MATH 2551-G Exam 1

Fall 2025

EXAM KEY

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to complete the problems.
- You may not use electronic devices of any kind during the exam. You may not use any reference materials other than your single page of hand-written notes you brought to the exam.
- The Learning Targets covered by this exam are listed below.
- Show your work. Answers without work shown will receive a **Not Yet**
- Good luck! Write yourself a message of encouragement on the front page!

Learning Targets

- **G1: Lines and Planes.** I can describe lines using the vector equation of a line. I can describe planes using the general equation of a plane. I can find the equations of planes using a point and a normal vector. I can find the intersections of lines and planes. I can describe the relationships of lines and planes to each other. I can solve problems with lines and planes.
- **G2: Calculus of Curves.** I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.
- **G3: Geometry of Curves.** I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.
- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.
- **G5: Parameterization.** I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.
- **D1: Computing Derivatives.** I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.

1. **[G1: Lines and Planes]** Consider the plane p with equation $2x - y + 4z = 5$.

(a) Give a line ℓ through the point $(10, 20, 30)$ which is orthogonal to p.

Solution. The direction vector of ℓ should be parallel to the normal vector of p for the line to be orthogonal to p. So one equation for this line is

$$\ell(t) = \langle 2, -1, 4\rangle t + \langle 10, 20, 30\rangle, \quad t \in \mathbb{R}.$$

(b) Give a different line through the point $(10, 20, 30)$ which is orthogonal to ℓ. Justify why your line is orthogonal to ℓ.

Solution. A line which is orthogonal to ℓ should have a direction vector which is orthogonal to $\langle 2, -1, 4\rangle$. There is a two-dimensional subspace of such vectors, spanned by $\langle 1, 2, 0\rangle$ and $\langle 2, 0, -1\rangle$. So there are many correct answers, e.g.

$$\langle 1, 2, 0\rangle t + \langle 10, 20, 30\rangle, \quad t \in \mathbb{R}.$$

(c) Give a plane which contains both of your lines from parts (a) and (b).

Solution. Your answer will depend on your choice in part (b). This plane's normal vector should be orthogonal to both direction vectors from (a) and (b) and it will contain the point $(10, 20, 30)$. For the choices above, we have

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix} = \langle -8, 4, 5\rangle.$$

So an equation for this plane is

$$-8(x - 10) + 4(y - 20) + 5(z - 30) = 0.$$

2. **[G2: Calculus of Curves]**

(a) **True/False:** If $\mathbf{r}(0) = \langle 3, 2, 1\rangle$ and $\mathbf{r}'(t) = \langle x(t), y(t), z(t)\rangle$ then

$$\mathbf{r}(t) = \int \langle x(t), y(t), z(t)\rangle \, dt + \langle 3, 2, 1\rangle.$$

False

(b) Give an equation for the tangent line to the curve parameterized by

$$\mathbf{r}(t) = \langle e^t, 4t - 1, t^2\rangle$$

at the point $(e, 3, 1)$.

Solution. First we need to find the derivative

$$\mathbf{r}'(t) = \langle e^t, 4, 2t\rangle.$$

Evaluating this at $t = 1$ (the parameter value corresponding to the point $(e, 3, 1)$) gives

$$\mathbf{r}'(1) = \langle e, 4, 2\rangle.$$

Thus an equation of the tangent line is given by

$$\ell(t) = \mathbf{r}(1) + \mathbf{r}'(1)t = \langle e, 3, 1\rangle + \langle e, 4, 2\rangle t.$$

3. **[G3: Geometry of Curves]**

(a) Order the following curves in increasing order of curvature.

A) A helix of curvature 1/100
B) A line segment of length 100
C) A circle of radius 2

B, A, C

(b) Find the coordinates of the point which lies a distance $2\sqrt{13}\pi/3$ along the helix

$$\mathbf{r}(t) = \langle 2\cos(t), -2\sin(t), 3t\rangle$$

in the direction of increasing parameter t from $(2, 0, 0)$.

Solution. First we compute the derivative

$$\mathbf{r}'(t) = \langle -2\sin(t), -2\cos(t), 3\rangle.$$

The speed is

$$\|\mathbf{r}'(t)\| = \sqrt{(-2\sin(t))^2 + (-2\cos(t))^2 + 3^2} = \sqrt{4+9} = \sqrt{13}.$$

The arc length function is therefore

$$s(t) = \int_0^t \sqrt{13}\, dt = \sqrt{13}t.$$

To find the parameter value corresponding to an arc length of $2\sqrt{13}\pi/3$, we solve

$$2\sqrt{13}\pi/3 = s(t) = \sqrt{13}t$$

to get $t = 2\pi/3$. Finally, we evaluate the position vector at this parameter value to find the desired point:

$$\mathbf{r}(2\pi/3) = \langle 2\cos(2\pi/3), -2\sin(2\pi/3), 3(2\pi/3)\rangle = \langle -1, -\sqrt{3}, 2\pi\rangle.$$

