MATH 2551-K FINAL EXAM PART 1

VERSION A
FALL 2023
COVERS SECTIONS 12.1-12.6, 13.1-13.4, 14.1

Full name: ______________________ **GT ID:** ______________

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- Please show your work.
- Good luck! Write yourself a message of encouragement on the front page!

For problems 1-3 choose whether each statement is true or false. If the statement is *always* true, pick true. If the statement is *ever* false, pick false. Be sure to neatly fill in the bubble corresponding to your answer choice.

1. (2 points) If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors in $\mathbb{R}^3$, then $|\mathbf{u} \times \mathbf{v}| = 1$.

   ◯ **TRUE**
   ◯ **FALSE**

2. (2 points) All surfaces in $\mathbb{R}^3$ are quadric surfaces.

   ◯ **TRUE**
   ◯ **FALSE**

3. (3 points) Level surfaces of the function $f(x, y, z) = \sqrt{x^2 + y^2}$ are:

   ◯ Circles centered at the origin
   ◯ Spheres centered at the origin
   ◯ Cylinders centered around the z-axis
   ◯ Upper hemispheres centered at the origin
   ◯ None of the above

4. (3 points) Which of the following planes is parallel to the plane $y = -2 - 2x + 4z$?

   ◯ $x + y + 2z = 1$
   ◯ $4x + 2y - 8z = -1$
   ◯ $2x + y + 4z = -2$
   ◯ $2x + 4z = -2$
   ◯ None of the above

5. (10 points) Find the length of the portion of the helix $\mathbf{r}(t) = \langle 2\sin(t), 5t, 2\cos(t)\rangle$ between $(0, 0, 2)$ and $(0, 5\pi, -2)$.

6. (a) (3 points) Find an equation of the plane perpendicular to $\mathbf{n} = \mathbf{i} + \mathbf{j}$ that passes through the point $P = (1, 2, 3)$.

(b) (3 points) Find an equation of the plane parallel to the line $\ell(t) = \langle t + 1, t - 2, 4 \rangle$ that passes through the point $Q = (2, 3, 4)$.

(c) (4 points) Use your work in parts (a) and (b) to explain why there is no plane with normal vector parallel to $\langle 1, 1, 0 \rangle$ that contains both the points P and Q.

7. Consider the curve parameterized by $\mathbf{r}(t) = \langle \sqrt{7}e^t, e^t\cos(t), e^t\sin(t)\rangle$ for $t \in \mathbb{R}$.

   (a) (5 points) Compute the unit tangent vector $\mathbf{T}(t)$.

   (b) (5 points) Compute the curvature $\kappa(t)$.

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FORMULA SHEET

- $\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$
- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta)$
- $\langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin(\theta)|$
- $L = \int_a^b |\mathbf{r}'(t)| \; dt$
- $s(t) = \int_{t_0}^t |\mathbf{r}'(T)| \; dT$
- $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{r}}{ds}$
- $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$
- $\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

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MATH 2551-K FINAL EXAM PART 2

VERSION A
FALL 2023
COVERS SECTIONS 14.2-14.8, 15.1-15.4

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1. (2 points) The total derivative of a function $f : \mathbb{R}^3 \to \mathbb{R}^5$ at the point (a, b, c) is a 5×3 matrix.

   ○ **TRUE**
   ○ **FALSE**

2. (2 points) Any surface that is a graph of a function of two variables $z = f(x, y)$ can be thought of as a level surface of a function of 3 variables.

   ○ **TRUE**
   ○ **FALSE**

3. (3 points) Compute the rate of change of the function $f(x, y, z) = 3xy + z^2$ at the point $(1, 0, 1)$ in the direction of $\langle 2, 1, 2\rangle$.

   ○ 5
   ○ 5/3
   ○ 7
   ○ 7/3
   ○ None of the above.

4. (3 points) Find the linearization of the function $f(x, y) = \sqrt{x^2 + y}$ at the point $(2, 5)$.

   ○ $L(x, y) = 3 + \frac{2}{3}(x - 2) + \frac{1}{6}(y - 5)$
   ○ $L(x, y) = \frac{2}{3}(x - 2) + \frac{1}{6}(y - 5)$
   ○ $L(x, y) = \frac{1}{6}(x - 2) + \frac{1}{6}(y - 5)$
   ○ $L(x, y) = 3 + \frac{x}{\sqrt{x^2 + y}}(x - 2) + \frac{1}{2\sqrt{x^2 + y}}(y - 5)$
   ○ None of the above.

5. (10 points) Find and classify the critical points of the function $f(x, y) = 5x^3 - 3xy + 5y^3$.

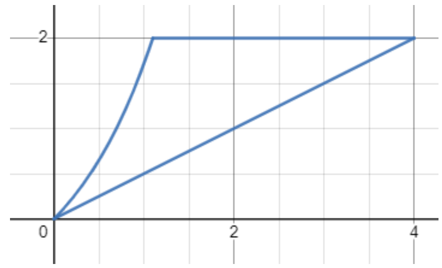
6. For each limit below, either compute its value or show that it does not exist.

