MATH 2551-G Exam 3

Fall 2025

EXAM KEY

Read all instructions carefully before beginning.

- Print your name and GT ID neatly above.
- You have 75 minutes to complete as many problems as you wish to attempt.
- You may not use electronic devices of any kind during the exam. You may not use any reference materials other than your single page of hand-written notes you brought to the exam.
- The Learning Targets covered by this exam are listed below.
- Show your work. Answers without work shown will receive a **Not Yet**
- Good luck! Write yourself a message of encouragement on the front page!

Learning Targets

- **G1: Lines and Planes.** I can describe lines using the vector equation of a line. I can describe planes using the general equation of a plane. I can find the equations of planes using a point and a normal vector. I can find the intersections of lines and planes. I can describe the relationships of lines and planes to each other. I can solve problems with lines and planes.
- **G2: Calculus of Curves.** I can compute tangent vectors to parametric curves and their velocity, speed, and acceleration. I can find equations of tangent lines to parametric curves. I can solve initial value problems for motion on parametric curves.
- **G3: Geometry of Curves.** I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.
- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.
- **G5: Parameterization.** I can find parametric equations for common curves, such as line segments, graphs of functions of one variable, circles, and ellipses. I can match given parametric equations to Cartesian equations and graphs. I can parameterize common surfaces, such as planes, quadric surfaces, and functions of two variables.
- **D1: Computing Derivatives.** I can compute partial derivatives, total derivatives, directional derivatives, and gradients. I can use the Chain Rule for multivariable functions to compute derivatives of composite functions.
- **D2: Tangent Planes and Linear Approximations.** I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.

- **D3: Optimization.** I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.
- **I1: Double & Triple Integrals.** I can set up double and triple integrals as iterated integrals over any region. I can sketch regions based on a given iterated integral.
- **I2: Iterated Integrals.** I can compute iterated integrals of two and three variable functions, including applying Fubini's Theorem to change the order of integration of an iterated integral.
- **I3: Change of Variables.** I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.
- **V1: Line Integrals.** I can set up and evaluate scalar and vector field line integrals in two and three dimensions.
- **V2: Conservative Vector Fields.** I can test for conservative vector fields and find potential functions. I can state and apply the Fundamental Theorem of Line Integrals.
- **V3: Generalizations of the FTC.** I can state and apply Green's Theorem, Stokes' Theorem and the Divergence Theorem to solve problems in two and three dimensions. I can choose which theorem is appropriate for different integrals. I can compute curl and divergence of vector fields.
- **V4: Surface Integrals.** I can set up and compute surface integrals for scalar and vector valued functions.
- **A1: Interpreting Derivatives.** I can interpret the meaning of a partial derivative, a gradient, or a directional derivative of a function at a given point in a specified direction, including in the context of a graph or a contour plot.
- **A2: Integral Applications.** I can use multiple integrals to solve physical problems, such as finding area, average value, volume, or the mass or center of mass of a lamina or solid. I can interpret mass, center of mass, work, flow, circulation, flux, and surface area in terms of line and/or surface integrals, as appropriate.

Tasks

1. **[G1: Lines and Planes]** Let p_1 be the plane defined by the equation $5x + 3y - 2z = 91$ and let Q be the point $(2, 1, -1)$.

   (a) Find an equation for the plane p_2 which contains the point Q and is parallel to p_1.

   Solution. The normal vector to p_2 needs to be parallel to the normal vector of p_1, so we can take the same vector. Since the plane must contain Q, we use that as the reference point:

   $$5(x-2) + 3(y-1) - 2(z+1) = 0 \quad \text{or} \quad 5x + 3y - 2z = 15.$$

   (b) Find an equation for the line ℓ which passes through Q and is orthogonal to both planes.

   Solution. Since ℓ is orthogonal to both planes, its direction vector is parallel to the normal of both planes, so we again take the same vector and use Q as our reference point.

   $$\ell(t) = \langle 5, 3, -2 \rangle t + \langle 2, 1, -1 \rangle.$$

   (c) Find the point R where the line ℓ intersects the plane p_1.

   Solution. To find R, we plug the line equation into the plane equation.

   $$\begin{aligned} 5(5t+2) + 3(3t+1) - 2(-2t-1) &= 91 \\ 25t + 10 + 9t + 3 + 4t + 2 &= 91 \\ 38t &= 76 \\ t &= 2 \end{aligned}$$

   So R is $\ell(2) = \langle 12, 7, -5 \rangle$.

