MATH 2551-K FINAL EXAM
PART 1
VERSION A
SPRING 2024
COVERS SECTIONS 12.1-12.6, 13.1-13.4, 14.1-14.2

Full name: _____ **GT ID:**_____

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- Please show your work.

- Good luck! Write yourself a message of encouragement on the front page!

Question	Points
1	2
2	2
3	8
4	9
5	10
6	9
Total:	40

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) The limit $\lim_{(x,y)\to(0,0)} f(x,y) = L$ if the limit along every line through the origin is L.

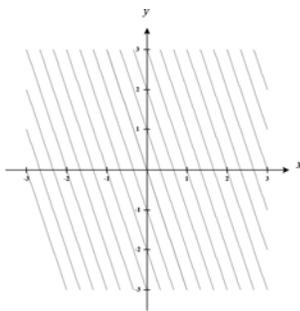   ○ **TRUE**   ○ **FALSE**

2. (2 points) For any vectors in $\mathbb{R}^3$, $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
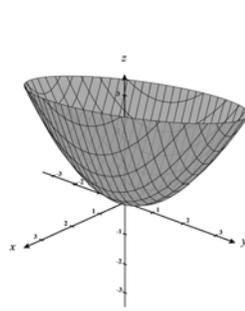
   ○ **TRUE**   ○ **FALSE**

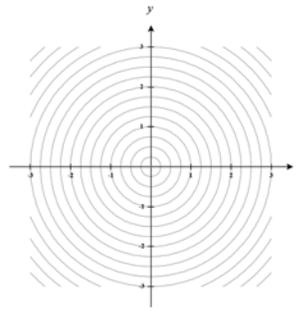3. (8 points) Match each function or quadric surface to its graph and contour plot.

Function	Graph	Contour Plot
$z = 3x + y$	_____	_____
$z = \sqrt{x^2 + y^2}$	_____	_____
$z = x^2 - y^2$	_____	_____
$z = x^2 + y^2/4$	_____	_____
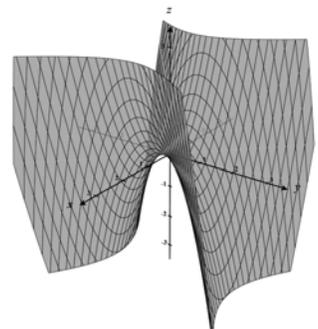
(A)   (B)   (C)   (D)
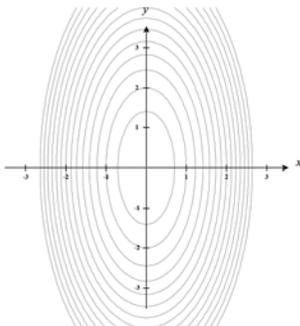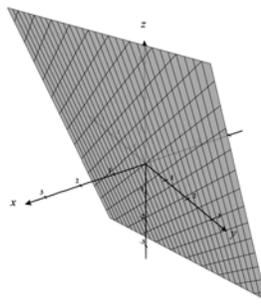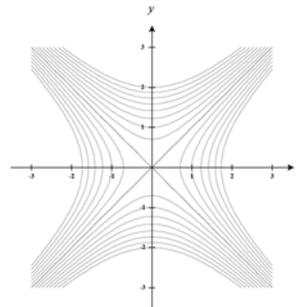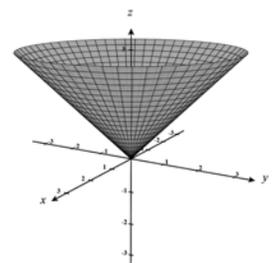

(E)   (F)   (G)   (H)

4. The sphere $x^2 + y^2 + z^2 = 6$ meets the paraboloid $z = x^2 + y^2$ in a circle.

(a) (5 points) Give a parameterization of this circle.

(b) (4 points) Find the arc length of the portion of this circle that lies between $(1, 1, 2)$ and $(-1, 1, 2)$.

5. In this problem you will work with the planes $2x - y + 5z = 2$ and $-x + 3y + z = 4$.

   (a) (6 points) Find the line of intersection of these planes.

   (b) (4 points) Find a plane orthogonal to both planes that passes through the origin.