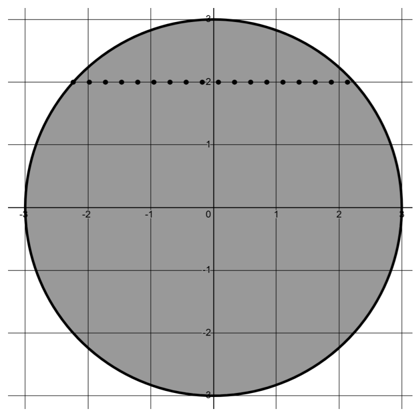
4. **[G4: Surfaces]**

(a) Find and sketch the domain of the function

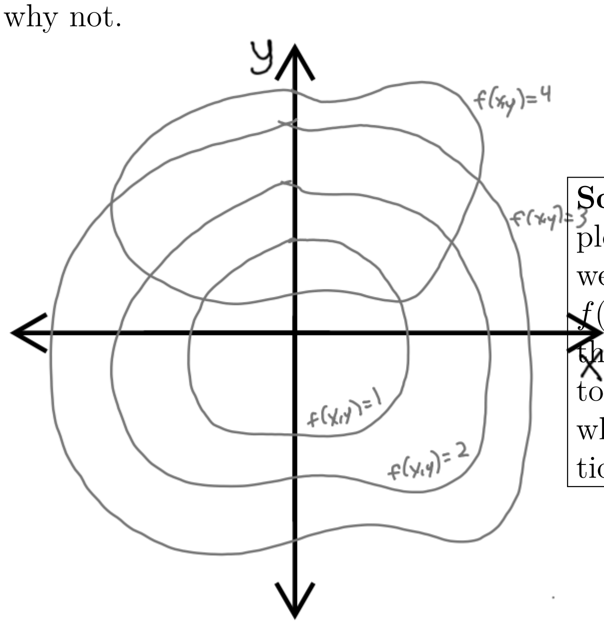
$$f(x, y) = \sqrt{9 - x^2 - y^2} + \frac{2x}{3y - 6}.$$

Clearly indicate any points which are not included.

Solution. The domain of f is the set of points (x, y) such that the expression under the square root is nonnegative and the denominator in the fraction is nonzero. So we need $9 - x^2 - y^2 \geq 0$ and $3y - 6 \neq 0$. The first condition is satisfied when $x^2 + y^2 \leq 9$, i.e in the disk of radius 3 centered at the origin. The second condition is satisfied when $y \neq 2$. So the domain of f is the disk of radius 3 centered at the origin, excluding the horizontal line $y = 2$. This domain is sketched below, with the boundary circle drawn solid since it is included in the domain and the line $y = 2$ drawn dotted since it is not included in the domain.



(b) The plot shown below cannot be the contour plot of any function $f(x, y)$. Explain why not.



Solution. The contours in the plot cross each other. If this were a contour plot of a function $f(x, y)$, then at the point where the contours cross, f would have to take on two different values, which is impossible for a function.

5. **[G5: Parameterization]** The sphere $x^2+y^2+z^2=8$ meets the paraboloid $2z=x^2+y^2$ in a circle. Give a parameterization of this circle. Be sure to specify a domain.

Solution. Since $x^2+y^2=2z$, we have $2z+z^2=8$ along the intersection of these surfaces. So $z^2+2z-8=(z+4)(z-2)=0$. Since $2z=x^2+y^2\geq 0$, the only solution is $z=2$. Hence the intersection is the circle $x^2+y^2=4$ in the plane $z=2$, which can be parameterized as

$$\mathbf{r}(t)=\langle 2\cos(t),2\sin(t),2\rangle,\qquad 0\leq t\leq 2\pi.$$

6. **[D1: Computing Derivatives]** Consider the function $f(x,y)=x^2y+xy^3+\cos(\pi y)$. Compute the following derivatives:

(a) $Df(1,-2)$

Solution.

$$Df(x,y)=\begin{bmatrix}f_x(x,y) & f_y(x,y)\end{bmatrix}=\begin{bmatrix}2xy+y^3 & x^2+3xy^2-\pi\sin(\pi y)\end{bmatrix}$$

so

$$Df(1,-2)=\begin{bmatrix}-12 & 13\end{bmatrix}.$$

(b) $\dfrac{\partial^2 f}{\partial x\partial y}(x,y)$

Solution.

$$\begin{aligned}\frac{\partial^2 f}{\partial x\partial y}(x,y)&=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right)\\&=\frac{\partial}{\partial x}\left(x^2+3xy^2-\pi\sin(\pi y)\right)\\&=2x+3y^2.\end{aligned}$$

(c) $f_{y^{10}x^3}(x,y)$
Hint: Is this the best order to compute this derivative in?

Solution. We can compute this derivative in the order $f_{x^3y^{10}}$ instead, since mixed partials are equal if the function is sufficiently smooth (which this one is). We have

$$f_x(x,y)=2xy+y^3,\quad f_{xx}(x,y)=2y,\quad f_{xxx}(x,y)=0.$$

So $f_{y^{10}x^3}(x,y)=0$.

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