   (d) Compute the distance between the two planes using your work in parts (a)-(c) above.

   Solution. The distance between the two planes is the length of any line segment orthogonal to both planes. Thus the segment of ℓ between Q and R gives the distance:

   $$|\vec{QR}| = \sqrt{(12-2)^2 + (7-1)^2 + (-5-(-1))^2} = \sqrt{100 + 36 + 16} = \sqrt{152}.$$

2. **[G2: Calculus of Curves]** For parts (a)-(c), determine whether the statement is true or false and write T or F in the box provided. For part (d), bubble in the multiple choice option that corresponds to your answer.

(a) **T/F:** A smooth curve in the plane that never crosses itself has a single tangent line at a given point.

T

(b) **T/F:** Let $\mathbf{r}(t)$ parameterize a curve C in space with $\mathbf{r}(0) = \langle 1, 2, 3 \rangle$. Then $\mathbf{r}'(t)$ gives a direction vector for the tangent line to the curve at the point $(1, 2, 3)$.

F

(c) **T/F:** The solution to the initial value problem

$$\mathbf{r}''(t) = \sin(t)\mathbf{i} - 2\mathbf{j} + 16e^{4t}\mathbf{k}, \quad -\infty < t < \infty, \quad \mathbf{r}(0) = \langle 0, 0, 0 \rangle, \mathbf{r}'(0) = \langle 0, 0, 0 \rangle$$

is

$$\mathbf{r}(t) = -\sin(t)\mathbf{i} - t^2\mathbf{j} + e^{4t}\mathbf{k}$$

F

(d) Which statement below about motion in space is **not** true?

○ A) A particle with no velocity at a given time may still be accelerating at that time.

○ B) A particle moving with zero acceleration must be moving at constant speed.

○ C) The path of a particle moving along a smooth curve can intersect itself.

✓ D) A particle moving at constant speed must have zero acceleration.

3. **[G3: Geometry of Curves]** For this problem, bubble in the multiple choice option that corresponds to your answer on each part.

(a) Which statement below about arc length is **not** true?

○ The length of a curve C is always positive.

○ The length of a curve C may be equal to the distance between its endpoints.

○ The length of a curve C parameterized by arc length can be greater than 1.

○ Every smooth curve can be parameterized by arc length.

✓ The length of a curve C depends on the parameterization we use to compute it.

(b) Which of the following expressions is the curvature of a curve $x = g(y)$ in the xy-plane?

○ A) $|g''(y)|$

✓ B) $\dfrac{|g''(y)|}{(1+(g'(y))^2)^{3/2}}$

○ C) $\sqrt{1+(g'(y))^2}$

○ D) $\dfrac{|g''(y)|}{\sqrt{1+(g'(y))^2}}$

(c) Which of the following vectors could be the principal unit normal to a curve at a point P where the tangent line is

$$\mathbf{r}(t) = P + \langle 7, 1, -3\rangle t?$$

✓ A) $\dfrac{1}{\sqrt{50}}\langle 1, -7, 0\rangle$

○ B) $\dfrac{1}{\sqrt{59}}\langle 3, 1, 7\rangle$

○ C) $\dfrac{1}{\sqrt{59}}\langle 7, 1, -3\rangle$

○ D) $\langle 0, 3, 1\rangle$

4. **[G4: Surfaces]** For this problem, bubble in the multiple choice option that corresponds to your answer on each part.

   (a) Which option below best describes the domain of the function $f(x, y) = x^2 + y^2 + \sqrt{x - 2 + y^2}$?

   ◯ A) The part of the xy-plane on one side of a vertical line

   ◯ B) All of the xy-plane

   ◯ C) All of the xy-plane except for a disk about the origin

   ✓ D) The part of the xy-plane on one side of a parabola

   ◯ E) The part of the xy-plane on one side of a horizontal line

   (b) The contours of the function $f(x, y) = \arccos(x^2 + y^2)$ are best described as:

   ◯ A) lines

   ✓ B) circles

   ◯ C) sinusoidal curves

   ◯ D) parabolas

   (c) The quadric surface defined by the equation $z^2 = x^2 + y^2 + 4$ is a:

   ◯ A) ellipsoid

   ◯ B) elliptic paraboloid

   ◯ C) hyperbolic paraboloid

   ◯ D) hyperboloid of one sheet

   ✓ E) hyperboloid of two sheets

   ◯ F) cone

5. [**G5: Parameterization**]

(a) Fill in the circle next to **all** of the parameterizations $\mathbf{r}(u,v)$ below corresponding to the surface which is the part of the elliptical paraboloid $x = 4y^2 + z^2$ with $0 \le x \le 4$.

