

# MATH 2551-G Exam 2

Fall 2025

## EXAM KEY

**Read all instructions carefully** before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to complete as many problems as you wish to attempt.
- You may not use electronic devices of any kind during the exam. You may not use any reference materials other than your single page of hand-written notes you brought to the exam.
- The Learning Targets covered by this exam are listed below.
- Show your work. Answers without work shown will receive a **Not Yet**
- Good luck! Write yourself a message of encouragement on the front page!

### Learning Targets

- **G1: Lines and Planes.** I can describe lines using the vector equation of a line. I can describe planes using the general equation of a plane. I can find the equations of planes using a point and a normal vector. I can find the intersections of lines and planes. I can describe the relationships of lines and planes to each other. I can solve problems with lines and planes.
- **G2: Calculus of Curves.** I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.
- **G3: Geometry of Curves.** I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.
- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.
- **G5: Parameterization.** I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.
- **D1: Computing Derivatives.** I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.

- **D2: Tangent Planes and Linear Approximations.** I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.
- **D3: Optimization.** I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.
- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **I2: Iterated Integrals.** I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.
- **I3: Change of Variables.** I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.
- **A1: Interpreting Derivatives.** I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.
- **A2: Integral Applications.** I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.

**Tasks**

## 1. [G1: Lines and Planes]

(a) Which plane given below is **not** parallel to the plane  $5x + 2y - 3z = 3$ ?

i.  $5x + 2y - 3z = 4$

ii.  $10x + 4y - 6z = 3$

iii. The plane through the origin which is orthogonal to  $\langle 5, 2, 3 \rangle$

iv. The plane through the origin which is orthogonal to  $\langle -5, -2, 3 \rangle$

v. The plane containing the points  $(3/5, 0, 0)$ ,  $(0, 3/2, 0)$ , and  $(0, 0, -1)$ .

iii.

(b) **T/F:** If two lines in  $\mathbb{R}^3$  are each orthogonal to a third line, then they must be parallel to each other.

F

(c) **T/F:** If two planes in  $\mathbb{R}^3$  are parallel to a third plane, then they must be parallel to each other.

T

(d) Which line given below is **not** the same line as the others?

i.  $\ell_1(t) = \langle 1, 2, 3 \rangle + t\langle 1, 0, -1 \rangle$

ii.  $\ell_2(t) = \langle 0, 2, 2 \rangle + t\langle 1, 0, -1 \rangle$

iii.  $\ell_3(s) = \langle 1, 2, 3 \rangle + s\langle 2, 0, -2 \rangle$

iv. The line containing the points  $(1, 2, 3)$  and  $(0, 2, 4)$

v. The line containing the points  $(1, 2, 3)$  and  $(2, 2, 2)$

ii.

## 2. [G2: Calculus of Curves]

- (a) A helicopter with velocity vector  $\mathbf{r}'(t)$  takes off from the top of a tower located at  $(0, 4, 100)$  (in meters) at time  $t = 0$  and flies around for 25 minutes. What is the height of the helicopter at time  $t = 25$  minutes if  $\int_0^{25} \mathbf{r}'(t) dt = \mathbf{0}$ ?

100 m

- (b) Solve the initial-value problem

$$\mathbf{r}'(t) = \langle te^{t^2+1}, \cos(2t), t^2 - 3 \rangle, \quad \mathbf{r}(0) = \langle 2, 3, 1 \rangle.$$

**Solution.** We have

$$\begin{aligned} \mathbf{r}(t) &= \langle 2, 3, 1 \rangle + \int_0^t \langle Te^{T^2+1}, \cos(2T), T^2 - 3 \rangle dT \\ &= \langle 2, 3, 1 \rangle + \left\langle \frac{1}{2}e^{T^2+1}, \frac{1}{2}\sin(2T), \frac{1}{3}T^3 - 3T \right\rangle \Big|_0^t \\ &= \langle 2, 3, 1 \rangle + \left\langle \frac{1}{2}e^{t^2+1} - \frac{1}{2}e, \frac{1}{2}\sin(2t), \frac{1}{3}t^3 - 3t \right\rangle \\ &= \left\langle \frac{1}{2}e^{t^2+1} + 2 - \frac{1}{2}e, \frac{1}{2}\sin(2t) + 3, \frac{1}{3}t^3 - 3t + 1 \right\rangle \end{aligned}$$

3. [G3: Geometry of Curves] Consider the curve parameterized by

$$\mathbf{r}(t) = \langle \sqrt{2}e^t, e^t \sin(t), e^t \cos(t) \rangle, \quad t \in \mathbb{R}.$$

- (a) Compute the unit tangent vector  $\mathbf{T}(t)$ .

