MATH 2551-K FINAL EXAM PART 1

VERSION A
FALL 2023
COVERS SECTIONS 12.1-12.6, 13.1-13.4, 14.1

Full name: ______________________________ **GT ID:**____________________

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For problems 1-3 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors in $\mathbb{R}^3$, then $|\mathbf{u} \times \mathbf{v}| = 1$.

   ✓ **TRUE**
   ◯ **FALSE**

2. All surfaces in $\mathbb{R}^3$ are quadric surfaces.

   ◯ **TRUE**
   ✓ **FALSE**

3. Level surfaces of the function $f(x, y, z) = \sqrt{x^2 + y^2}$ are:
   ◯ Circles centered at the origin
   ◯ Spheres centered at the origin
   ✓ **Cylinders centered around the z-axis**
   ◯ Upper hemispheres centered at the origin
   ◯ None of the above

4. Which of the following planes is parallel to the plane $y = -2 - 2x + 4z$?
   ◯ $x + y + 2z = 1$
   ✓ $4x + 2y - 8z = -1$
   ◯ $2x + y + 4z = -2$
   ◯ $2x + 4z = -2$
   ◯ None of the above

5. Find the length of the portion of the helix $\mathbf{r}(t) = \langle 2\sin(t), 5t, 2\cos(t)\rangle$ between $(0,0,2)$ and $(0, 5\pi, -2)$.

Solution: We need to compute $\|\mathbf{r}'(t)|$ to find this arc length. We have $\mathbf{r}'(t) = \langle 2\cos(t), 5, -2\sin(t)\rangle$, so $|\mathbf{r}'(t)| = \sqrt{(2\cos(t))^2 + 5^2 + (-2\sin(t))^2} = \sqrt{29}$. The given parameterization reaches $(0,0,2)$ when $t = 0$ and $(0, 5\pi, -2)$ when $t = \pi$.

Thus we have

$$\begin{aligned} \text{length} &= \int_0^\pi |\mathbf{r}'(t)|\ dt \\ &= \int_0^\pi \sqrt{29}\ dt \\ &= \sqrt{29}\pi. \end{aligned}$$

6. (a) Find an equation of the plane perpendicular to $\mathbf{n} = \mathbf{i} + \mathbf{j}$ that passes through the point $P = (1, 2, 3)$.

Solution: This plane has normal vector $\langle 1, 1, 0\rangle$ and passes through $(1, 2, 3)$, so it has the equation

$$(x-1)+(y-2)=0 \qquad \text{or} \qquad x+y=3.$$

(b) Find an equation of the plane parallel to the line $\ell(t) = \langle t+1, t-2, 4\rangle$ that passes through the point $Q = (2, 3, 4)$.

Solution: This plane has normal vector $\langle 1, 1, 0\rangle$ and passes through $(2, 3, 4)$, so it has the equation

$$(x-2)+(y-3)=0 \qquad \text{or} \qquad x+y=5.$$

(c) Use your work in parts (a) and (b) to explain why there is no plane with normal vector parallel to $\langle 1, 1, 0\rangle$ that contains both the points P and Q.

Solution: The standard form of such a plane is $x + y = d$ for some $d \in \mathbb{R}$. We see in part (a) that to contain the point P, $d = 3$, while in part (b) we see that to contain the point Q, $d = 5$. Since d can't be both 3 and 5, no such plane exists.

7. Consider the curve parameterized by $\mathbf{r}(t) = \langle \sqrt{7}e^t, e^t\cos(t), e^t\sin(t)\rangle$ for $t \in \mathbb{R}$.

(a) Compute the unit tangent vector $\mathbf{T}(t)$.