   (a) (2 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+3y^4+1}$

   (b) (4 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^2+3y^4}$
   Hint: Try converting to polar coordinates and taking the limit as $r \to 0$

   (c) (4 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+3y^4}$

7. (10 points) The region R bounded by $y = e^x - 1, y = 2$, and $y = x/2$ is shown to the right. Write an iterated integral or sum of iterated integrals for the double integral $\iint_R e^x \, dA$, using either order of integration, then compute your integral. Fully simplify your answer.



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- Total Derivative: For $\mathbf{f}(x_1, \dots, x_n) = \langle f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)\rangle$

$$D\mathbf{f} = \begin{bmatrix} (f_1)_{x_1} & (f_1)_{x_2} & \dots & (f_1)_{x_n} \\ (f_2)_{x_1} & (f_2)_{x_2} & \dots & (f_2)_{x_n} \\ \vdots & \ddots & \dots & \vdots \\ (f_m)_{x_1} & (f_m)_{x_2} & \dots & (f_m)_{x_n} \end{bmatrix}$$

- Linearization: Near $\mathbf{a}$, $L(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

- Chain Rule: If $h = g(f(\mathbf{x}))$ then $Dh(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$

- Implicit Differentiation: $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z}$.

- Directional Derivative: If $\mathbf{u}$ is a unit vector, $D_{\mathbf{u}}f(P) = Df(P)\mathbf{u} = \nabla f(P) \cdot \mathbf{u}$

- The tangent line to a level curve of $f(x, y)$ at (a, b) is $0 = \nabla f(a, b) \cdot \langle x - a, y - b\rangle$

- The tangent plane to a level surface of $f(x, y, z)$ at (a, b, c) is

$$0 = \nabla f(a, b, c) \cdot \langle x - a, y - b, z - c\rangle.$$

- Hessian Matrix: For $f(x, y)$, $Hf(x, y) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

- Second Derivative Test: If (a, b) is a critical point of $f(x, y)$ then
  1. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) < 0$ then f has a local maximum at (a, b)
  2. If $\det(Hf(a, b)) > 0$ and $f_{xx}(a, b) > 0$ then f has a local minimum at (a, b)
  3. If $\det(Hf(a, b)) < 0$ then f has a saddle point at (a, b)
  4. If $\det(Hf(a, b)) = 0$ the test is inconclusive

- Area/volume: $\text{area}(R) = \iint_R dA, \quad \text{volume}(D) = \iiint_D dV$

- Trig identities: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x)), \quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- Average value: $f_{avg} = \dfrac{\iint_R f(x, y)dA}{\text{area of } R}$

- Polar coordinates: $x = r\cos(\theta), \quad y = r\sin(\theta), \quad dA = r\ dr\ d\theta$

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MATH 2551-K FINAL EXAM PART 3

VERSION A
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COVERS SECTIONS 15.1-15.8, 16.1-16.8

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1. (2 points) Every constant vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is conservative.

   ◯ **TRUE**
   ◯ **FALSE**

2. (2 points) If $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = 0$, then $\mathbf{F} = 0$.

   ◯ **TRUE**
   ◯ **FALSE**

3. (3 points) Let $\mathbf{F}(x, y, z) = \langle 3x, -3y, 0 \rangle$ and S be the surface which is the part of the cylinder $y^2 + z^2 = 4$ between $x = -1$ and $x = 10$, oriented away from the x-axis. $\mathbf{F}$ is the curl of a vector field $\mathbf{F}$. Which of the theorems below would be appropriate to use to compute the flux of $\mathbf{F}$ across S?

   ◯ Fundamental Theorem of Line Integrals
   ◯ Green's Theorem (circulation)
   ◯ Green's Theorem (flux)
   ◯ Stokes' Theorem
   ◯ Divergence Theorem

4. (3 points) Let $\mathbf{F}(x, y) = \langle 3x, -3y \rangle$ and C be a simple closed curve surrounding the origin with positive orientation. Which of the theorems below would be appropriate to use to compute the flux of $\mathbf{F}$ across C?

   ◯ Fundamental Theorem of Line Integrals
   ◯ Green's Theorem (circulation)
   ◯ Green's Theorem (flux)
   ◯ Stokes' Theorem
   ◯ Divergence Theorem

5. (14 points) Let $\mathbf{F}(x, y) = \langle y - 3x^2, 2\rangle$ and $\mathbf{F}(x, y) = \langle y^2 e^{xy}, (1 + xy)e^{xy}\rangle$. In this problem you will work with these vector fields and the curve C that is the portion of the parabola $y = 4 - x^2$ starting at $(0, 4)$ and ending at $(2, 0)$.