6. Consider the curve parameterized by $\mathbf{r}(t) = \langle e^t + e^{-t}, 2t \rangle$ for $t \in \mathbb{R}$.

   (a) (5 points) Compute the unit tangent vector $\mathbf{T}(t)$. Fully simplify your answer.

   (b) (2 points) Sketch the principal unit normal vector to the solid curve drawn below at the marked point. The drawn arrow is the tangent vector at that point.
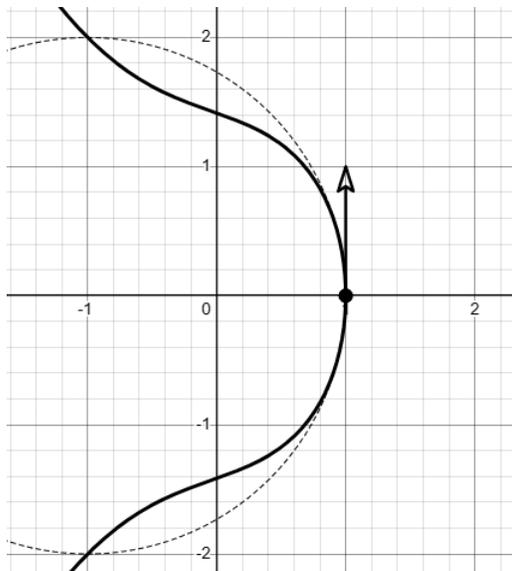
   (c) (2 points) The dashed circle drawn in the graph above has radius 2 and exactly matches the curve at the marked point. What can you conclude about the curvature of the curve at the marked point?

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FORMULA SHEET

- Dot product: $\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$

- Dot product: $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta)$

- Cross product: $\langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

- Cross product magnitude: $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin(\theta)|$

- Arc length: $L = \int_a^b |\mathbf{r}'(t)| \, dt$

- Arc length function: $s(t) = \int_{t_0}^t |\mathbf{r}'(T)| \, dT$

- Unit tangent: $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{d\mathbf{r}}{ds}$

- Curvature: $\kappa = \left| \dfrac{d\mathbf{T}}{ds} \right| = \dfrac{1}{|\mathbf{v}|} \left| \dfrac{d\mathbf{T}}{dt} \right| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

- Principal unit normal: $\mathbf{N} = \dfrac{1}{\kappa} \dfrac{d\mathbf{T}}{ds} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

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MATH 2551-K FINAL EXAM
PART 2
VERSION A
SPRING 2024
COVERS SECTIONS 14.3-14.8, 15.1-15.4

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Question	Points
1	2
2	2
3	6
4	10
5	10
6	10
Total:	40

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) The total derivative of a function $f : \mathbb{R}^5 \to \mathbb{R}^2$ at the point (a, b, c, d, e) is a 2×5 matrix.

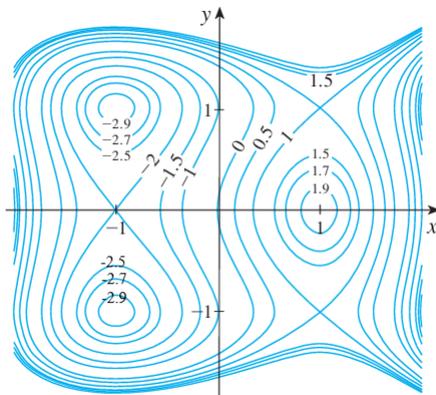   ○ **TRUE**   ○ **FALSE**

2. (2 points) There exists a function $f(x, y)$ with continuous partial derivatives of all orders where $f_x(x, y) = 2xy = f_y(x, y)$.

   ○ **TRUE**   ○ **FALSE**

3. (6 points) Based on the contour plot below for a function f, identify and classify all critical points of the function.

4. Let $f(x, y, z) = yz + x^2 e^{z-y}$.

    (a) (5 points) Find an equation of the tangent plane to the level surface $f = 6$ at the point $P = (\dfrac{2}{\sqrt{e}}, 1, 2)$.

    (b) (3 points) Find the linearization $L(x, y, z)$ of f at P.

    (c) (2 points) Use the linearization you found to approximate the value of $f(\dfrac{2}{\sqrt{e}}, 1.1, 2.1)$.

5. (10 points) Find the extreme values of the function $f(x, y) = x^2 + (y - 1)^2$ on or inside the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$.