○ A) $\mathbf{r}(u,v) = \langle u, v, 4u^2 + v^2 \rangle$, $0 \le 4u^2 + v^2 \le 4$

○ B) $\mathbf{r}(u,v) = \langle u\cos(v), u\sin(v), 4u^2\cos^2(v) + u^2\sin^2(v) \rangle$, $0 \le u \le 4, 0 \le v \le 2\pi$

✓ C) $\mathbf{r}(u,v) = \langle u, \frac{\sqrt{u}}{2}\cos(v), \sqrt{u}\sin(v) \rangle$, $0 \le u \le 4, 0 \le v \le 2\pi$

✓ D) $\mathbf{r}(u,v) = \langle 4v^2, v\cos(u), 2v\sin(u) \rangle$, $0 \le u \le 2\pi, 0 \le v \le 1$

✓ E) $\mathbf{r}(u,v) = \langle 4u^2 + v^2, u, v \rangle, 0 \le 4u^2 + v^2 \le 4$

(b) Give a parameterization of the boundary of the surface in part (a) that is oriented clockwise around the positive x-axis when viewed looking down the positive x-axis toward the yz-plane. Be sure to give a domain.

Solution. The boundary of the surface is the ellipse $4 = 4y^2 + z^2$ in the plane $x = 4$. A parameterization of this ellipse with clockwise orientation is

$$\mathbf{r}(t) = \langle 4, \sin(t), 2\cos(t) \rangle, 0 \le t \le 2\pi.$$

6. **[D1: Computing Derivatives]** In this problem, you will work with the function $g : \mathbb{R}^3 \to \mathbb{R}$ given by $g(x, y, z) = x^4 + y^3 + z^2$ and the point $P = (2, -4, 4)$ in the domain of g.

   (a) Suppose that you are only able to travel away from P in one of the following directions. Which direction (assuming you move with unit speed) will yield the greatest instantaneous decrease in g?

   - ○ parallel to the x-axis, with x increasing
   - ○ parallel to the y-axis, with y increasing
   - ○ parallel to the z-axis, with z increasing
   - ✓ directly away from the origin

   (b) Justify your answer to part (a).

Solution. This problem is asking in which of the given directions is the directional derivative of g most negative. So we compute.

$$Dg(x, y, z) = \begin{bmatrix} 4x^3 & 3y^2 & 2z \end{bmatrix}, \text{ so } Dg(P) = \begin{bmatrix} 32 & 48 & 8 \end{bmatrix}.$$

We also need a unit vector in each direction; for (A)-(C) these are the standard unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and for (D) it is the vector $\mathbf{u} = \vec{OP}/|\vec{OP}| = \frac{1}{6}\langle -2, 4, -4\rangle$. We then have:

$$\begin{aligned} D_{\mathbf{i}}g(P) &= Dg(P)\mathbf{i} = 32 \\ D_{\mathbf{j}}g(P) &= Dg(P)\mathbf{j} = 48 \\ D_{\mathbf{k}}g(P) &= Dg(P)\mathbf{k} = 8 \\ D_{\mathbf{u}}g(P) &= Dg(P)\mathbf{u} = \frac{1}{6}(2(32) - 4(48) + (4)(8)) = -16 \end{aligned}$$

Of these, 48 is the smallest value, so the direction (iv) yields the greatest instantaneous decrease in g.

7. [**D2: Tangent Planes and Linear Approximations**] Let $f(x, y, z) = yz + x^2 e^{z-y}$.

(a) Find an equation of the tangent plane to the level surface $f = 7$ at the point $P = (\sqrt{e}, 3, 2)$.

Solution. The equation of a tangent plane to a level surface of a function of three variables at a point $P = (x_0, y_0, z_0)$ is $\nabla f(P) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$.

$$\nabla f = \langle 2xe^{z-y}, z - x^2 e^{z-y}, y + x^2 e^{z-y} \rangle,$$

so $\nabla f(P) = \langle 2/\sqrt{e}, 1, 4 \rangle$.
Thus an equation of the tangent plane is

$$\frac{2}{\sqrt{e}}(x - \sqrt{e}) + (y - 3) + 4(z - 2) = 0$$

or

$$\frac{2}{\sqrt{e}}x + y + 4z = 13$$

(b) Find the linearization $L(x, y, z)$ of f at P.

