

MATH 2551-K FINAL EXAM
PART 1
VERSION A
SPRING 2024
COVERS SECTIONS 12.1-12.6, 13.1-13.4, 14.1-14.2

EXAM KEY

Honor code statement: I will abide strictly by the Georgia Tech honor code at all times. I will not use a calculator. I will not reference any website, application, or other CAS-enabled service. I will not consult with my notes or anyone during this exam. I will not provide aid to anyone else during this exam.

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- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

| Question | Points |
|----------|--------|
| 1 | 2 |
| 2 | 2 |
| 3 | 8 |
| 4 | 9 |
| 5 | 10 |
| 6 | 9 |
| Total: | 40 |

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) The limit $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = L$ if the limit along every line through the origin is L .

TRUE

FALSE

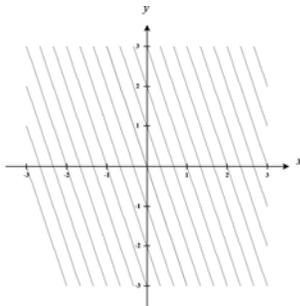
2. (2 points) For any vectors in \mathbb{R}^3 , $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

TRUE

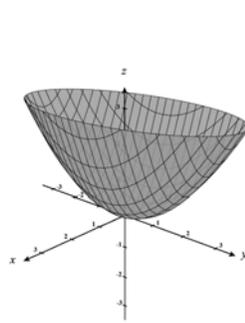
FALSE

3. (8 points) Match each function or quadric surface to its graph and contour plot.

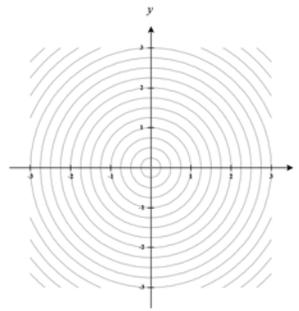
| Function | Graph | Contour Plot |
|------------------------|------------|--------------|
| $z = 3x + y$ | <u>(F)</u> | <u>(A)</u> |
| $z = \sqrt{x^2 + y^2}$ | <u>(H)</u> | <u>(C)</u> |
| $z = x^2 - y^2$ | <u>(D)</u> | <u>(G)</u> |
| $z = x^2 + y^2/4$ | <u>(B)</u> | <u>(E)</u> |



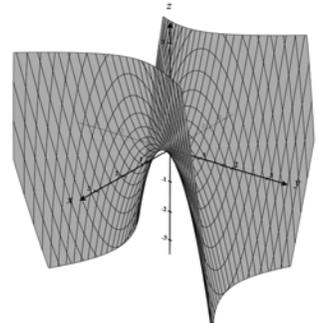
(A)



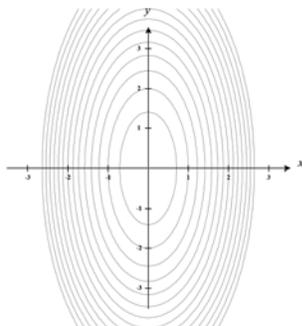
(B)



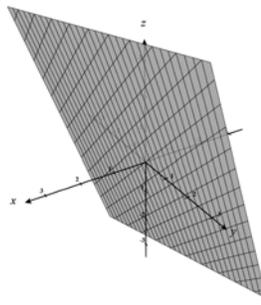
(C)



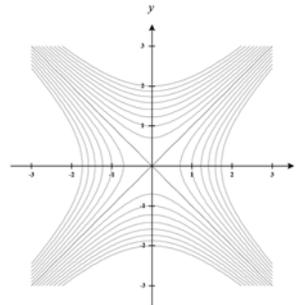
(D)



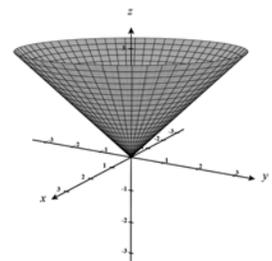
(E)



(F)



(G)



(H)

4. The sphere $x^2 + y^2 + z^2 = 6$ meets the paraboloid $z = x^2 + y^2$ in a circle.
- (a) (5 points) Give a parameterization of this circle.