Solution: We have $\mathbf{r}(t) = e^t\langle\sqrt{7}, \cos(t), \sin(t)\rangle$, so by the product rule

$$\begin{aligned}\mathbf{r}'(t) &= e^t\langle\sqrt{7}, \cos(t), \sin(t)\rangle + e^t\langle 0, -\sin(t), \cos(t)\rangle\\ &= e^t\langle\sqrt{7}, \cos(t)-\sin(t), \cos(t)+\sin(t)\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}|\mathbf{r}'(t)| &= e^t\sqrt{7+(\cos(t)-\sin(t))^2+(\cos(t)+\sin(t))^2}\\ &= e^t\sqrt{7+\cos^2(t)-2\cos(t)\sin(t)+\sin^2(t)+\cos^2(t)+2\cos(t)\sin(t)+\sin^2(t)}\\ &= e^t\sqrt{7+2\cos^2(t)+2\sin^2(t)}\\ &= e^t\sqrt{9}\\ &= 3e^t\end{aligned}$$

Combining these, we get that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{3}\langle\sqrt{7}, \cos(t)-\sin(t), \cos(t)+\sin(t)\rangle.$$

(b) Compute the curvature $\kappa(t)$.

Solution: First we compute $\mathbf{T}'(t)$:

$$\mathbf{T}'(t) = \frac{1}{3}\langle 0, -\sin(t)-\cos(t), -\sin(t)+\cos(t)\rangle$$

Then we have

$$\begin{aligned}|\mathbf{T}'(t)| &= \frac{1}{3}\sqrt{0^2+(-\sin(t)-\cos(t))^2+(-\sin(t)+\cos(t))^2}\\ &= \frac{1}{3}\sqrt{\cos^2(t)-2\cos(t)\sin(t)+\sin^2(t)+\cos^2(t)+2\cos(t)\sin(t)+\sin^2(t)}\\ &= \frac{1}{3}\sqrt{2\cos^2(t)+2\sin^2(t)}\\ &= \frac{\sqrt{2}}{3}.\end{aligned}$$

Therefore

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{2}/3}{3e^t} = \frac{\sqrt{2}}{9e^t}.$$

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FORMULA SHEET

- $\langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle = u_1v_1 + u_2v_2 + u_3v_3$
- $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$
- $\langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin(\theta)|$
- $L = \int_a^b |\mathbf{r}'(t)|\ dt$
- $s(t) = \int_{t_0}^t |\mathbf{r}'(T)|\ dT$
- $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{r}}{ds}$
- $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$
- $\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

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MATH 2551-K FINAL EXAM PART 2

VERSION A
FALL 2023
COVERS SECTIONS 14.2-14.8, 15.1-15.4

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1. (2 points) The total derivative of a function $f : \mathbb{R}^3 \to \mathbb{R}^5$ at the point (a, b, c) is a 5×3 matrix.

   ✓ **TRUE**
   ○ **FALSE**

2. (2 points) Any surface that is a graph of a function of two variables $z = f(x, y)$ can be thought of as a level surface of a function of 3 variables.

   ✓ **TRUE**
   ○ **FALSE**

3. (3 points) Compute the rate of change of the function $f(x, y, z) = 3xy + z^2$ at the point $(1, 0, 1)$ in the direction of $\langle 2, 1, 2\rangle$.

   ○ 5
   ○ 5/3
   ○ 7
   ✓ 7/3
   ○ None of the above.

4. (3 points) Find the linearization of the function $f(x, y) = \sqrt{x^2 + y}$ at the point $(2, 5)$.

   ✓ $L(x, y) = 3 + \frac{2}{3}(x - 2) + \frac{1}{6}(y - 5)$
   ○ $L(x, y) = \frac{2}{3}(x - 2) + \frac{1}{6}(y - 5)$
   ○ $L(x, y) = \frac{1}{6}(x - 2) + \frac{1}{6}(y - 5)$
   ○ $L(x, y) = 3 + \frac{x}{\sqrt{x^2 + y}}(x - 2) + \frac{1}{2\sqrt{x^2 + y}}(y - 5)$
   ○ None of the above.

5. (10 points) Find and classify the critical points of the function $f(x, y) = 5x^3 - 3xy + 5y^3$.