Solution. The linearization $L(x, y, z)$ at P is

$$L(x, y, z) = f(P) + \nabla f(P) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle.$$

$f(P) = 7$ and we computed the rest of this in part (a).
So the linearization is

$$L(x, y, z) = 7 + \frac{2}{\sqrt{e}}(x - \sqrt{e}) + (y - 3) + 4(z - 2).$$

(c) Use the linearization you found to approximate the value of $f(\sqrt{e}, 3.1, 2.1)$.

Solution.

$$f(\sqrt{e}, 3.1, 2.1)) \approx L(\sqrt{e}, 3.1, 2.1)) = 7 + \frac{2}{\sqrt{e}}(0) + (.1) + 4(.1) = 7.5$$

8. [**D3: Optimization**] Determine the largest value of the function $f(x, y) = xy$ such that $x^3 + y^3 = 2$. Explain why f does not achieve a minimum subject to this constraint.

Solution. We can answer this problem using the method of Lagrange multipliers. Our objective function is $f(x, y) = xy$ and our constraint is $g(x, y) = x^3 + y^3 = 2$. We have $\nabla f = \langle y, x\rangle$ and $\nabla g = \langle 3x^2, 3y^2\rangle$. Equating $\nabla f = \lambda \nabla g$, we get the system of equations

$$\begin{cases} y = 3x^2\lambda \\ x = 3y^2\lambda \\ x^3 + y^3 = 2. \end{cases}$$

The nicest variable to isolate is λ. Doing so in the first equation yields two cases: either $x = 0$ or $\lambda = \dfrac{y}{3x^2}$.

<u>Case 1:</u> $x = 0$. From equation 1, we also have $y = 0$. But then equation 3 is false: $0^3 + 0^3 \neq 2$. So this is impossible.

<u>Case 2:</u> $\lambda = \dfrac{y}{3x^2}$. Substitution into equation 2 gives

$$x = 3y^2 \frac{y}{3x^2}.$$

Simplifying, and multiplying by x^2 to clear the fractions gives $x^3 = y^3$ or $x = y$. So equation 3 becomes $2x^3 = 2$, i.e. $x^3 = 1$, i.e. $x = 1$. Then $y = 1$ also and this is the only solution point.

The largest value of $f(x, y)$ subject to this constraint is therefore $f(1, 1) = 1$. f cannot attain a minimum value subject to this constraint because as $x \to -\infty$ along the constraint, $y \to \infty$ and so $f = xy \to -\infty$. Hence there is no minimum value.

9. **[I1: Double & Triple Integrals]** Write an integral for the volume of the finite region bounded by the planes

$$z = x, \quad x + z = 5, \quad z = y, \quad y = 10, \quad \text{and } z = 0.$$

Solution. The Rules for Triple Integrals tell us that this region will be easiest to describe using either x or y as the first variable of integration and z as the last variable. Following the rules generates the following equivalent integrals:

$$\int_0^{5/2} \int_z^{5-z} \int_z^{10} dy\, dx\, dz$$

or

$$\int_0^{5/2} \int_0^{x} \int_z^{10} dy\, dz\, dx + \int_{5/2}^{5} \int_0^{5-x} \int_z^{10} dy\, dz\, dx$$

or

$$\int_0^{5/2} \int_z^{10} \int_z^{5-z} dx\, dy\, dz$$

or

$$\int_0^{5/2} \int_0^{y} \int_z^{5-z} dx\, dz\, dy + \int_{5/2}^{10} \int_0^{5/2} \int_z^{5-z} dx\, dz\, dy$$

or

$$\int_0^{5/2} \int_y^{5-y} \int_0^{y} dz\, dx\, dy + \int_0^{5/2} \int_x^{10} \int_0^{y} dz\, dx\, dy + \int_{5/2}^{5} \int_{5-x}^{10} \int_0^{5-x} dz\, dx\, dy$$

10. **[I2: Iterated Integrals]** Credit for this learning target may be earned either by completing the problem below or by completing successfully the V4 problem.