6. (10 points) Consider the integral

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3 + 1} \, dy \, dx.$$

Sketch the region of integration and then find the value of this integral.

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FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n) \rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \cdots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \cdots & (f_2)_{x_n} \\ \vdots & \ddots & \cdots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \cdots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b \rangle$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
    1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
    2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
    3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
    4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA, \quad \text{volume}(D) = \iiint_D dV$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y)dA}{\text{area of } R}$

- Polar coordinates: $x = r\cos(\theta), \quad y = r\sin(\theta), \quad dA = r \ dr \ d\theta$

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MATH 2551-K FINAL EXAM
PART 3
VERSION A
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COVERS SECTIONS 15.5-15.8, 16.1-16.8

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Question	Points
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2	2
3	3
4	3
5	8
6	10
7	12
Total:	40

For problems 1-2 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) The Jacobian determinant for a change of variables is always constant.

   ○ **TRUE**   ○ **FALSE**

2. (2 points) Every non-constant vector field has either nonzero curl or nonzero divergence.

   ○ **TRUE**   ○ **FALSE**

3. (3 points) Let $\mathbf{F}(x, y, z) = \langle 3x, -3y, z \rangle$ and S be the surface which is the part of the cylinder $x^2 + y^2 = 4$ between $z = -2$ and $z = 4$, oriented away from the z-axis. $\mathbf{F}$ is not the curl of a vector field $\mathbf{G}$ since it has positive divergence.

   Which of the theorems or methods below would be appropriate to use to compute the flux of $\mathbf{F}$ across S? Choose all that apply (there may be only one).

   ○ Direct calculation
   ○ Fundamental Theorem of Line Integrals
   ○ Green's Theorem
   ○ Stokes' Theorem
   ○ Divergence Theorem

4. (3 points) Let $\mathbf{F}(x, y) = \langle 3x, 2x - 3y \rangle$ and C be a simple closed curve surrounding the origin with positive orientation.

   Which of the theorems or methods below would be appropriate to use to compute the circulation of $\mathbf{F}$ around C? Choose all that apply (there may be only one).

   ○ Direct calculation
   ○ Fundamental Theorem of Line Integrals
   ○ Green's Theorem
   ○ Stokes' Theorem
   ○ Divergence Theorem

5. Let $\mathbf{F}(x, y) = \langle ye^{xy} - 3x^2, 2 + xe^{xy} \rangle$. In this problem you will work with this vector field and the curve C given by $\mathbf{r}(t) = \langle \sin(t)\cos(t), \sin(t) + t\cos(t) \rangle$, $0 \leq t \leq \pi/2$.

   (a) (5 points) Show that $\mathbf{F}$ is conservative using the curl test. Then find a potential function f for $\mathbf{F}$.

   (b) (3 points) Compute the work done by $\mathbf{F}$ along the curve C. Fully simplify your answer.

6. Consider the region D in $\mathbb{R}^3$ given by $\sqrt{x^2 + y^2} \leq z \leq \sqrt{4 - x^2 - y^2}$ and suppose a solid occupying this region has mass density $\delta(x, y, z) = x^2 + y^2 + z^2$.

(a) (5 points) Write an integral in cylindrical coordinates for the mass of the solid. **Do not evaluate your integral.**

(b) (5 points) Write an integral in spherical coordinates for the mass of the solid. **Do not evaluate your integral.**

7. In this problem, you will use Stokes' Theorem to evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F} = -yz\mathbf{i} + (4y^{\cos(y^3)} + 1)\mathbf{j} + xy\mathbf{k}$$
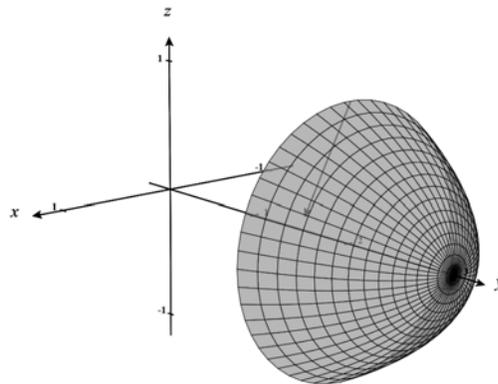
and C is the circle of radius 2 in the plane $y = 2$ centered at $(0, 2, 0)$, oriented clockwise around the y-axis (moving from the $\mathbf{i}$ direction towards the $\mathbf{k}$ direction).