Solution: To find the critical points, we set $Df(x, y) = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

$$Df(x, y) = \begin{bmatrix} 15x^2 - 3y & -3x + 15y^2 \end{bmatrix},$$

so we have $15x^2 - 3y = 0$ and $-3x + 15y^2 = 0$. Simplifying gives $y = 5x^2$ and $x = 5y^2$. We now substitute the first equation into the second.

$$\begin{aligned} x &= 5(5x^2)^2 \\ x &= 125x^4 \\ 0 &= x(1 - 125x^3) \end{aligned}$$

Now either $x = 0$ or $(1 - 125x^3) = 0$. In the first case, we have $y = 5(0)^2 = 0$ and in the second case we have $x = 1/5$ and $y = 5(1/5)^2 = 1/5$. So the two critical points of f are $(0, 0)$ and $(1/5, 1/5)$.

We now classify the critical points.

$$Hf(x, y) = \begin{bmatrix} 30x & -3 \\ -3 & 30y \end{bmatrix}.$$

Thus at $(0, 0)$ we have

$$\det(Hf(0, 0)) = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0,$$

so by the Second Derivative Test $(0, 0)$ is the location of a saddle point of f.

At $(1/5, 1/5)$ we have

$$\det(Hf(1/5, 1/5)) = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 27 > 0$$

and $f_{xx} = 6 > 0$, so by the Second Derivative Test $(1/5, 1/5)$ is the location of a local minimum of f.

6. For each limit below, either compute its value or show that it does not exist.

(a) (2 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+3y^4+1}$

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+3y^4+1} = \frac{0(0)^2}{0^2+3(0)^4+1} = 0$$

(b) (4 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^2+3y^4}$

Hint: Try converting to polar coordinates and taking the limit as $r \to 0$

Solution: After converting to polar coordinates and taking the limit as $r \to 0$, we have

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^2+3y^4} &= \lim_{r\to 0} \frac{r^3\cos^3(\theta)r^2\sin^2(\theta)}{r^2\cos^2(\theta)+3r^4\sin^4(\theta)} \\
&= \lim_{r\to 0} \frac{r^3\cos^3(\theta)\sin^2(\theta)}{\cos^2(\theta)+3r^2\sin^4(\theta)} \\
&= \frac{(0)\cos^3(\theta)\sin^2(\theta)}{\cos^2(\theta)+3(0)^2\sin^4(\theta)} \\
&= \frac{0}{\cos^2(\theta)+0} \\
&= 0
\end{aligned}$$

(c) (4 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+3y^4}$

Solution: Along the line $x = 0$ through $(0,0)$, we have

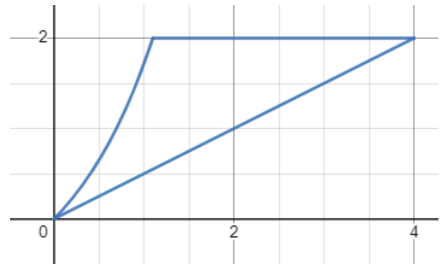
$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+3y^4} = \lim_{(0,y)\to(0,0)} \frac{xy^2}{x^2+3y^4} = \lim_{y\to 0} \frac{0}{0+3y^4} = 0.$$

Along the parabola $x = y^2$ through $(0,0)$, we have

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+3y^4} = \lim_{(y^2,y)\to(0,0)} \frac{xy^2}{x^2+3y^4} = \lim_{y\to 0} \frac{y^4}{y^4+3y^4} = \lim_{y\to 0} \frac{1}{4} = \frac{1}{4}.$$

Since these limits are different, the overall limit does not exist by the two-path test.

7. (10 points) The region R bounded by $y = e^x - 1, y = 2$, and $y = x/2$ is shown to the right. Write an iterated integral or sum of iterated integrals for the double integral $\iint_R e^x \, dA$, using either order of integration, then compute your integral. Fully simplify your answer.