Compute
$$\int_0^3 \int_0^1 \int_0^{\sqrt{y}} 2xyze^{x^2}\ dx\ dy\ dz.$$

Solution.

$$\begin{aligned}
\int_0^3 \int_0^1 \int_0^{\sqrt{y}} 2xyze^{x^2}\ dx\ dy\ dz &= \int_0^3 \int_0^1 2yz \left(\int_0^{\sqrt{y}} xe^{x^2}\ dx \right) dy\ dz \quad (\text{Let } u = x^2) \\
&= \int_0^3 \int_0^1 2yz \left(\frac{1}{2} e^u \right) \Big|_{u=0}^{u=y} dy\ dz \\
&= \int_0^3 \int_0^1 yz(e^y - 1) dy\ dz \\
& u = y \quad dv = (e^y - 1)dy \\
& du = dy \quad v = e^y - y \\
&= \int_0^3 z \left(y(e^y - y)\big|_{y=0}^{y=1} - \int_0^1 e^y - y\ dy \right)\ dz \\
&= \int_0^3 z \left(ye^y - e^y - \frac{1}{2} y^2 \Big|_{y=0}^{y=1} \right)\ dz \\
&= \int_0^3 z \left(e - e - \frac{1}{2} - 0 + 1 + 0 \right)\ dz \\
&= \int_0^3 \frac{1}{2} z\ dz \\
&= \frac{1}{4} z^2 \Big|_0^3 \\
&= \frac{9}{4}
\end{aligned}$$

11. **[I3: Change of Variables]** Make an appropriate linear change of variables $(x, y) = T(u, v)$ to rewrite

$$\iint_R (9x^2 + 24xy + 16y^2)(x - 5y)\; dx\; dy$$

as an integral in u and v where R is the parallelogram in the fourth quadrant bounded by the lines

$$3x + 4y = 17, \quad 3x + 4y = 77, \quad x - 5y = 227, \quad x - 5y = 314.$$

Do **not** evaluate the integral.

Solution. Either $u = 3x + 4y$ and $v = x - 5y$ or $u = x - 5y$ and $v = 3x + 4y$ will work here. The only change to the final integral will be to swap the names of the variables of integration. We will use the second option.

From this and the given line equations, we see that the region R in the xy-plane corresponds to the rectangular region G in the uv-plane given by $227 \leq u \leq 314$ and $17 \leq v \leq 77$.

The integrand is also nicely expressed in terms of u and v already, since we have

$$(9x^2 + 24xy + 16y^2)(x - 5y) = (3x + 4y)^2(x - 5y) = uv^2.$$

Finally, since the transformation is linear, we can easily find the Jacobian by computing the inverse of the determinant of the matrix of coefficients of this inverse transformation:

$$|\det(DT(u, v))| = \frac{1}{|\det(DT^{-1}(x, y))|} = \frac{1}{\left|\det \begin{bmatrix} 1 & -5 \\ 3 & 4 \end{bmatrix}\right|} = \frac{1}{|4 + 15|} = \frac{1}{19}.$$

Putting all of this together gives the final integral:

$$\int_{17}^{77} \int_{227}^{314} \frac{uv^2}{19}\; du\; dv$$

or, with the other choice of variables,

$$\int_{227}^{314} \int_{17}^{77} \frac{u^2 v}{19}\; du\; dv.$$

12. **[V1: Line Integrals]** Consider the curve C which is the line segment from the point $(1,1)$ to the point $(4,2)$. Use any method you like to answer the following questions.

(a) Compute $\int_C x - y\, ds$.

Solution. A parameterization of C is

$$\mathbf{r}(t) = \langle 1,1\rangle + \langle 3,1\rangle t, \quad 0 \le t \le 1.$$

Then

$$\|\mathbf{r}'(t)\| = \sqrt{(3)^2 + 1^2} = \sqrt{10}.$$

So we have

$$\begin{aligned}\int_C x - y\, ds &= \int_0^1 (3t + 1 - (t+1))\sqrt{10}\, dt \\ &= \sqrt{10}\int_0^1 2t\, dt \\ &= \sqrt{10}\, t^2\Big|_0^1 \\ &= \sqrt{10}\end{aligned}$$

(b) Compute $\int_C (x\mathbf{i} - y\mathbf{j}) \cdot d\mathbf{r}$.