   (a) Is $\mathbf{F}$ conservative? If so, find a function $f : \mathbb{R}^2 \to \mathbb{R}$ so that $\mathbf{F} = \nabla f$.

   (b) Is $\mathbf{F}$ conservative? If so, find a function $g : \mathbb{R}^2 \to \mathbb{R}$ so that $\mathbf{F} = \nabla g$.

   (c) Compute the work done by $\mathbf{F}$ along the curve C. Fully simplify your answer.

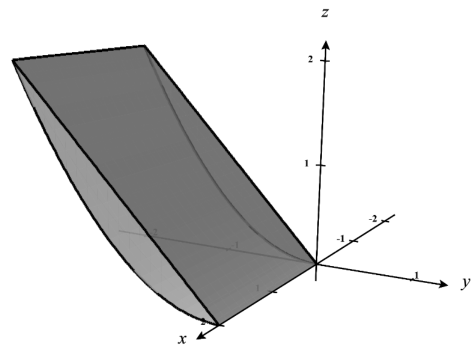
   (d) Compute the work done by $\mathbf{F}$ along the curve C. Fully simplify your answer.

6. Let D be the sphere of radius 3 centered at the origin in $\mathbb{R}^3$. The volume of this sphere is 36π. Suppose that the density of a liquid filling this sphere is $\delta(x, y, z) = 1.2x$ kilograms per cubic meter.

   (a) (4 points) Write an integral expression for the mass of this sphere filled with liquid. **Do not evaluate your integral expression.**

   (b) (4 points) Write an integral expression for the average distance of a point in this sphere from the origin. **Do not evaluate your integral expression.**

7. Consider the volume D bounded by the planes $x = 0, x = 2, z = -y$ and the surface $z = y^2/2$.



(a) (4 points) Write an integral for the volume of D using Cartesian coordinates in the order $dz\ dy\ dx$. Show your work, including a sketch of the shadow of the region. **Do not evaluate your integral**.

(b) (4 points) Write an integral for the volume of D using Cartesian coordinates in the order $dx\ dy\ dz$. Show your work, including a sketch of the shadow of the region. **Do not evaluate your integral**.

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- Trig identities: $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$, $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$
- Volume$(D) = \iiint_D dV$, $\quad f_{avg} = \dfrac{\iiint_D f(x,y,z)dV}{\text{volume of } D}$, $\quad$ Mass: $M = \iiint_D \delta\ dV$
- Cylindrical coordinates: $x = r\cos(\theta)$, $\quad y = r\sin(\theta)$, $z = z$, $dV = r\ dz\ dr\ d\theta$
- Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$
- First moments (3D solid): $M_{yz} = \iiint_D x\delta\ dV$, $M_{xz} = \iiint_D y\delta\ dV$, $M_{xy} = \iiint_D z\delta\ dV$
- Center of mass (3D solid): $(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$
- Substitution for double integrals: If R is the image of G under a coordinate transformation $\mathbf{T}(u,v) = \langle x(u,v), y(u,v)\rangle$ then

$$\iint_R f(x,y)\ dx\ dy = \iint_G f(\mathbf{T}(u,v))|\det D\mathbf{T}(u,v)|\ du\ dv.$$

- Scalar line integral: $\int_C f(x,y,z)\ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- Tangential vector line integral: $\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt$
- Normal vector line integral: $\int_C \mathbf{F}(x,y)\cdot\mathbf{n}\ ds = \int_C P\ dy - Q\ dx = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\langle y'(t), -x'(t)\rangle\ dt$.
- Fundamental Theorem of Line Integrals: $\int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$ if C is a path from A to B
- Curl (Mixed Partials) Test: $\mathbf{F} = \nabla f$ if curl $\mathbf{F} = 0 \Leftrightarrow P_z = R_x, Q_z = R_y$, and $Q_x = P_y$.
- $\nabla = \langle\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$ $\qquad$ div $\mathbf{F} = \nabla\cdot\mathbf{F}$ $\qquad$ curl $\mathbf{F} = \nabla\times\mathbf{F}$
- Green's Theorem: If C is a simple closed curve with positive orientation and R is the simply-connected region it encloses, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\ dA \qquad \int_C \mathbf{F}\cdot\mathbf{n}\ ds = \iint_R (\nabla\cdot\mathbf{F})\ dA.$$

- Surface Area$=\iint_S 1\ d\sigma$
- Scalar surface integral: $\iint_S f(x,y,z)\ d\sigma = \iint_R f(\mathbf{r}(u,v))\ |\mathbf{r}_u\times\mathbf{r}_v|dA$
- Flux surface integral: $\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma = \iint_S \mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\ dA$
- Stokes' Theorem: If S is a piecewise smooth oriented surface bounded by a piecewise smooth curve C and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\ ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ d\sigma.$$

- Divergence Theorem: If S is a piecewise smooth closed oriented surface enclosing a volume D and $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on D, then
$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV.$$

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