**Solution.** We have  $\mathbf{r}(t) = e^t \langle \sqrt{2}, \sin(t), \cos(t) \rangle$ , so by the product rule

$$\begin{aligned} \mathbf{r}'(t) &= e^t \langle \sqrt{2}, \sin(t), \cos(t) \rangle + e^t \langle 0, \cos(t), -\sin(t) \rangle \\ &= e^t \langle \sqrt{2}, \sin(t) + \cos(t), \cos(t) - \sin(t) \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} \|\mathbf{r}'(t)\| &= e^t \sqrt{2 + (\cos(t) + \sin(t))^2 + (\cos(t) - \sin(t))^2} \\ &= e^t \sqrt{2 + \cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t) + \cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t)} \\ &= e^t \sqrt{2 + 2\cos^2(t) + 2\sin^2(t)} \\ &= e^t \sqrt{4} = 2e^t \end{aligned}$$

Combining these, we get that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{2} \langle \sqrt{2}, \sin(t) + \cos(t), \cos(t) - \sin(t) \rangle.$$

- (b) Compute the curvature  $\kappa(t)$ .

**Solution.** First we compute  $\mathbf{T}'(t) = \frac{1}{2} \langle 0, -\sin(t) + \cos(t), -\sin(t) - \cos(t) \rangle$ .

Then we have

$$\begin{aligned} \|\mathbf{T}'(t)\| &= \frac{1}{2} \sqrt{(-\sin(t) + \cos(t))^2 + (-\sin(t) - \cos(t))^2} \\ &= \frac{1}{2} \sqrt{\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t) + \cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t)} \\ &= \frac{1}{2} \sqrt{2\cos^2(t) + 2\sin^2(t)} \\ &= \frac{\sqrt{2}}{2}. \end{aligned}$$

Therefore

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{2}/2}{2e^t} = \frac{\sqrt{2}}{4e^t}.$$

## 4. [G4: Surfaces]

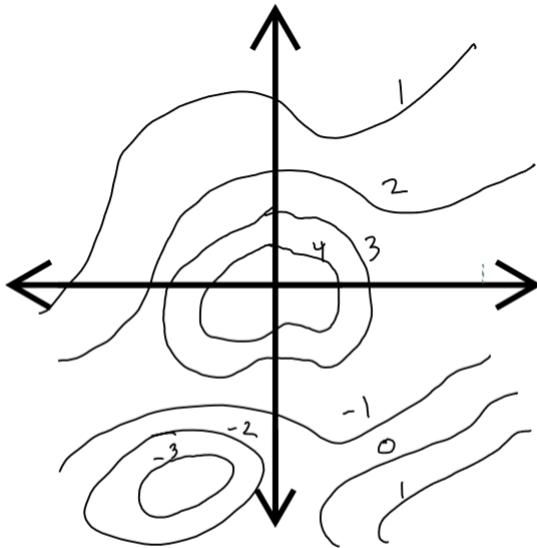
- (a) Find and sketch the domain of the function

$$f(x, y) = \frac{\sqrt{4 - (x - 1)^2 - (y - 1)^2}}{x^2 e^{y-5}}.$$

Be sure to clearly indicate which points are included or excluded.

**Solution.** The domain is  $\{(x, y) \mid (x - 1)^2 + (y - 1)^2 \leq 4, x \neq 0\}$ . The numerator requires that  $(x - 1)^2 + (y - 1)^2 \leq 4$ , which is the disk of radius 2 centered at  $(1, 1)$ , and the denominator requires that  $x \neq 0$ . Thus the domain is the disk of radius 2 centered at  $(1, 1)$  with the  $y$ -axis removed. The boundary circle is included since the numerator can be zero.

- (b) Decide whether the plot shown below could be the contour plot of a function
- $f(x, y)$
- . Justify your answer.



**Solution.** Answers will vary. This plot could be a contour plot of a function; no points shown violate the “vertical line test.” The function may be discontinuous in the middle of the region based on the drawn contours.

5. [**G5: Parameterization**] Find a parameterization of the curve that lies on the paraboloid  $z = 1000 - x^2 - y^2$  above the points with  $0 = 2x + 8y - 4y^3$  for  $-1 \leq y \leq 1$ . Be sure to give a domain.