Solution. We can either use the same parameterization as in (a) or notice that we have $\nabla(\frac{1}{2}x^2 - \frac{1}{2}y^2) = x\mathbf{i} - y\mathbf{j}$ and use the Fundamental Theorem of Line Integrals.
Via parameterization:

$$\begin{aligned}\int_C (x\mathbf{i} - y\mathbf{j}) \cdot d\mathbf{r} &= \int_0^1 \langle 3t+1, -t-1\rangle \cdot \langle 3,1\rangle\, dt \\ &= \int_0^1 3(3t+1) - (t+1)\, dt \\ &= \int_0^1 8t + 2\, dt \\ &= 4t^2 + 2t\Big|_0^1 \\ &= 6\end{aligned}$$

Via the Fundamental Theorem of Line Integrals:

$$\int_C (x\mathbf{i} - y\mathbf{j}) \cdot d\mathbf{r} = \frac{1}{2}(x^2 - y^2)\Big|_{(1,1)}^{(4,2)} = \frac{1}{2}(16 - 4) = 6$$

13. **[V2: Conservative Vector Fields]** Let

$$\mathbf{F}(x,y) = \langle 2x\sin(y) + y\cos(x), x^2\cos(y) + \sin(x)\rangle \qquad \mathbf{G}(x,y) = \langle x^2\sin(y), y\cos(x)\rangle.$$

In this problem you will work with these vector fields and the line segement C beginning $(\pi, \pi/2)$ and ending at $(\pi/2, \pi)$.

(a) One of these two fields is conservative. Identify which one and find a potential function for that field. Clearly explain how you know the other field is not conservative.

Solution. Both fields are defined on all of $\mathbb{R}^2$, which is an open, simply connected domain. Therefore we can use the curl test to determine whether each field is conservative. We have

$$\operatorname{curl}\mathbf{F} = \langle 0, 0, 2x\cos(y) + \cos(x) - (2x\cos(y) + \cos(x))\rangle = \langle 0,0,0\rangle,$$

so $\mathbf{F}$ is conservative. On the other hand,

$$\operatorname{curl}\mathbf{G} = \langle 0, 0, -y\sin(x) - x^2\cos(y)\rangle,$$

which is not the zero vector everywhere. Therefore $\mathbf{G}$ is not conservative.
Now we find a potential function for $\mathbf{F}$. We need to find $f(x,y)$ such that

$$f_x = 2x\sin(y) + y\cos(x) \quad \text{and} \quad f_y = x^2\cos(y) + \sin(x).$$

Integrating the first equation with respect to x gives

$$f(x,y) = x^2\sin(y) + y\sin(x) + h(y),$$

for some function $h(y)$. Differentiating this with respect to y gives

$$f_y = x^2\cos(y) + \sin(x) + h'(y).$$

Setting this equal to the second equation above gives $h'(y) = 0$, so $h(y)$ is a constant. Therefore one potential function for $\mathbf{F}$ is

$$f(x,y) = x^2\sin(y) + y\sin(x).$$

(b) Compute the work done by the conservative field along the curve C. Fully simplify your answer.

Solution. The work done is

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= f(\pi/2, \pi) - f(\pi, \pi/2) \\ &= (0 + \pi) - (\pi^2 + 0) \\ &= \pi - \pi^2.\end{aligned}$$

14. **[V3: Generalizations of the FTC]** Consider the vector field

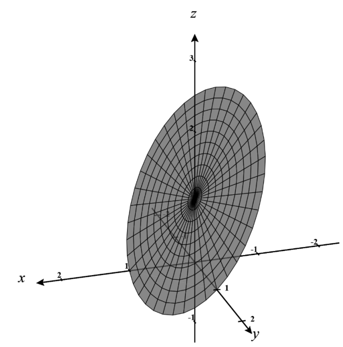
$$\mathbf{F}(x,y,z) = \langle z, x, y\rangle,$$

the surface S which the part of the plane $x+y+z=1$ inside the cylinder $x^2+y^2=1$, oriented with normal in the $\langle 1,1,1\rangle$ direction, and the curve C which is the boundary of S. The curve and surface are pictured below.

(a) Give a parameterization of C which is oriented compatibly with S.

Solution. Since S is oriented with normal away from the origin, we need C to be oriented counterclockwise around the z-axis viewed from above the xy-plane for a compatible orientation. Therefore one compatible parameterization of C is

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), 1-\cos(t)-\sin(t)\rangle, \quad 0 \le t \le 2\pi.$$



(b) Apply Stokes' Theorem with your parameterization from (a) to compute the flux of the curl of $\mathbf{F}$ across S.