Solution: This region is horizontally but not vertically simple, so we integrate in the $dx\ dy$ order. At each fixed y, a slice is bounded on the left by $y = e^x - 1$ and on the right by $y = x/2$. We need bounds on x, so we solve to get $x = \ln(y+1)$ and $x = 2y$, respectively. The y-values in the region vary from 0 to 2, so we have

$$\begin{aligned}
\iint_r e^x \ dA &= \int_0^2 \int_{\ln(y+1)}^{2y} e^x \ dx \ dy \\
&= \int_0^2 e^{2y} - e^{\ln(y+1)} \ dy \\
&= \int_0^2 e^{2y} - y - 1 \ dy \\
&= \frac{1}{2}e^{2y} - \frac{1}{2}y^2 - y|_0^2 \\
&= \frac{1}{2}e^4 - 2 - 2 - \frac{1}{2} + 0 + 0 \\
&= \frac{1}{2}(e^4 - 9)
\end{aligned}$$

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FORMULA SHEET

- Total Derivative: For $\mathbf{f}(x_1, \ldots, x_n) = \langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n) \rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \ldots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \ldots & (f_2)_{x_n} \\ \vdots & \ddots & \ldots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \ldots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b \rangle$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c \rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$
- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA, \quad \text{volume}(D) = \iiint_D dV$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y)dA}{\text{area of } R}$

- Polar coordinates: $x = r\cos(\theta), \quad y = r\sin(\theta), \quad dA = r\ dr\ d\theta$

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MATH 2551-K FINAL EXAM PART 3

VERSION A
FALL 2023
COVERS SECTIONS 15.1-15.8, 16.1-16.8

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1. (2 points) Every constant vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is conservative.

   ✓ **TRUE**

   ◯ **FALSE**

2. (2 points) If $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = 0$, then $\mathbf{F} = 0$.

   ◯ **TRUE**

   ✓ **FALSE**

3. (3 points) Let $\mathbf{F}(x, y, z) = \langle 3x, -3y, 0 \rangle$ and S be the surface which is the part of the cylinder $y^2 + z^2 = 4$ between $x = -1$ and $x = 10$, oriented away from the x-axis. $\mathbf{F}$ is the curl of a vector field $\mathbf{F}$. Which of the theorems below would be appropriate to use to compute the flux of $\mathbf{F}$ across S?

   ◯ Fundamental Theorem of Line Integrals

   ◯ Green's Theorem (circulation)

   ◯ Green's Theorem (flux)

   ✓ **Stokes' Theorem**

   ◯ Divergence Theorem

4. (3 points) Let $\mathbf{F}(x, y) = \langle 3x, -3y \rangle$ and C be a simple closed curve surrounding the origin with positive orientation. Which of the theorems below would be appropriate to use to compute the flux of $\mathbf{F}$ across C?

   ◯ Fundamental Theorem of Line Integrals

   ◯ Green's Theorem (circulation)

   ✓ **Green's Theorem (flux)**

   ◯ Stokes' Theorem

   ◯ Divergence Theorem

5. (14 points) Let $\mathbf{F}(x,y) = \langle y - 3x^2, 2\rangle$ and $\mathbf{F}(x,y) = \langle y^2 e^{xy}, (1+xy)e^{xy}\rangle$. In this problem you will work with these vector fields and the curve C that is the portion of the parabola $y = 4 - x^2$ starting at $(0,4)$ and ending at $(2,0)$.

(a) Is $\mathbf{F}$ conservative? If so, find a function $f : \mathbb{R}^2 \to \mathbb{R}$ so that $\mathbf{F} = \nabla f$.

Solution: We have $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \dfrac{\partial}{\partial x}(2) - \dfrac{\partial}{\partial y}(y - 3x^2) = 0 - 1 \neq 0$, so $\mathbf{F}$ is not conservative.

(b) Is $\mathbf{F}$ conservative? If so, find a function $g : \mathbb{R}^2 \to \mathbb{R}$ so that $\mathbf{F} = \nabla g$.