Solution: Since $x^2 + y^2 = z$, we have $z + z^2 = 6$ along the intersection of these surfaces. So $z^2 + z - 6 = (z + 3)(z - 2) = 0$. Since $z = x^2 + y^2 \geq 0$, the only solution is $z = 2$. Hence the intersection is the circle $x^2 + y^2 = 2$ in the plane $z = 2$, which can be parameterized as

$$\mathbf{r}(t) = \langle \sqrt{2} \cos(t), \sqrt{2} \sin(t), 2 \rangle, \quad 0 \leq t \leq 2\pi.$$

- (b) (4 points) Find the arc length of the portion of this circle that lies between $(1, 1, 2)$ and $(-1, 1, 2)$.

Solution: First, we find the parameter values t_0 and t_1 at which the parameterization passes through these points. At $(1, 1, 2)$ we have $\sqrt{2} \cos(t_0) = 1 = \sqrt{2} \sin(t_0)$, so $t_0 = \pi/4$. At $(-1, 1, 2)$ we have $-\sqrt{2} \cos(t_1) = 1 = \sqrt{2} \sin(t_1)$, so $t_1 = 3\pi/4$.

Now we compute

$$\begin{aligned} L &= \int_{\pi/4}^{3\pi/4} |\mathbf{r}'(t)| \, dt \\ &= \int_{\pi/4}^{3\pi/4} | \langle -\sqrt{2} \sin(t), \sqrt{2} \cos(t), 0 \rangle | \, dt \\ &= \int_{\pi/4}^{3\pi/4} \sqrt{2} \, dt \\ &= \frac{\pi\sqrt{2}}{2} \end{aligned}$$

5. In this problem you will work with the planes $2x - y + 5z = 2$ and $-x + 3y + z = 4$.

(a) (6 points) Find the line of intersection of these planes.

Solution: The line of intersection lies in both planes, so its direction vector is orthogonal to both planes' normal vectors. We have $\mathbf{n}_1 = \langle 2, -1, 5 \rangle$ and $\mathbf{n}_2 = \langle -1, 3, 1 \rangle$, so a direction vector for the plane is

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 5 \\ -1 & 3 & 1 \end{vmatrix} = \langle -1 - 15, -(2 + 5), 6 - 1 \rangle = \langle -16, -7, 5 \rangle.$$

Any point on the line of intersection also lies on both planes; we will have one free variable, so choosing the point with $z = 0$ means $2x - y = 2$ and $-x + 3y = 4$. Solving this system gives $x = 2, y = 2$, so we use $P = (2, 2, 0)$ as our point.

Hence an equation of the line of intersection is $\mathbf{r}(t) = \langle 2 - 16t, 2 - 7t, 5t \rangle, t \in \mathbb{R}$.

(b) (4 points) Find a plane orthogonal to both planes that passes through the origin.

Solution: A plane orthogonal to both planes has normal vector orthogonal to both planes' normal vectors. So we take $\mathbf{n} = \langle -16, -7, 5 \rangle$. If the plane passes through the origin then we must have $d = 0$ in the standard plane equation and so we get

$$-16x - 7y + 5z = 0.$$

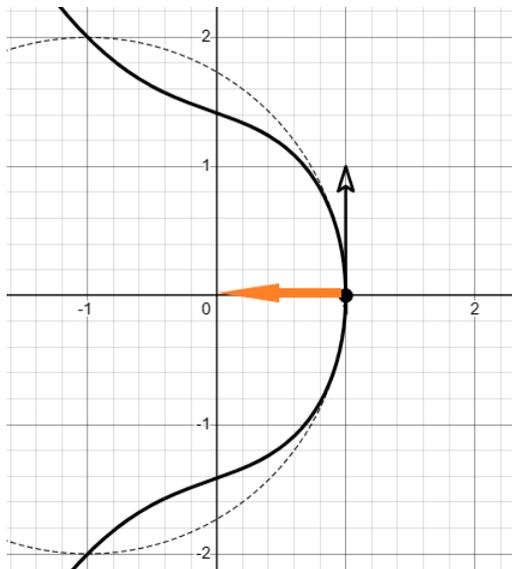
6. Consider the curve parameterized by $\mathbf{r}(t) = \langle e^t + e^{-t}, 2t \rangle$ for $t \in \mathbb{R}$.