Solution. We will need $\mathbf{r}'(t)$ and $\mathbf{F}(\mathbf{r}(t))$ to apply Stokes' Theorem here. These are

$$\mathbf{r}'(t) = \langle -\sin(t), \cos(t), \sin(t)-\cos(t)\rangle \qquad \mathbf{F}(\mathbf{r}(t)) = \langle 1-\cos(t)-\sin(t), \cos(t), \sin(t)\rangle.$$

Now we can apply Stokes' Theorem:

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F})\cdot \mathbf{n}\, d\sigma &= \int_C \mathbf{F}\cdot d\mathbf{r} \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\, dt \\
&= \int_0^{2\pi} \langle 1-\cos(t)-\sin(t), \cos(t), \sin(t)\rangle \cdot \langle -\sin(t), \cos(t), \sin(t)-\cos(t)\rangle\, dt \\
&= \int_0^{2\pi} -\sin(t) + \sin(t)\cos(t) + \sin^2(t) + \cos^2(t) + \sin^2(t) - \sin(t)\cos(t)\, dt \\
&= \int_0^{2\pi} 1 - \sin(t) + \sin^2(t)\, dt \\
&= \int_0^{2\pi} \frac{3}{2} - \sin(t) - \frac{1}{2}\cos(2t)\, dt \\
&= \frac{3}{2}t + \cos(t) - \frac{1}{4}\sin(2t)\Big|_0^{2\pi} \\
&= 3\pi.
\end{aligned}$$

15. [**V4: Surface Integrals**] Compute the flux of the vector field $\mathbf{F} = \langle zx, z, zy \rangle$ across the surface S consisting of the portion of the paraboloid $z = 1 - x^2 - y^2$ with $z \geq 0$, oriented with normal vectors away from the origin. Use any method you like.

Solution. Please don't use $\mathbf{r}_3$; it's a pain. We can either use the definition of flux integrals directly with $\mathbf{r}_1$ or $\mathbf{r}_2$ or use the Divergence Theorem with the closed surface consisting of S and the disk $x^2 + y^2 \leq 1$ in the xy-plane since the flux of the field on this disk is zero.

Using a parameterization: The simplest of these parameterizations to use is $\mathbf{r}_2$, since it will have a nice domain of integration ($\mathbf{r}_1$ will eventually lead to polar coordinates anyway). To use our pullback formula, we need $\mathbf{r}_r \times \mathbf{r}_\theta$:

$$\mathbf{r}_r = \langle \cos(\theta), \sin(\theta), -2r \rangle \qquad \mathbf{r}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0 \rangle.$$

Then taking the cross product gives

$$\mathbf{r}_r \times \mathbf{r}_\theta = \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r \rangle.$$

We now apply the pullback formula:

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \iint_R \mathbf{F}(\mathbf{r}_2(r,\theta)) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 (1-r^2)\langle r\cos(\theta), 1, r\sin(\theta) \rangle \cdot \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r \rangle\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 (1-r^2)(2r^3\cos^2(\theta) + 3r^2\sin(\theta))\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 (2r^3 - 2r^5)\cos^2(\theta) + (3r^2 - 3r^4)\sin(\theta)\, dr\, d\theta \\
&= \int_0^{2\pi} (\frac{1}{2}r^4 - \frac{1}{3}r^6)\cos^2(\theta) + (r^3 - \frac{3}{5}r^5)\sin(\theta)\Big|_{r=0}^{r=1}\, d\theta \\
&= \int_0^{2\pi} \frac{1}{6}\cos^2(\theta) + \frac{2}{5}\sin(\theta)\, d\theta \\
&= \int_0^{2\pi} \frac{1}{12}(1 + \cos(2\theta)) + \frac{2}{5}\sin(\theta)\, d\theta \\
&= \frac{1}{12}\theta + \frac{1}{24}\sin(2\theta) - \frac{2}{5}\cos(\theta)\Big|_0^{2\pi} \\
&= \frac{\pi}{6}.
\end{aligned}$$

16. [**A1: Interpreting Derivatives**] In this problem, you will analyze the derivatives of a height function $h(x, y)$ measured in inches. Here positive x is east and positive y is north, both measured in feet.A contour plot for h is shown below.