Solution: We have

$$\begin{aligned}(\nabla \times \mathbf{F}) \cdot \mathbf{k} &= \frac{\partial}{\partial x}((1+xy)e^{xy}) - \frac{\partial}{\partial y}(y^2 e^{xy}) \\ &= ye^{xy} + y(1+xy)e^{xy} - (2ye^{xy} + xy^2 e^{xy}) \\ &= 0\end{aligned}$$

So $\mathbf{F}$ is conservative. To find a potential g, we take an antiderivative:

$$g(x,y) = \int g_x(x,y)\, dx = \int y^2 e^{xy}\, dx = ye^{xy} + h(y).$$

We can determine $h(y)$ by comparing y-partial derivatives:

$$g_y(x,y) = (1+xy)e^{xy} = e^{xy} + xye^{xy} + h'(y),$$

so $h'(y) = 0$ and so $h(y) = C$.
Therefore a potential function g for $\mathbf{F}$ is $g(x,y) = ye^{xy} + C$.

(c) Compute the work done by $\mathbf{F}$ along the curve C. Fully simplify your answer.

Solution: We parameterize and apply our formula for work done. C can be parameterized by $\mathbf{r}(t) = \langle t, 4 - t^2 \rangle$ for $0 \leq t \leq 2$. Then $\mathbf{r}'(t) = \langle 1, -2t \rangle$ and $\mathbf{F}(\mathbf{r}(t)) = \langle 4 - t^2 - 3t^2, 2 \rangle = \langle 4 - 4t^2, 2 \rangle$.
Thus we have

$$\begin{aligned}
\text{work done} &= \int_C \mathbf{F} \cdot \mathbf{T}\, ds \\
&= \int_0^2 \langle 4 - 4t^2, 2 \rangle \cdot \langle 1, -2t \rangle\, dt \\
&= \int_0^2 4 - 4t - 4t^2\, dt \\
&= 4t - 2t^2 - \frac{4}{3}t^3 |_0^2 \\
&= 8 - 8 - \frac{32}{3} = -\frac{32}{3}
\end{aligned}$$

(d) Compute the work done by $\mathbf{F}$ along the curve C. Fully simplify your answer.

Solution: By the Fundamental Theorem of Line Integrals,

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathbf{T}\, ds &= g(2, 0) - g(0, 4) \\
&= 0e^0 - 4e^0 \\
&= -4
\end{aligned}$$

6. Let D be the sphere of radius 3 centered at the origin in $\mathbb{R}^3$. The volume of this sphere is 36π. Suppose that the density of a liquid filling this sphere is $\delta(x, y, z) = 1.2x$ kilograms per cubic meter.

(a) (4 points) Write an integral expression for the mass of this sphere filled with liquid. **Do not evaluate your integral expression.**

Solution: The mass is

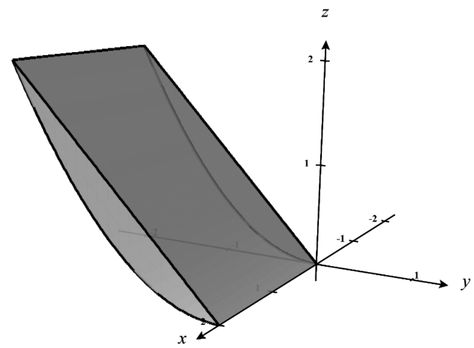
$$\begin{aligned}\iiint_D \delta(x,y,z)\,dV &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} 1.2x\;dz\;dy\;dx \\ &= \int_0^{2\pi}\int_0^{3}\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} 1.2r^2\cos(\theta)\;dz\;dr\;d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{3} 1.2\rho^3\sin^2(\varphi)\cos(\theta)\;d\rho\;d\varphi\;d\theta\end{aligned}$$

(b) (4 points) Write an integral expression for the average distance of a point in this sphere from the origin. **Do not evaluate your integral expression.**