(a) (5 points) Compute the unit tangent vector $\mathbf{T}(t)$. Fully simplify your answer.

Solution: We have

$$\begin{aligned} \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \\ &= \frac{\langle e^t - e^{-t}, 2 \rangle}{|\langle e^t - e^{-t}, 2 \rangle|} \\ &= \frac{\langle e^t - e^{-t}, 2 \rangle}{\sqrt{(e^t - e^{-t})^2 + 4}} \\ &= \frac{\langle e^t - e^{-t}, 2 \rangle}{\sqrt{e^{2t} + 2 + e^{-2t}}} \\ &= \frac{\langle e^t - e^{-t}, 2 \rangle}{e^t + e^{-t}} \end{aligned}$$

(b) (2 points) Sketch the principal unit normal vector to the solid curve drawn below at the marked point. The drawn arrow is the tangent vector at that point.



(c) (2 points) The dashed circle drawn in the graph above has radius 2 and exactly matches the curve at the marked point. What can you conclude about the curvature of the curve at the marked point?

Solution: The curvature of a circle of radius 2 is $1/2$, so since the curve and the circle match at the marked point, the curvature of the curve at the point must also be $1/2$.

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FORMULA SHEET

- Dot product: $\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1v_1 + u_2v_2 + u_3v_3$
- Dot product: $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$
- Cross product: $\langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- Cross product magnitude: $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin(\theta)$
- Arc length: $L = \int_a^b |\mathbf{r}'(t)| dt$
- Arc length function: $s(t) = \int_{t_0}^t |\mathbf{r}'(T)| dT$
- Unit tangent: $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{r}}{ds}$
- Curvature: $\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$
- Principal unit normal: $\mathbf{N} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

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MATH 2551-K FINAL EXAM
PART 2
VERSION A
SPRING 2024
COVERS SECTIONS 14.3-14.8, 15.1-15.4

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| 1 | 2 |
| 2 | 2 |
| 3 | 6 |
| 4 | 10 |
| 5 | 10 |
| 6 | 10 |
| Total: | 40 |

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) The total derivative of a function $f : \mathbb{R}^5 \rightarrow \mathbb{R}^2$ at the point (a, b, c, d, e) is a 2×5 matrix.

TRUE

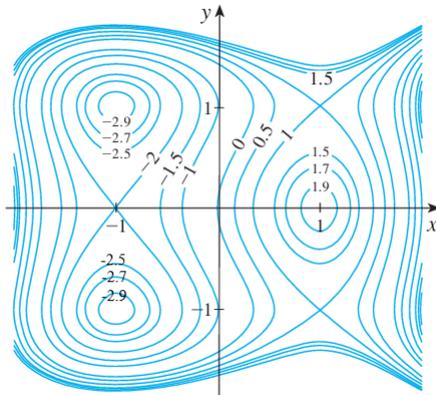
FALSE

2. (2 points) There exists a function $f(x, y)$ with continuous partial derivatives of all orders where $f_x(x, y) = 2xy = f_y(x, y)$.

TRUE

FALSE

3. (6 points) Based on the contour plot below for a function f , identify and classify all critical points of the function.



Solution: We have critical points at the centers of the concentric curves around $(1, 0)$, $(-1, -1)$, and $(-1, 1)$. These are a local max, a local min, and a local min, respectively, based on the values of the contours.

There are also saddle points at all of the places where contour lines cross: $(-1, 0)$, $(1, -1)$, and $(1, 1)$.

4. Let $f(x, y, z) = yz + x^2e^{z-y}$.

- (a) (5 points) Find an equation of the tangent plane to the level surface $f = 6$ at the point $P = (\frac{2}{\sqrt{e}}, 1, 2)$.

