which is the disk of radius 2 centered at $(1,1)$, and the denominator requires that $x \neq 0$. Thus the domain is the disk of radius 2 centered at $(1,1)$ with the y-axis removed. The boundary circle is included since the numerator can be zero.

(b) Decide whether the plot shown below could be the contour plot of a function $f(x,y)$. Justify your answer.



Solution. Answers will vary. This plot could be a contour plot of a function; no points shown violate the "vertical line test." The function may be discontinuous in the middle of the region based on the drawn contours.

5. **[G5: Parameterization]** Find a parameterization of the curve that lies on the paraboloid $z = 1000 - x^2 - y^2$ above the points with $0 = 2x + 8y - 4y^3$ for $-1 \le y \le 1$. Be sure to give a domain.

Solution. We have $z = f(x, y)$ and we can rearrange the constraint about the points in the xy-plane to give x as a function of y. So let $y = t, -1 \le t \le 1$ and then we get the parameteriation

$$\mathbf{r}(t) = \langle 2t^3 - 4t, t, 1000 - (2t^3 - 4t)^2 - t^2 \rangle, \quad -1 \le t \le 1.$$

6. **[D1: Computing Derivatives]** Suppose $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ parameterizes a curve in the plane with

$$\mathbf{r}(t) = \langle -3t + \cos(t), t^3 + 4t \rangle.$$

If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable function with

$$Df(\mathbf{r}(0)) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

what is the direction of the tangent vector to the curve parameterized by $f \circ \mathbf{r}$ at $(f \circ \mathbf{r})(0)$? Your answer should be a unit vector.

Solution. We apply the Chain Rule. A tangent vector to the curve parameterized by $f \circ \mathbf{r}$ at the point $(f \circ \mathbf{r})(0)$ is $(f \circ \mathbf{r})'(0)$, and by the Chain Rule this is

$$\begin{aligned}
(f \circ \mathbf{r})'(0) &= D(f \circ \mathbf{r})(0) \\
&= Df(\mathbf{r}(0))D\mathbf{r}(0) \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -3 - \sin(t) \\ 3t^2 + 4 \end{bmatrix} \Bigg|_{t=0} \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 3 \end{bmatrix}
\end{aligned}$$

Now we normalize to get our unit vector direction:

$$\mathbf{T} = \frac{1}{5}\langle 4, 3 \rangle.$$

7. **[D2: Tangent Planes and Linear Approximations]** Suppose that $f(x, y, z)$ is a differentiable function and the plane $2x - y + 2z = 3$ is tangent to the level surface of f containing the point $(1, 1, 1)$. Use this information to answer the following questions.

   (a) Which one of these vectors could be $\nabla f(1, 1, 1)$?

   i. $\langle 1, 1, 1\rangle$
   ii. $\langle 1, 0, -1\rangle$
   iii. $\langle 4, -2, 4\rangle$
   iv. $\langle 3, -1, 3\rangle$

iii

   (b) Suppose we know that the linear approximation of f at $(1, 1, 1)$ can be used to estimate $f(1.1, 0.9, 1) \approx 2.8$. Use your answer to (a) to determine $f(1, 1, 1)$ if ∇f is the vector you chose.

Solution. There are four possible correct answers depending on the vector chosen for ∇f in (a). The general solution is as follows. From the linearization formula, we have:

$f(1.1, 0.9, 1) \approx 2.8 = L(1.1, 0.9, 1) = f(1, 1, 1) + \nabla f(1, 1, 1) \cdot \langle 1.1 - 1, 0.9 - 1, 1 - 1\rangle.$

Now we solve for $f(1, 1, 1)$ to get:

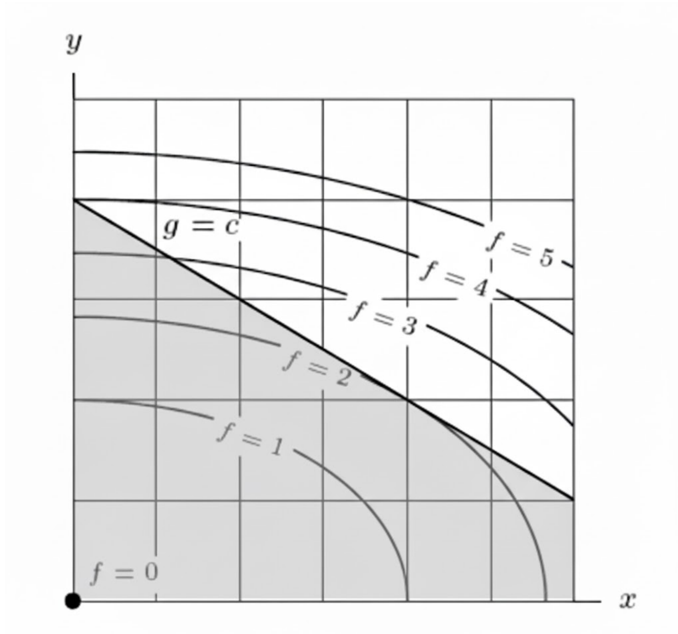
$$f(1, 1, 1) = 2.8 - \nabla f(1, 1, 1) \cdot \langle 0.1, -0.1, 0\rangle.$$