**Solution.** We have  $z = f(x, y)$  and we can rearrange the constraint about the points in the  $xy$ -plane to give  $x$  as a function of  $y$ . So let  $y = t, -1 \leq t \leq 1$  and then we get the parameteriation

$$\mathbf{r}(t) = \langle 2t^3 - 4t, t, 1000 - (2t^3 - 4t)^2 - t^2 \rangle, \quad -1 \leq t \leq 1.$$

6. [D1: Computing Derivatives] Suppose  $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^2$  parameterizes a curve in the plane with

$$\mathbf{r}(t) = \langle -3t + \cos(t), t^3 + 4t \rangle.$$

If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a differentiable function with

$$Df(\mathbf{r}(0)) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

what is the direction of the tangent vector to the curve parameterized by  $f \circ \mathbf{r}$  at  $(f \circ \mathbf{r})(0)$ ?

Your answer should be a unit vector.

**Solution.** We apply the Chain Rule. A tangent vector to the curve parameterized by  $f \circ \mathbf{r}$  at the point  $(f \circ \mathbf{r})(0)$  is  $(f \circ \mathbf{r})'(0)$ , and by the Chain Rule this is

$$\begin{aligned} (f \circ \mathbf{r})'(0) &= D(f \circ \mathbf{r})(0) \\ &= Df(\mathbf{r}(0))D\mathbf{r}(0) \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -3 - \sin(t) \\ 3t^2 + 4 \end{bmatrix} \Big|_{t=0} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 3 \end{bmatrix} \end{aligned}$$

Now we normalize to get our unit vector direction:

$$\mathbf{T} = \frac{1}{5} \langle 4, 3 \rangle.$$

7. [D2: Tangent Planes and Linear Approximations] Suppose that  $f(x, y, z)$  is a differentiable function and the plane  $2x - y + 2z = 3$  is tangent to the level surface of  $f$  containing the point  $(1, 1, 1)$ . Use this information to answer the following questions.

(a) Which one of these vectors could be  $\nabla f(1, 1, 1)$ ?

- i.  $\langle 1, 1, 1 \rangle$
- ii.  $\langle 1, 0, -1 \rangle$
- iii.  $\langle 4, -2, 4 \rangle$
- iv.  $\langle 3, -1, 3 \rangle$

iii

(b) Suppose we know that the linear approximation of  $f$  at  $(1, 1, 1)$  can be used to estimate  $f(1.1, 0.9, 1) \approx 2.8$ . Use your answer to (a) to determine  $f(1, 1, 1)$  if  $\nabla f$  is the vector you chose.

**Solution.** There are four possible correct answers depending on the vector chosen for  $\nabla f$  in (a). The general solution is as follows. From the linearization formula, we have:

$$f(1.1, 0.9, 1) \approx 2.8 = L(1.1, 0.9, 1) = f(1, 1, 1) + \nabla f(1, 1, 1) \cdot \langle 1.1 - 1, 0.9 - 1, 1 - 1 \rangle.$$

Now we solve for  $f(1, 1, 1)$  to get:

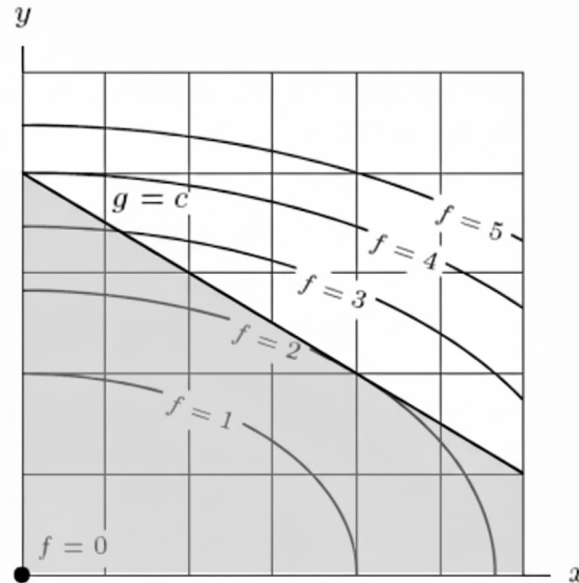
$$f(1, 1, 1) = 2.8 - \nabla f(1, 1, 1) \cdot \langle 0.1, -0.1, 0 \rangle.$$

Using each possible  $\nabla f(1, 1, 1)$  from (a):

- For  $\langle 1, 1, 1 \rangle$ , we have  $f(1, 1, 1) = 2.8$ .
- For  $\langle 1, 0, -1 \rangle$ , we have  $f(1, 1, 1) = 2.7$ .
- For  $\langle 4, -2, 4 \rangle$ , we have  $f(1, 1, 1) = 2.2$ .
- For  $\langle 3, -1, 3 \rangle$ , we have  $f(1, 1, 1) = 2.4$ .