(a) (2 points) Sketch C in $\mathbb{R}^3$, including an orientation.
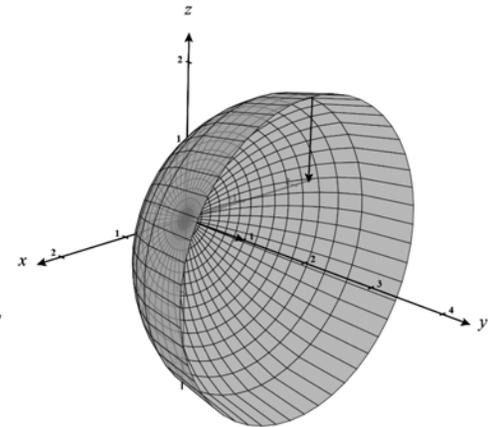
(b) (2 points) Which surface pictured below can be used to apply Stokes' Theorem here? Be careful with orientations.
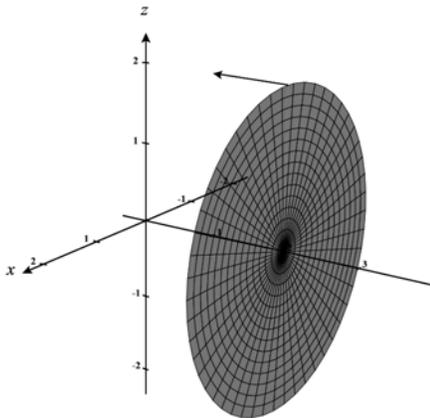
○ $S_1 : y = 2; x^2 + z^2 \leq 4$, oriented in the $-\mathbf{j}$ direction

○ $S_2 : y = 3 - x^2 - z^2; y \geq 2$, oriented towards the y-axis

○ $S_3 : x^2 + (y - 2)^2 + z^2 = 4$; $y \leq 2$, oriented towards the y-axis

(c) (2 points) Compute $\nabla \times \mathbf{F}$.

7. continued.

(d) (6 points) Apply Stokes' Theorem to find the circulation using the surface you chose in (b).

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FORMULA SHEET

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Volume$(D) = \iiint_D dV$,　　$f_{avg} = \dfrac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$,　　Mass: $M = \iiint_D \delta \, dV$

- Cylindrical coordinates: $x = r\cos(\theta)$,　$y = r\sin(\theta)$, $z = z$, $dV = r \, dz \, dr \, d\theta$

- Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$, $dV = \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$

- First moments (3D solid): $M_{yz} = \iiint_D x\delta \, dV$, $M_{xz} = \iiint_D y\delta \, dV$, $M_{xy} = \iiint_D z\delta \, dV$

- Center of mass (3D solid): $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right)$

- Substitution for double integrals: If R is the image of G under a coordinate transformation $\mathbf{T}(u,v) = \langle x(u,v), y(u,v) \rangle$ then

$$\iint_R f(x,y) \, dx \, dy = \iint_G f(\mathbf{T}(u,v))|\det D\mathbf{T}(u,v)| \, du \, dv.$$

- Scalar line integral: $\int_C f(x,y,z) \, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$

- Tangential vector line integral: $\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$

- Normal vector line integral: $\int_C \mathbf{F}(x,y) \cdot \mathbf{n} \, ds = \int_C P \, dy - Q \, dx = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t) \rangle \, dt$.

- Fundamental Theorem of Line Integrals: $\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B

- Curl (Mixed Partials) Test: $\mathbf{F} = \nabla f$ if curl $\mathbf{F} = \mathbf{0} \Leftrightarrow P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.

- $\nabla = \langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \rangle$　　　　div $\mathbf{F} = \nabla \cdot \mathbf{F}$　　　　curl $\mathbf{F} = \nabla \times \mathbf{F}$

- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA \qquad \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R (\nabla \cdot \mathbf{F}) \, dA.$$

- Surface Area$= \iint_S 1 \, d\sigma$
- Scalar surface integral: $\iint_S f(x,y,z) \, d\sigma = \iint_R f(\mathbf{r}(u,v)) \, |\mathbf{r}_u \times \mathbf{r}_v| dA$
- Flux surface integral: $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S \mathbf{F} \cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$
- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV.$$

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