Solution: The equation of a tangent plane to a level surface of a function of three variables at a point $P = (x_0, y_0, z_0)$ is $\nabla f(P) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$.

$\nabla f = \langle 2xe^{z-y}, z - x^2e^{z-y}, y + x^2e^{z-y} \rangle$, so $\nabla f(P) = \langle 4\sqrt{e}, -2, 5 \rangle$.

Thus an equation of the tangent plane is

$$4\sqrt{e}\left(x - \frac{2}{\sqrt{e}}\right) - 2(y - 1) + 5(z - 2) = 0$$

or

$$(4\sqrt{e})x - 2y + 5z = 16$$

- (b) (3 points) Find the linearization $L(x, y, z)$ of f at P .

Solution: The linearization $L(x, y, z)$ at P is $L(x, y, z) = f(P) + \nabla f(P) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$. $f(P) = 6$ and we computed the rest of this in part (a).

So the linearization is

$$L(x, y, z) = 6 + 4\sqrt{e}\left(x - \frac{2}{\sqrt{e}}\right) - 2(y - 1) + 5(z - 2).$$

- (c) (2 points) Use the linearization you found to approximate the value of $f(\frac{2}{\sqrt{e}}, 1.1, 2.1)$.

Solution:

$$f\left(\frac{2}{\sqrt{e}}, 1.1, 2.1\right) \approx L\left(\frac{2}{\sqrt{e}}, 1.1, 2.1\right) = 6 + 4\sqrt{e}(0) - 2(.1) + 5(.1) = 6.3$$

5. (10 points) Find the extreme values of the function $f(x, y) = x^2 + (y - 1)^2$ on or inside the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$.

Solution: First we find the local extreme values inside the triangle. $Df(x, y) = [2x \ 2(y - 1)] = [0 \ 0]$. So there is a critical point at $(0, 1)$, which is inside the triangle.

Now we find critical points on each boundary segment.

On $x = 0$: We have $f(0, y) = (y - 1)^2$, so $f'(y) = 2(y - 1) = 0$ gives a critical point at $(0, 1)$ again.

On $y = 0$: We have $f(x, 0) = x^2 + 1$, so $f'(x) = 2x = 0$ gives a critical point at $(0, 0)$.

On $y = 2 - x$: We have $f(x, 2 - x) = x^2 + (1 - x)^2$, so $f'(x) = 2x - 2(1 - x) = 4x - 2 = 0$ yields a critical point at $(1/2, 3/2)$.

Finally, we include the endpoints $(2, 0)$ and $(0, 2)$ that have not already come up and test points.

| | | | | | |
|-----------|----------|----------|--------------|----------|----------|
| (x, y) | $(0, 1)$ | $(0, 0)$ | $(1/2, 3/2)$ | $(2, 0)$ | $(0, 2)$ |
| $f(x, y)$ | 0 | 1 | 1/2 | 5 | 1 |

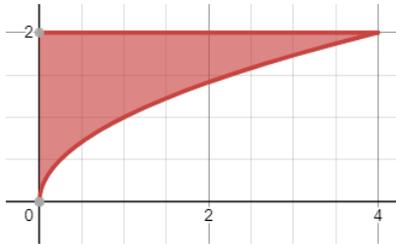
Thus the maximum value of f on this region is 5 achieved at $(2, 0)$ and the minimum value is 0 achieved at $(0, 1)$.

6. (10 points) Consider the integral

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3 + 1} dy dx.$$

Sketch the region of integration and then find the value of this integral.

Solution: We have $\sqrt{x} \leq y \leq 2, 0 \leq x \leq 4$, sketched below.