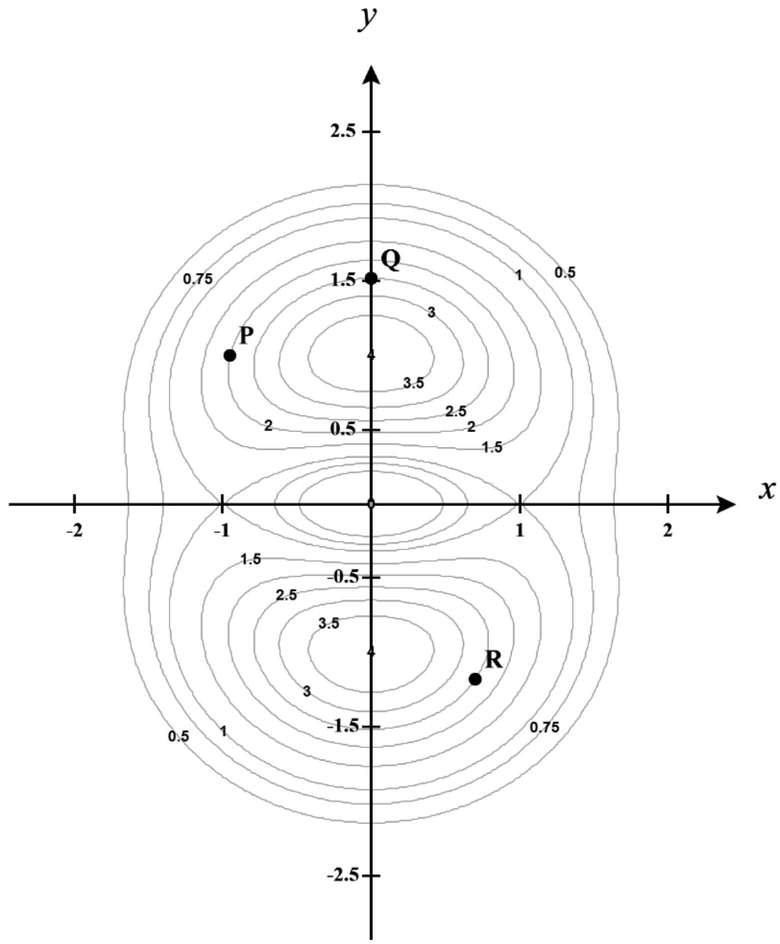
(a) Do you move up, down, or stay level moving north from the point P?

down

(b) Determine the sign $(+, -, 0)$ of the rate of change of h at Q towards $(-1, 0)$.

$+$

(c) Draw a vector which points in the direction of greatest rate of change of height at the point R.



(d) Explain the meaning of the fact that

$$D_{\mathbf{u}}h(-2, 0) \approx 0.422$$

if $\mathbf{u}$ is the unit vector in the $\langle 1, 1 \rangle$ direction.

Solution. At the point $(-2, 0)$, the rate of change of h in the direction $\mathbf{u}$ is approximately 0.422. This means that if you move northeast from that point, you will be moving uphill at a rate of about 0.422 inches per foot moved northeast.

17. [**A2: Integral Applications**]

(a) Give a possible interpretation of the statement

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = -200$$

where S is the surface of a net. Be specific about the physical meaning you choose.

Solution. Many possible solutions. For example, this could represent the net flow of water (in cubic meters per second) through a fishing net submerged in a river, where $\mathbf{F}$ represents the velocity field of the water. The negative sign could then indicate that there is a net flow of water into the net.

(b) Suppose that $\delta(x, y) = x^2 + (2 - y) + 100$ is the density (in kg/m^2) of a lamina occupying the region R in the xy-plane bounded by the lines $x = 0$, $x = 4$, $y = 0$, and $y = 2$.

Explain which quarter of the rectangle contains the center of mass of the lamina. Justify your answer.

Solution. The density is largest when x is large and y is small, since δ increases with x^2 and decreases with y. Therefore, the center of mass will be pulled towards the region where x is largest and y is smallest, which is the lower right quarter of the rectangle (where $2 \le x \le 4$ and $0 \le y \le 1$).

(c) A drone moves through the air (position measured in meters) and experiences a drag force in Newtons given by $\mathbf{F}(x, y)$. We compute that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 50 \text{ J}$$

where C is a closed path taken by the drone. Explain the meaning of this calculation, including units.

Solution. The line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ represents the total work done by the drag force $\mathbf{F}$ on the drone as it moves along the closed path C. The result of 50 J (joules) indicates that the drag force has done 50 joules of work on the drone during its motion along this path. Since the path is closed, this work represents energy lost due to drag forces acting against the drone's motion.

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