Solution: The average distance of a point in this sphere from the origin is:

$$\begin{aligned}d_{avg} &= \frac{1}{V_{sphere}}\iiint_D d_{(0,0,0)}(x,y,z)\;dV \\ &= \frac{1}{36\pi}\iiint_D \sqrt{x^2+y^2+z^2}\;dV \\ &= \frac{1}{36\pi}\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\;dz\;dy\;dx \\ &= \frac{1}{36\pi}\int_0^{2\pi}\int_0^{3}\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r\sqrt{r^2+z^2}\;dz\;dr\;d\theta \\ &= \frac{1}{36\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{3} \rho^3\sin(\varphi)\;d\rho\;d\varphi\;d\theta\end{aligned}$$

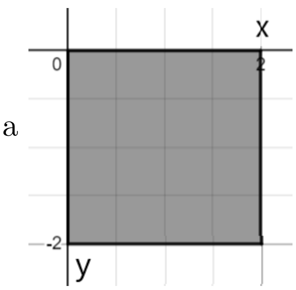
7. Consider the volume D bounded by the planes $x = 0, x = 2, z = -y$ and the surface $z = y^2/2$.



(a) (4 points) Write an integral for the volume of D using Cartesian coordinates in the order $dz\, dy\, dx$. Show your work, including a sketch of the shadow of the region. **Do not evaluate your integral**.

Solution:

The shadow of this region in the xy-plane is a square:



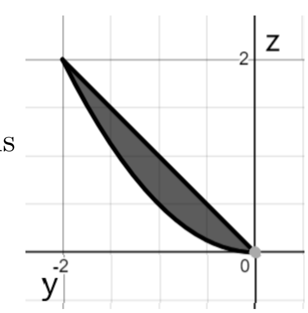
We have

$$V = \int_0^2 \int_{-2}^0 \int_{y^2/2}^{-y} dz\, dy\, dx.$$

(b) (4 points) Write an integral for the volume of D using Cartesian coordinates in the order $dx\, dy\, dz$. Show your work, including a sketch of the shadow of the region. **Do not evaluate your integral**.

Solution:

The shadow of this region in the yz-plane is shown to the right:



We have

$$V = \int_0^2 \int_{-\sqrt{2z}}^{-z} \int_0^2 dx\, dy\, dz.$$

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FORMULA SHEET

- Trig identities: $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$, $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$
- Volume$(D) = \iiint_D dV$, $\quad f_{avg} = \dfrac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$, $\quad$ Mass: $M = \iiint_D \delta\ dV$
- Cylindrical coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $z = z$, $dV = r\ dz\ dr\ d\theta$
- Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$
- First moments (3D solid): $M_{yz} = \iiint_D x\delta\ dV$, $M_{xz} = \iiint_D y\delta\ dV$, $M_{xy} = \iiint_D z\delta\ dV$
- Center of mass (3D solid): $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$
- Substitution for double integrals: If R is the image of G under a coordinate transformation $\mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle$ then

$$\iint_R f(x,y)\ dx\ dy = \iint_G f(\mathbf{T}(u,v))|\det D\mathbf{T}(u,v)|\ du\ dv.$$

- Scalar line integral: $\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- Tangential vector line integral: $\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt$
- Normal vector line integral: $\int_C \mathbf{F}(x,y)\cdot\mathbf{n}\ ds = \int_C P\ dy - Q\ dx = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\langle y'(t), -x'(t)\rangle\ dt$.
- Fundamental Theorem of Line Integrals: $\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Curl (Mixed Partials) Test: $\mathbf{F} = \nabla f$ if curl $\mathbf{F} = 0 \Leftrightarrow P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.
- $\nabla = \langle\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$ $\qquad$ div $\mathbf{F} = \nabla\cdot\mathbf{F}$ $\qquad$ curl $\mathbf{F} = \nabla\times\mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dA \qquad \int_C \mathbf{F}\cdot\mathbf{n}\ ds = \iint_R (\nabla\cdot\mathbf{F})\ dA.$$

- Surface Area=$\iint_S 1\ d\sigma$
- Scalar surface integral: $\iint_S f(x,y,z)\ d\sigma = \iint_R f(\mathbf{r}(u,v))\ |\mathbf{r}_u\times\mathbf{r}_v|dA$
- Flux surface integral: $\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iint_S \mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\ dA$
- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then
$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$

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