To compute the integral, we will change the order of integration.

$$\begin{aligned} \int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3 + 1} dy dx &= \int_0^2 \int_0^{y^2} \frac{1}{y^3 + 1} dx dy \\ &= \int_0^2 \frac{y^2}{y^3 + 1} dy \\ &= \frac{1}{3} \ln |y^3 + 1| \Big|_0^2 \\ &= \frac{1}{3} \ln(9) \end{aligned}$$

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FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \dots, x_n) = \langle f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n) \rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near \mathbf{a} , $L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$
- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$
- Implicit Differentiation: $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$ and $\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$.
- Directional Derivative: If \mathbf{u} is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$
- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b \rangle$
- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$
- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
 1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
 2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
 3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
 4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA$, $\text{volume}(D) = \iiint_D dV$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{\text{avg}} = \frac{\iint_R f(x, y) dA}{\text{area of } R}$

- Polar coordinates: $x = r \cos(\theta)$, $y = r \sin(\theta)$, $dA = r dr d\theta$

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MATH 2551-K FINAL EXAM
PART 3
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1. (2 points) The Jacobian determinant for a change of variables is always constant.

TRUE

FALSE

2. (2 points) Every non-constant vector field has either nonzero curl or nonzero divergence.

TRUE

FALSE

3. (3 points) Let $\mathbf{F}(x, y, z) = \langle 3x, -3y, z \rangle$ and S be the surface which is the part of the cylinder $x^2 + y^2 = 4$ between $z = -2$ and $z = 4$, oriented away from the z -axis. \mathbf{F} is not the curl of a vector field \mathbf{G} since it has positive divergence.

Which of the theorems or methods below would be appropriate to use to compute the flux of \mathbf{F} across S ? Choose all that apply (there may be only one).

Direct calculation

Fundamental Theorem of Line Integrals

Green's Theorem

Stokes' Theorem

Divergence Theorem

4. (3 points) Let $\mathbf{F}(x, y) = \langle 3x, 2x - 3y \rangle$ and C be a simple closed curve surrounding the origin with positive orientation.

Which of the theorems or methods below would be appropriate to use to compute the circulation of \mathbf{F} around C ? Choose all that apply (there may be only one).

Direct calculation

Fundamental Theorem of Line Integrals

Green's Theorem

Stokes' Theorem

Divergence Theorem

5. Let $\mathbf{F}(x, y) = \langle ye^{xy} - 3x^2, 2 + xe^{xy} \rangle$. In this problem you will work with this vector field and the curve C given by $\mathbf{r}(t) = \langle \sin(t) \cos(t), \sin(t) + t \cos(t) \rangle$, $0 \leq t \leq \pi/2$.

(a) (5 points) Show that \mathbf{F} is conservative using the curl test. Then find a potential function f for \mathbf{F} .

Solution: We have

$$\begin{aligned} \nabla \times \mathbf{F} &= \left\langle 0, 0, \frac{\partial}{\partial x}(2 + xe^{xy}) - \frac{\partial}{\partial y}(ye^{xy} - 3x^2) \right\rangle \\ &= \langle 0, 0, (0 + e^{xy} + xye^{xy}) - (e^{xy} + yxe^{xy} - 0) \rangle \\ &= \mathbf{0}, \end{aligned}$$

so \mathbf{F} is conservative.

To find a potential function for f , we take the antiderivatives of each component with respect to the corresponding variable and compare terms.

$$\begin{aligned} f(x, y) &= \int ye^{xy} - 3x^2 \, dx = e^{xy} - x^3 + g(y) \\ f(x, y) &= \int 2 + xe^{xy} \, dy = 2y + e^{xy} + h(x) \end{aligned}$$

From this we see that $g(y) = 2y$, $h(x) = -x^3$, and so a potential function for \mathbf{F} is

$$f(x, y) = e^{xy} - x^3 + 2y.$$

(b) (3 points) Compute the work done by \mathbf{F} along the curve C . Fully simplify your answer.