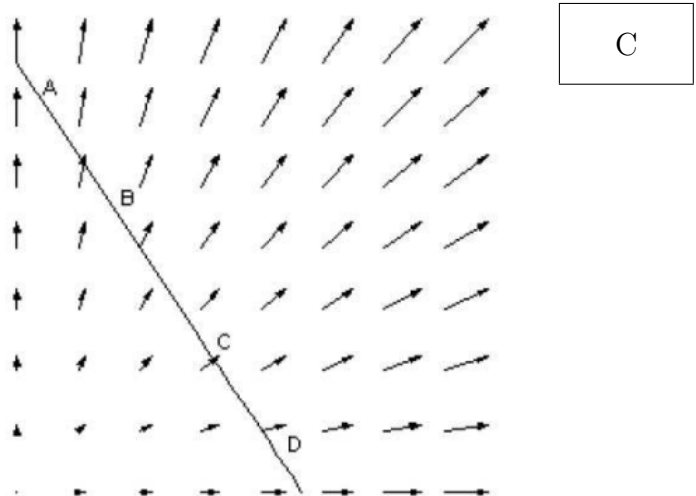
8. [D3: Optimization]

- (a) Pictured below is the contour plot of a smooth function  $f(x, y)$  along with a linear constraint  $g = c$ . Count the number of critical points inside the region and on the boundary, then find the maximum and minimum values of  $f(x, y)$  on the trapezoidal region below  $g = c$  in the first quadrant, shaded in gray.



# interior critical points  # boundary critical points   
 Minimum:  Maximum:

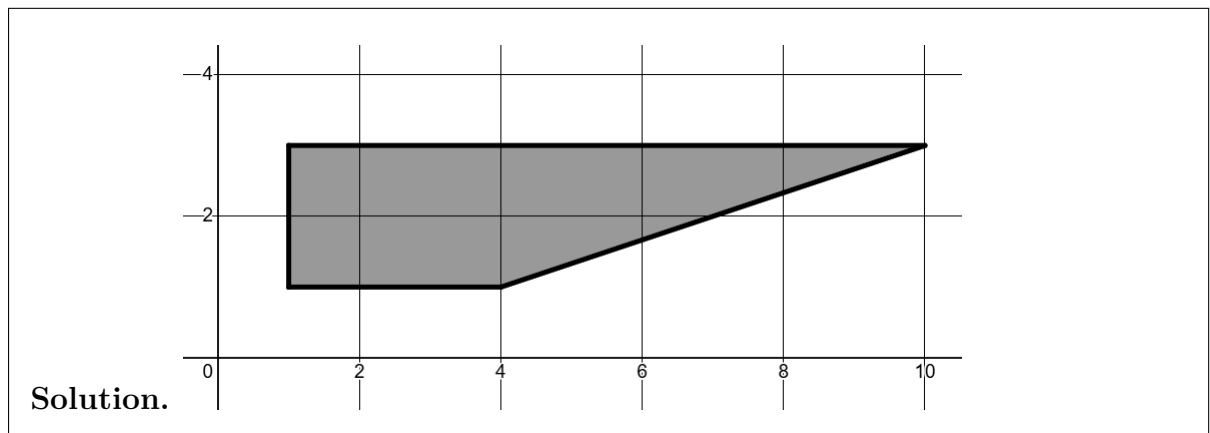
- (b) This plot shows the gradient vectors for a function  $f(x, y)$  and a linear constraint. Which point given is closest to the global min of  $f(x, y)$  subject to the constraint?



9. [I1: Double & Triple Integrals] In this problem you will work with the region  $R$  with  $1 \leq x \leq 1 + 3y$  and  $1 \leq y \leq 3$  to set up a double integral

$$I = \iint_R 4y + x^2 \, dA.$$

- (a) Sketch this region.



- (b) Set up the integral in the order  $dy \, dx$ . You may need to split the integral.

**Solution.** We can see this region is not vertically simple, so we need to split the integral at  $x = 4$  where the lower bound shifts. We have

$$I = \int_1^4 \int_1^3 4y + x^2 \, dy \, dx + \int_4^{10} \int_{(x-1)/3}^3 4y + x^2 \, dy \, dx.$$

- (c) Set up the integral in the order  $dx \, dy$ . You may need to split the integral.