Using each possible $\nabla f(1, 1, 1)$ from (a):

- For $\langle 1, 1, 1\rangle$, we have $f(1, 1, 1) = 2.8$.
- For $\langle 1, 0, -1\rangle$, we have $f(1, 1, 1) = 2.7$.
- For $\langle 4, -2, 4\rangle$, we have $f(1, 1, 1) = 2.2$.
- For $\langle 3, -1, 3\rangle$, we have $f(1, 1, 1) = 2.4$.

8. [**D3: Optimization**]

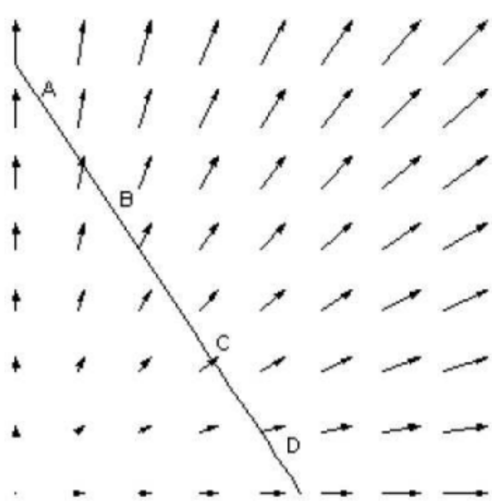
(a) Pictured below is the contour plot of a smooth function $f(x, y)$ along with a linear constraint $g = c$. Count the number of critical points inside the region and on the boundary, then find the maximum and minimum values of $f(x, y)$ on the trapezoidal region below $g = c$ in the first quadrant, shaded in gray.



 # interior critical points 0 **# boundary critical points** 1/5(corners)

Minimum: 0 **Maximum:** 4

(b) This plot shows the gradient vectors for a function $f(x, y)$ and a linear constraint. Which point given is closest to the global min of $f(x, y)$ subject to the constraint?



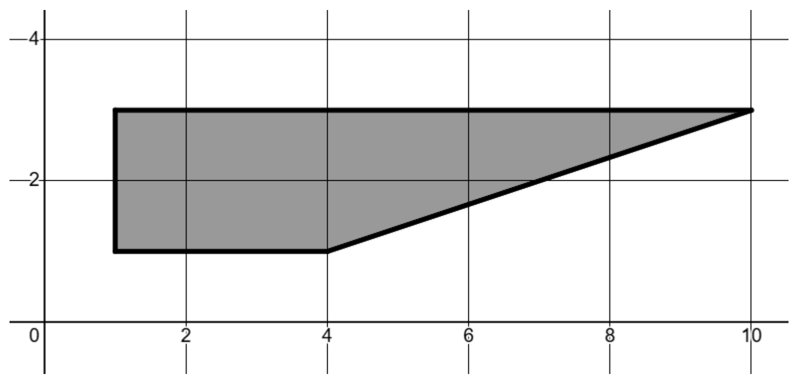
C

9. **[I1: Double & Triple Integrals]** In this problem you will work with the region R with $1 \leq x \leq 1+3y$ and $1 \leq y \leq 3$ to set up a double integral

$$I = \iint_R 4y + x^2 \, dA.$$

(a) Sketch this region.

Solution.



(b) Set up the integral in the order $dy\,dx$. You may need to split the integral.

Solution. We can see this region is not vertically simple, so we need to split the integral at $x = 4$ where the lower bound shifts. We have

$$I = \int_1^4 \int_1^3 4y + x^2 \, dy \, dx + \int_4^{10} \int_{(x-1)/3}^3 4y + x^2 \, dy \, dx.$$

(c) Set up the integral in the order $dx\,dy$. You may need to split the integral.