Solution: The endpoints of this curve are $\mathbf{r}(0) = \langle 0, 0 \rangle$ and $\mathbf{r}(\pi/2) = \langle 0, 1 \rangle$. Then by the Fundamental Theorem of Line Integrals,

$$\begin{aligned} \text{work done} &= \int_C \mathbf{F} \cdot \mathbf{T} \, ds \\ &= f(0, 1) - f(0, 0) \\ &= (e^0 - 0 + 2) - (e^0 - 0 + 0) \\ &= 2 \end{aligned}$$

6. Consider the region D in \mathbb{R}^3 given by $\sqrt{x^2 + y^2} \leq z \leq \sqrt{4 - x^2 - y^2}$ and suppose a solid occupying this region has mass density $\delta(x, y, z) = x^2 + y^2 + z^2$.
- (a) (5 points) Write an integral in cylindrical coordinates for the mass of the solid. **Do not evaluate your integral.**

Solution: The cone $z = r$ intersects the upper hemisphere $z = \sqrt{4 - r^2}$ when $r^2 = 4 - r^2$, i.e. $r = \sqrt{2}$. The density is $\delta = r^2 + z^2$. Therefore we have

$$\text{mass} = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} (r^2 + z^2)r \, dz \, dr \, d\theta.$$

- (b) (5 points) Write an integral in spherical coordinates for the mass of the solid. **Do not evaluate your integral.**

Solution: In spherical coordinates, the hemisphere is the part of $\rho = 2$ with $0 \leq \varphi \leq \pi/2$ and the region above the cone $z \geq \sqrt{x^2 + y^2}$ is the region with $\varphi \leq \pi/4$. The density is $\delta = \rho^2$. Therefore we have

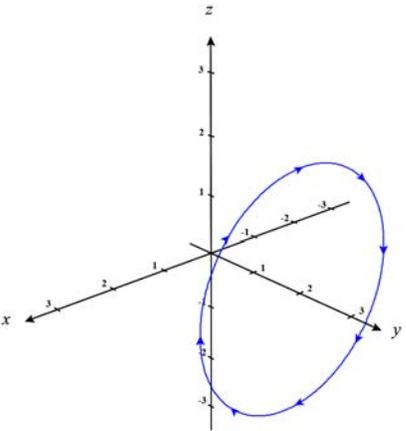
$$\text{mass} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^4 \sin(\varphi) \, d\rho \, d\varphi \, d\theta.$$

7. In this problem, you will use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{x}$, where

$$\mathbf{F} = -yz\mathbf{i} + (4y^{\cos(y^3)} + 1)\mathbf{j} + xy\mathbf{k}$$

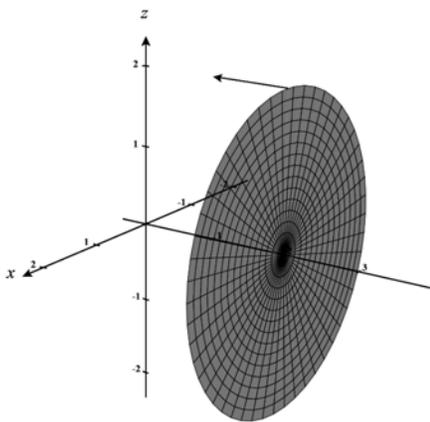
and C is the circle of radius 2 in the plane $y = 2$ centered at $(0, 2, 0)$, oriented clockwise around the y -axis (moving from the \mathbf{i} direction towards the \mathbf{k} direction).

(a) (2 points) Sketch C in \mathbb{R}^3 , including an orientation.

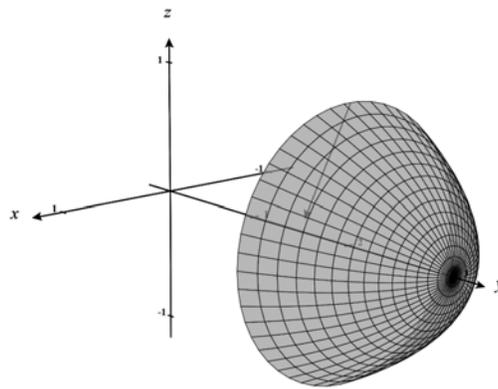


Solution:

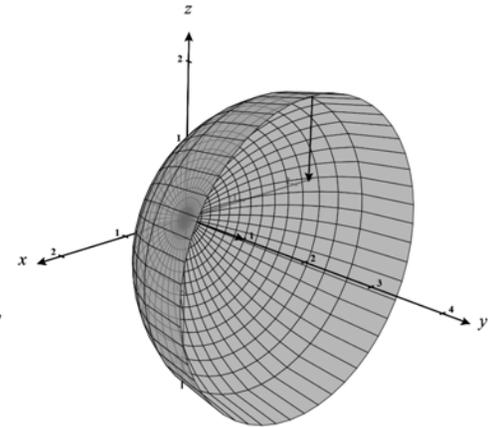
(b) (2 points) Which surface pictured below can be used to apply Stokes' Theorem here? Be careful with orientations.