**Solution.** On the other hand, the region is horizontally simple, and our bounds come straight from the original description of the region.

$$I = \int_1^3 \int_1^{1+3y} 4y + x^2 \, dx \, dy.$$

10. [I2: Iterated Integrals] Evaluate the integral

$$\int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xze^{zy^2} dx dy dz.$$

**Solution.**

$$\begin{aligned} \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xze^{zy^2} dx dy dz &= \int_0^1 \int_0^1 \left[ 6x^2ye^{zy^2} \right]_{x=0}^{x=\sqrt{y}} dy dz \\ &= \int_0^1 \int_0^1 6yze^{zy^2} dy dz \\ &= \int_0^1 \int_0^z 3e^u du dy \quad (\text{Let } u = yz^2, du = 2yz dy) \\ &= \int_0^1 3e^y - 3 dy \\ &= 3e^y - 3y \Big|_0^1 \\ &= 3e - 6 \end{aligned}$$

## 11. [I3: Change of Variables]

(a) Which of the following regions resembles a quarter of a doughnut?

i.  $0 \leq r \leq 5, 0 \leq \theta \leq \pi/2$

ii.  $3 \leq r \leq 5, 0 \leq \theta \leq 2\pi$

iii.  $2 \leq r \leq 5, \pi \leq \theta \leq 2\pi$

iv.  $1 \leq r \leq 5, \pi \leq \theta \leq 3\pi/2$

iv.

(b) Which of the following integrals is equivalent to

$$\int_0^3 \int_{\pi}^{2\pi} r \, d\theta \, dr?$$

i.  $\int_0^3 \int_{-\sqrt{9-x^2}}^0 \sqrt{x^2 + y^2} \, dy \, dx$

ii.  $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} 1 \, dy \, dx$

iii.  $\int_{-3}^0 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} 1 \, dx \, dy$

iv.  $\int_{-3}^3 \int_0^{\sqrt{9-y^2}} 1 \, dx \, dy$

v.  $\int_{-3}^0 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \sqrt{x^2 + y^2} \, dx \, dy$

iii.

(c) What geometric shape is described by the equation  $r = \theta$ ?

i. line

ii. circle

iii. ellipse

iv. spiral

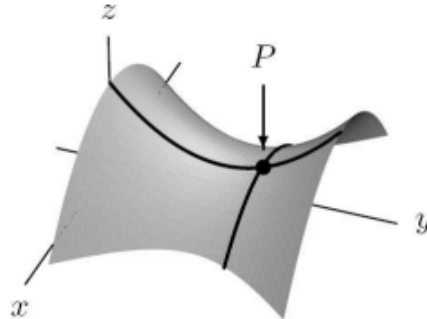
v. parabola

iv.

## 12. [A1: Interpreting Derivatives]

- (a) The figure below shows the surface  $z = f(x, y)$  along with a point  $P$  on the surface. What are the signs of  $f_{xx}(P)$  and  $f_{yy}(P)$ ?

- i.  $f_{xx}(P) > 0, f_{yy}(P) \approx 0$
- ii.  $f_{xx}(P) < 0, f_{yy}(P) > 0$
- iii.  $f_{xx}(P) \approx 0, f_{yy}(P) \approx 0$
- iv.  $f_{xx}(P) > 0, f_{yy}(P) > 0$



ii.

- (b) The function  $F(C, S)$  gives the amount of fuel (in gallons) used by a boat rental company each day, where  $C$  is the number of catamarans rented that day and  $S$  is the number of speedboats rented that day. What is the meaning of the statement

$$F_C(2, 6) = 37?$$

Be as specific as possible, including units.

**Solution.** Answers may vary in form. The statement means that when the company is renting out 2 catamarans and 6 speedboats in a day, the increase in fuel consumption from adding another catamaran rental is 37 gallons per day.

- (c) **T/F:** A differentiable function  $f(x, y)$  has gradient  $\nabla f$  at the point  $(a, b)$ . Then the vector  $\nabla f(a, b)$  is perpendicular to the surface  $z = f(x, y)$  at the point  $(a, b, f(a, b))$ .

F

## 13. [A2: Integral Applications]

- (a) Write an integral expression for the area of the region in the first quadrant with  $0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}$ .

**Solution.**

$$\text{Area} = \iint_R 1 \, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} dy \, dx$$

- (b) Write an integral expression for the  $x$ -coordinate of the center of mass of the flat plate occupying the same region as in part (a) with density function  $\delta(x, y) = 1 + y$ .

**Solution.**

$$\bar{x} = \frac{M_y}{\text{Mass}} = \frac{\iint_R x\delta(x, y) \, dA}{\iint_R \delta(x, y) \, dA} = \frac{\int_0^1 \int_{x^2}^{\sqrt{x}} x(1+y) \, dy \, dx}{\int_0^1 \int_{x^2}^{\sqrt{x}} 1+y \, dy \, dx}$$

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