Solution. On the other hand, the region is horizontally simple, and our bounds come straight from the original description of the region.

$$I = \int_1^3 \int_1^{1+3y} 4y + x^2 \, dx \, dy.$$

10. **[I2: Iterated Integrals]** Evaluate the integral

$$\int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xze^{zy^2}\ dx\ dy\ dz.$$

Solution.

$$\begin{aligned}
\int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xze^{zy^2}\ dx\ dy\ dz &= \int_0^1 \int_0^1 \left[6x^2ye^{zy^2}\right]_{x=0}^{x=\sqrt{y}}\ dy\ dz \\
&= \int_0^1 \int_0^1 6yze^{zy^2}\ dy\ dz \\
&= \int_0^1 \int_0^z 3e^u\ du\ dy \quad (\text{Let } u = yz^2, du = 2yz\ dy) \\
&= \int_0^1 3e^y - 3\ dy \\
&= 3e^y - 3y\Big|_0^1 \\
&= 3e - 6
\end{aligned}$$

11. **[I3: Change of Variables]**

(a) Which of the following regions resembles a quarter of a doughnut?

i. $0 \leq r \leq 5, 0 \leq \theta \leq \pi/2$
ii. $3 \leq r \leq 5, 0 \leq \theta \leq 2\pi$
iii. $2 \leq r \leq 5, \pi \leq \theta \leq 2\pi$
iv. $1 \leq r \leq 5, \pi \leq \theta \leq 3\pi/2$

iv.

(b) Which of the following integrals is equivalent to

$$\int_0^3 \int_\pi^{2\pi} r \, d\theta \, dr?$$

i. $\int_0^3 \int_{-\sqrt{9-x^2}}^0 \sqrt{x^2+y^2} \, dy \, dx$
ii. $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} 1 \, dy \, dx$
iii. $\int_{-3}^0 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} 1 \, dx \, dy$
iv. $\int_{-3}^3 \int_0^{\sqrt{9-y^2}} 1 \, dx \, dy$
v. $\int_{-3}^0 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \sqrt{x^2+y^2} \, dx \, dy$

iii.

(c) What geometric shape is described by the equation $r = \theta$?

i. line
ii. circle
iii. ellipse
iv. spiral
v. parabola

iv.

12. **[A1: Interpreting Derivatives]**

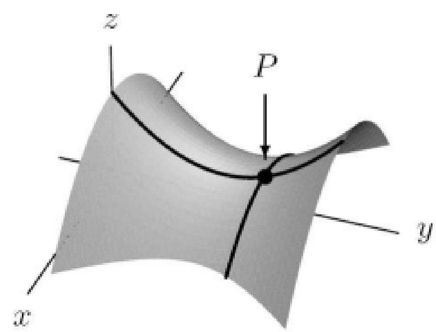
(a) The figure below shows the surface $z = f(x, y)$ along with a point P on the surface. What are the signs of $f_{xx}(P)$ and $f_{yy}(P)$?

i. $f_{xx}(P) > 0, f_{yy}(P) \approx 0$

ii. $f_{xx}(P) < 0, f_{yy}(P) > 0$

iii. $f_{xx}(P) \approx 0, f_{yy}(P) \approx 0$

iv. $f_{xx}(P) > 0, f_{yy}(P) > 0$



ii.

(b) The function $F(C, S)$ gives the amount of fuel (in gallons) used by a boat rental company each day, where C is the number of catamarans rented that day and S is the number of speedboats rented that day. What is the meaning of the statement

$$F_C(2, 6) = 37?$$

Be as specific as possible, including units.

Solution. Answers may vary in form. The statement means that when the company is renting out 2 catamarans and 6 speedboats in a day, the increase in fuel consumption from adding another catamaran rental is 37 gallons per day.

(c) **T/F:** A differentiable function $f(x, y)$ has gradient ∇f at the point (a, b). Then the vector $\nabla f(a, b)$ is perpendicular to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$.

F

13. [**A2: Integral Applications**]

(a) Write an integral expression for the area of the region in the first quadrant with $0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}$.

Solution.

$$\text{Area} = \iint_R 1 \; dA = \int_0^1 \int_{x^2}^{\sqrt{x}} \; dy \; dx$$

(b) Write an integral expression for the x-coordinate of the center of mass of the flat plate occupying the same region as in part (a) with density function $\delta(x, y) = 1 + y$.

Solution.

$$\bar{x} = \frac{M_y}{\text{Mass}} = \frac{\iint_R x\delta(x,y) \; dA}{\iint_R \delta(x,y) \; dA} = \frac{\displaystyle\int_0^1 \int_{x^2}^{\sqrt{x}} x(1+y) \; dy \; dx}{\displaystyle\int_0^1 \int_{x^2}^{\sqrt{x}} 1 + y \; dy \; dx}$$

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