$S_1 : y = 2; x^2 + z^2 \leq 4,$
oriented in the $-\mathbf{j}$ direction



$S_2 : y = 3 - x^2 - z^2; y \geq 2,$
oriented towards the y -axis



$S_3 : x^2 + (y - 2)^2 + z^2 = 4;$
 $y \leq 2,$
oriented towards the y -axis

(c) (2 points) Compute $\nabla \times \mathbf{F}$.

Solution:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yz & 4y^{\cos(y^3)} + 1 & xy \end{vmatrix} = \langle x, -2y, z \rangle.$$

7. continued.

- (d) (6 points) Apply Stokes' Theorem to find the circulation using the surface you chose in (b).

Solution: Note that the unit normal to this surface is $-\mathbf{j}$ at all points and $y = 2$ at all points on the surface.

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T} &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma \\ &= \iint_S \langle x, -2y, z \rangle \cdot \langle 0, -1, 0 \rangle \, d\sigma \\ &= \iint_S 2y \, d\sigma \\ &= \iint_S 4 \, d\sigma \\ &= 4 \cdot \text{Area}(S) \\ &= (4)\pi(2)^2 \\ &= 16\pi.\end{aligned}$$

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FORMULA SHEET

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- Volume(D) = $\iiint_D dV$, $f_{avg} = \frac{\iiint_D f(x, y, z) dV}{\text{volume of } D}$, Mass: $M = \iiint_D \delta dV$
- Cylindrical coordinates: $x = r \cos(\theta)$, $y = r \sin(\theta)$, $z = z$, $dV = r dz dr d\theta$
- Spherical coordinates: $x = \rho \sin(\phi) \cos(\theta)$, $y = \rho \sin(\phi) \sin(\theta)$, $z = \rho \cos(\phi)$,
 $dV = \rho^2 \sin(\phi) d\rho d\phi d\theta$
- First moments (3D solid): $M_{yz} = \iiint_D x\delta dV$, $M_{xz} = \iiint_D y\delta dV$, $M_{xy} = \iiint_D z\delta dV$
- Center of mass (3D solid): $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$
- Substitution for double integrals: If R is the image of G under a coordinate transformation $\mathbf{T}(u, v) = \langle x(u, v), y(u, v) \rangle$ then

$$\iint_R f(x, y) dx dy = \iint_G f(\mathbf{T}(u, v)) |\det D\mathbf{T}(u, v)| du dv.$$

- Scalar line integral: $\int_C f(x, y, z) ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt$
- Tangential vector line integral: $\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$
- Normal vector line integral: $\int_C \mathbf{F}(x, y) \cdot \mathbf{n} ds = \int_C P dy - Q dx = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t) \rangle dt$.
- Fundamental Theorem of Line Integrals: $\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Curl (Mixed Partials) Test: $\mathbf{F} = \nabla f$ if $\text{curl } \mathbf{F} = \mathbf{0} \Leftrightarrow P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.
- $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ $\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$ $\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA \qquad \int_C \mathbf{F} \cdot \mathbf{n} ds = \iint_R (\nabla \cdot \mathbf{F}) dA.$$

- Surface Area = $\iint_S 1 d\sigma$
- Scalar surface integral: $\iint_S f(x, y, z) d\sigma = \iint_R f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA$
- Flux surface integral: $\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_S \mathbf{F} \cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$
- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and \mathbf{F} is a vector field whose components have continuous partial derivatives on an open region containing S , then

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and \mathbf{F} is a vector field whose components have continuous partial derivatives on D , then

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \mathbf{F} dV.$$

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